On the stability of a nonmultiplicative type sum form functional equation

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Abstract. Our purpose is to obtain all possible general solutions of a sum form functional equation containing two unknown mappings and also discuss criteria for stability of the same.
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1. Introduction

For $n = 1, 2, \ldots$; let

$$\Gamma_n = \left\{ (p_1, \ldots, p_n); p_i \geq 0, i = 1, \ldots, n; \sum_{i=1}^{n} p_i = 1 \right\}$$

denote the set of all n-component discrete probability distributions. Let $\mathbb{R}$ denote the set of real numbers; I denote the unit closed interval $[0, 1]$, i.e. $I = [0, 1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ and I^* denote the interval $]0, 1]$, i.e. $I^* =]0, 1] = \{x \in \mathbb{R} : 0 < x \leq 1\}$.

One of the captivating branches of research work in the field of functional equations with reference to information theory is to study those functional equations that are used to characterize various entropies. For a probability distribution $(p_1, \ldots, p_n) \in \Gamma_n$, the Shannon entropy [15] is defined as follows:

$$H_n(p_1, \ldots, p_n) = -\sum_{i=1}^{n} p_i \log_2 p_i \tag{1.1}$$

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where H_n is a real valued mapping with domain Γ_n, $n = 1, 2, \dots$ and convention $0 \log_2 0 := 0$ is adopted. The functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} f(p_i) + \sum_{j=1}^{m} f(q_j) \tag{1.2}$$

where $f : I \to \mathbb{R}$; $(p_1, \dots, p_n) \in \Gamma_n$, $(q_1, \dots, q_m) \in \Gamma_m$ plays a vital role in characterizing Shannon entropies given by (1.1). Functional equation (1.2) was first observed by Chaundy and McLeod [3] with reference to some statistical thermodynamical problem.

Behara and Nath [2] provided a generalization of the Shannon entropy (1.1) by introducing the phenomenon of entropies of type (α, β). For a probability distribution $(p_1, \dots, p_n) \in \Gamma_n$, an entropy of type (α, β) is defined as follows:

$$H_n^{(\alpha,\beta)}(p_1,\dots,p_n) = \begin{cases} (2^{1-\alpha} - 2^{1-\beta})^{-1}\left(\sum_{i=1}^{n} p_i^{\alpha} - \sum_{i=1}^{n} p_i^{\beta}\right), & \text{if } \alpha \neq \beta \\ -2^{\beta-1}\sum_{i=1}^{n} p_i^{\beta}\log_2 p_i, & \text{if } \alpha = \beta \end{cases} \tag{1.3}$$

where $H_n^{(\alpha,\beta)} : \Gamma_n \to \mathbb{R}$, $n = 1, 2, \dots$; $0 < \alpha \in \mathbb{R}$, $0 < \beta \in \mathbb{R}$ and conventions $0^{\alpha} := 0$, $0^{\beta} := 0$, $1^{\alpha} := 1$, $1^{\beta} := 1$, $0^{\beta} \log_2 0 := 0$ are adopted. An axiomatic characterization of entropies of type (α, β) given by (1.3) leads to the study of the functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} p_i^{\alpha} \sum_{j=1}^{m} f(q_j) + \sum_{j=1}^{m} q_j^{\beta} \sum_{i=1}^{n} f(p_i), \tag{1.4}$$

where f is a real valued mapping with domain I; $(p_1, \dots, p_n) \in \Gamma_n$, $(q_1, \dots, q_m) \in \Gamma_m$; $0 < \alpha \in \mathbb{R}$, $0 < \beta \in \mathbb{R}$, $\alpha \neq 1$, $\beta \neq 1$, $0^{\alpha} := 0$, $0^{\beta} := 0$, $1^{\alpha} := 1$ and $1^{\beta} := 1$.

An extensive literature is available on entropies of type (α, β) which are characterized by functional equation (1.4). Many authors (Behara and Nath [2], Kannappan [6], [7]) studied the functional equation (1.4) and explored its solutions by presuming some conditions on the mapping $f : I \to \mathbb{R}$. Finally without presuming any regularity condition on the mapping $f : I \to \mathbb{R}$, Losonczi and Maksa [9] obtained the general solutions of (1.4) if $\alpha \neq 1$, $\beta \neq 1$ and $n \geq 3$, $m \geq 2$ be fixed integers. Recently few generalizations of (1.4) were considered by Nath and Singh [12], [13]; Singh and Dass [16] who obtained their general solutions for fixed integers $n \geq 3, m \geq 3$.

Also, motivated by the notion of stability Kocsis and Maksa [8] examined for the stability of the functional equation (1.4). For the problem of stability of functional equations we refer to the survey paper of Hyers and Rassias [5].

Intrigued by the functional equation (1.4), we intend to obtain the general solutions of

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) - \sum_{i=1}^{n} p_i^{\alpha} \sum_{j=1}^{m} f(q_j) - \sum_{j=1}^{m} q_j^{\beta} \sum_{i=1}^{n} h(p_i) = 0, \tag{1.5}$$

where $f : I \to \mathbb{R}$, $h : I \to \mathbb{R}$ are unknown mappings; $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3$, $m \geq 2$ be fixed integers; $0 < \alpha \in \mathbb{R}$, $0 < \beta \in \mathbb{R}$, $\alpha \neq 1$, $\beta \neq 1$, $0^\alpha := 0$, $0^\beta := 0$, $1^\alpha := 1$ and $1^\beta := 1$. Also, for fixed integers $n \geq 3$, $m \geq 3$ we discuss the problem of stability of the nonmultiplicative type sum form functional equation (1.5) which is given along the following lines:

Let $n \geq 3$, $m \geq 3$ and $0 \leq \varepsilon \in \mathbb{R}$ be fixed. Find all mappings $f : I \to \mathbb{R}$ and $h : I \to \mathbb{R}$ satisfying the functional inequality

$$\left| \sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i q_j) - \sum_{i=1}^{n} p_i^\alpha \sum_{j=1}^{m} f(q_j) - \sum_{j=1}^{m} q_j^\beta \sum_{i=1}^{n} h(p_i) \right| \leq \varepsilon \tag{1.6}$$

valid for all $(p_1, \ldots, p_n) \in \Gamma_n$ and $(q_1, \ldots, q_m) \in \Gamma_m$.

This paper is divided into four sections. In section 2, we present some auxiliary results which will be used in the subsequent sections. In section 3, we obtain the general solutions of the functional equation (1.5) for $n \geq 3$, $m \geq 2$ being fixed integers. In section 4, we examine the stability of the functional equation (1.5) for fixed integers $n \geq 3$, $m \geq 3$.

2. Auxiliary results

In this section we state few widely known definitions and results.

A mapping $a : I \to \mathbb{R}$ is said to be additive on I if it satisfies the equation $a(x + y) = a(x) + a(y)$ for all $x \in I$, $y \in I$. A mapping $A : \mathbb{R} \to \mathbb{R}$ is said to be additive on $\mathbb{R}$ if it satisfies the equation $A(x + y) = A(x) + A(y)$ for all $x \in \mathbb{R}$, $y \in \mathbb{R}$. Daróczy and Losonczi [4] proved that an additive mapping $a : I \to \mathbb{R}$ has a unique additive extension to the set of real numbers, i.e. there exists a unique mapping $A : \mathbb{R} \to \mathbb{R}$, additive on $\mathbb{R}$ and $A(x) = a(x)$ for all $x \in I$.

A mapping $\ell : I \to \mathbb{R}$ is said to be logarithmic on I if it satisfies $\ell(0) = 0$ and $\ell(xy) = \ell(x) + \ell(y)$ for all $x \in I^*$, $y \in I^*$.

Result 2.1 ([10])**.** Let $n \geq 3$ be a fixed integer. Suppose a mapping $\phi : I \to \mathbb{R}$ satisfies the functional equation $\sum_{i=1}^{n} \phi(p_i) = c$ for all $(p_1, \ldots, p_n) \in \Gamma_n$; c a given real constant. Then there exists an additive mapping $a_1 : \mathbb{R} \to \mathbb{R}$ such that $\phi(p) = a_1(p) - \frac{1}{n} a_1(1) + \frac{c}{n}$ for all $p \in I$.

Result 2.2 ([9])**.** Let $n \geq 3$, $m \geq 2$ be fixed integers and $g : I \to \mathbb{R}$ be a mapping which satisfies the functional equation

$$\sum_{i=1}^{n} \sum_{j=1}^{m} g(p_i q_j) = \sum_{i=1}^{n} g(p_i) \sum_{j=1}^{m} q_j^\beta \tag{A}$$

for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; β being a fixed positive real power with $0^\beta := 0$, $1^\beta := 1$ and $\beta \neq 1$. If $g(0) = 0$, then any general solution of (A) is of the form $g(p) = g(1)p^\beta + a_2(p)$ for all $p \in I$; $a_2 : \mathbb{R} \to \mathbb{R}$ is an additive mapping with $a_2(1) = 0$.

Result 2.3 ([9])**.** Let $H : I \to \mathbb{R}$ be a solution of the functional equation

$$\text{(B)} \qquad H(u)+H(v)=(u+v)^{\beta}\left[H\left(\frac{u}{u+v}\right)+H\left(\frac{v}{u+v}\right)\right]+\frac{u^{\beta}+v^{\beta}}{(u+v)^{\beta}}H(u+v)$$

where $u \geq 0$, $v \geq 0$, $0 < u+v \leq 1$; β being a fixed positive real power satisfying $0^{\beta} := 0$, $1^{\beta} := 1$ and $\beta \neq 1$. If $H(0) = H(1) = 0$, then $H(p) = p^{\beta}\ell(p)$ for all $p \in I$; $\ell : I \to \mathbb{R}$ is a logarithmic mapping.

Result 2.4 ([11])**.** Let $n \geq 3$ be a fixed integer and ε be a fixed positive real number. Suppose a mapping $\psi : I \to \mathbb{R}$ satisfies the functional inequality $|\sum_{i=1}^{n} \psi(p_i)| \leq \varepsilon$ for all $(p_1, \ldots, p_n) \in \Gamma_n$. Then, there exists an additive mapping $a_3 : \mathbb{R} \to \mathbb{R}$ and a bounded mapping $b : \mathbb{R} \to \mathbb{R}$ with $b(0) = 0$, $|b(p)| \leq 18\varepsilon$ such that $\psi(p) - \psi(0) = a_3(p) + b(p)$ for all $p \in I$.

Result 2.5 ([8, 17])**.** Let ε be a fixed positive real number. Suppose a mapping $H : I \to \mathbb{R}$ satisfies the functional inequality

$$\text{(C)} \qquad |H(pq) - p^{\beta}H(q) - q^{\beta}H(p)| \leq \varepsilon$$

for all $p \in I$, $q \in I$; β be a fixed positive real power such that $0^{\beta} := 0$, $1^{\beta} := 1$ and $\beta \neq 1$. Then any solution of (C) is of the form $H(p) = p^{\beta}\ell(p) + b_1(p)$ for all $p \in I$; $\ell : I \to \mathbb{R}$ is a logarithmic mapping and $b_1 : \mathbb{R} \to \mathbb{R}$ is a bounded mapping.

3. The general solution of functional equation (1.5)

Theorem 3.1. *Let $n \geq 3$, $m \geq 2$ be fixed integers and $f : I \to \mathbb{R}$, $h : I \to \mathbb{R}$ be mappings which satisfy the functional equation (1.5) for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; α and β being fixed positive real powers such that $0^{\alpha} := 0$, $1^{\alpha} := 1$, $0^{\beta} := 0$, $1^{\beta} := 1$, $\alpha \neq 1$ and $\beta \neq 1$. Then, for all $p \in I$, any general solution (f, h) of (1.5) is of the form*

$$(\alpha_1) \quad \left.\begin{array}{lll} \text{(i)} & f(p) = f(1)p^{\beta} + p^{\beta}\ell(p) + a_0(p), \quad a_0(1) = 0 & \text{if } \alpha = \beta \\ \text{(ii)} & h(p) = p^{\beta}\ell(p) + \bar{a}_0(p) + h(0), \quad \bar{a}_0(1) = -nh(0) & \end{array}\right\}$$

or

$$(\alpha_2) \quad \left.\begin{array}{lll} \text{(i)} & f(p) = c_1'p^{\alpha} + c_2'p^{\beta} + a_1(p), \quad a_1(1) = 0 & \text{if } \alpha \neq \beta \\ \text{(ii)} & h(p) = c_2'(p^{\beta} - p^{\alpha}) + \bar{a}_1(p) + h(0), \quad \bar{a}_1(1) = -nh(0) & \end{array}\right\}$$

where $a_0 : \mathbb{R} \to \mathbb{R}$, $\bar{a}_0 : \mathbb{R} \to \mathbb{R}$, $a_1 : \mathbb{R} \to \mathbb{R}$ and $\bar{a}_1 : \mathbb{R} \to \mathbb{R}$ are additive mappings; $\ell : I \to \mathbb{R}$ is a logarithmic mapping; c_i' $(i = 1, 2)$ are arbitrary real constants.

Proof. Substituting $q_1 = 1$, $q_2 = \cdots = q_m = 0$ in (1.5), we obtain

$$\sum_{i=1}^{n}\{f(p_i) - (f(1) + (m-1)f(0))p_i^{\alpha} - h(p_i)\} = n(1-m)f(0).$$

By Result 2.1, there exists an additive mapping $a : \mathbb{R} \to \mathbb{R}$ such that

$$h(p) = f(p) - (f(1) + (m-1)f(0))p^{\alpha} - a(p) - f(0) + h(0) \tag{3.1}$$

with

$$a(1) = n(h(0) - mf(0)). \tag{3.2}$$

From equations (1.5), (3.1) and (3.2), we have

$$\begin{aligned} &\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) - \sum_{i=1}^{n} p_i^{\alpha} \sum_{j=1}^{m} f(q_j) - \sum_{j=1}^{m} q_j^{\beta} \sum_{i=1}^{n} f(p_i) \\ &+ (f(1) + (m-1)f(0)) \sum_{i=1}^{n} p_i^{\alpha} \sum_{j=1}^{m} q_j^{\beta} - n(m-1)f(0) \sum_{j=1}^{m} q_j^{\beta} = 0. \end{aligned} \tag{3.3}$$

Now, choosing $p_1 = 1, p_2 = \cdots = p_n = 0$ in (3.3). We obtain $m(n-1)f(0)\big(1 - \sum_{j=1}^{m} q_j^{\beta}\big) = 0$. In view of our presumption that $n \geq 3$, $m \geq 2$, it follows that either $f(0) = 0$ or $1 - \sum_{j=1}^{m} q_j^{\beta}$ vanish identically on Γ_m. Suppose $1 - \sum_{j=1}^{m} q_j^{\beta} = 0$ for all $(q_1, \ldots, q_m) \in \Gamma_m$. Then in particular for a probability distribution $\big(\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0\big) \in \Gamma_m$, we have $\big(\frac{1}{2}\big)^{\beta} = \frac{1}{2}$. This hold only if $\beta = 1$, whereas throughout the paper we have assumed β to be a fixed positive real power such that $\beta \neq 1$. Consequently we arrive at a contradiction and thus have $f(0) = 0$. Thus equation (3.3) reduces to

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) - \sum_{i=1}^{n} p_i^{\alpha} \sum_{j=1}^{m} f(q_j) - \sum_{j=1}^{m} q_j^{\beta} \sum_{i=1}^{n} f(p_i) + f(1) \sum_{i=1}^{n} p_i^{\alpha} \sum_{j=1}^{m} q_j^{\beta} = 0. \tag{3.4}$$

By Result 2.1, there exists a mapping $A : \mathbb{R} \times \Gamma_m \to \mathbb{R}$, additive in the first variable such that

$$\sum_{j=1}^{m} f(pq_j) - p^{\alpha} \sum_{j=1}^{m} f(q_j) - \sum_{j=1}^{m} q_j^{\beta} f(p) + f(1)p^{\alpha} \sum_{j=1}^{m} q_j^{\beta} = A(p; q_1, \ldots, q_m) \tag{3.5}$$

with $A(1; q_1, \ldots, q_m) = 0$. Let $x \in I$ and $(r_1, \ldots, r_m) \in \Gamma_m$. Now replacing $p = xr_t$, $t = 1, \ldots, m$ successively in (3.5); summing the resulting m equations so obtained and then substituting the expression $\sum_{t=1}^{m} f(xr_t)$ calculated from (3.5). We get

$$\begin{aligned} &\sum_{t=1}^{m}\sum_{j=1}^{m} f(xr_t q_j) - (f(x) - f(1)x^{\alpha}) \sum_{t=1}^{m} r_t^{\beta} \sum_{j=1}^{m} q_j^{\beta} = A(x; q_1, \ldots, q_m) \\ &+ x^{\alpha} \left\{ \sum_{t=1}^{m} r_t^{\alpha} \sum_{j=1}^{m} f(q_j) - f(1) \sum_{t=1}^{m} r_t^{\alpha} \sum_{j=1}^{m} q_j^{\beta} + \sum_{j=1}^{m} q_j^{\beta} \sum_{t=1}^{m} f(r_t) \right\} + A(x; r_1, \ldots, r_m) \sum_{j=1}^{m} q_j^{\beta}. \end{aligned}$$

The left hand side of the above equation is symmetric in r_t and q_j, $t = 1, \dots, m$; $j = 1, \dots, m$. So should be its right hand side. Thus, we obtain

$$\begin{aligned} & A(x; q_1, \dots, q_m) \left[1 - \sum_{t=1}^{m} r_t^{\beta}\right] - A(x; r_1, \dots, r_m) \left[1 - \sum_{j=1}^{m} q_j^{\beta}\right] \\ & = x^{\alpha} \left\{ \sum_{t=1}^{m} f(r_t) \left(\sum_{j=1}^{m} q_j^{\alpha} - \sum_{j=1}^{m} q_j^{\beta} \right) - \sum_{j=1}^{m} f(q_j) \left(\sum_{t=1}^{m} r_t^{\alpha} - \sum_{t=1}^{m} r_t^{\beta} \right) \right. \\ & \left. + f(1) \left(\sum_{t=1}^{m} r_t^{\alpha} \sum_{j=1}^{m} q_j^{\beta} - \sum_{j=1}^{m} q_j^{\alpha} \sum_{t=1}^{m} r_t^{\beta} \right) \right\}. \end{aligned} \tag{3.6}$$

Case 1. $\alpha = \beta$.

In this case, equation (3.6) reduces to

$$A(x; q_1, \dots, q_m) \left[1 - \sum_{t=1}^{m} r_t^{\beta}\right] = A(x; r_1, \dots, r_m) \left[1 - \sum_{j=1}^{m} q_j^{\beta}\right].$$

As explained earlier in this section that $1 - \sum_{t=1}^{m} r_t^{\beta} \not\equiv 0$ for $\beta \neq 1$, $0 < \beta \in \mathbb{R}$, there exists a probability distribution $(r_1^*, \dots, r_m^*) \in \Gamma_m$ so that $1 - \sum_{t=1}^{m} r_t^{*\beta} \neq 0$. As a consequence, we have

$$A(x; q_1, \dots, q_m) = a_0(x) \left[1 - \sum_{j=1}^{m} q_j^{\beta}\right] \tag{3.7}$$

where $a_0 : \mathbb{R} \to \mathbb{R}$ defined as $a_0(x) = A(x; r_1^*, \dots, r_m^*) \left[1 - \sum_{t=1}^{m} r_t^{*\beta}\right]^{-1}$ is an additive mapping with $a_0(1) = 0$. Now, from equations (3.5) and (3.7), we get

$$\sum_{j=1}^{m} H(pq_j) - p^{\beta} \sum_{j=1}^{m} H(q_j) - \sum_{j=1}^{m} q_j^{\beta} H(p) = 0 \tag{3.8}$$

where $H : I \to \mathbb{R}$ is a mapping defined as

$$H(x) = f(x) - f(1)x^{\beta} - a_0(x) \tag{3.9}$$

for all $x \in I$. Replacing $x = 0$ and $x = 1$ respectively in (3.9) and using $f(0) = 0$; $a_0(1) = 0$, it follows that $H(0) = 0$ and $H(1) = 0$.

Let $u \geq 0$, $v \geq 0$, $0 < u + v \leq 1$, substituting $p = u + v$, $q_1 = \frac{u}{u+v}$, $q_2 = 1 - q_1$, $q_3 = \cdots = q_m = 0$ in equation (3.8) and using $H(0) = 0$; $H(1) = 0$ equation (B) follows. Hence by applying Result 2.3, $H(u) = u^{\beta}\ell(u)$ for all $u \in I$, where $\ell : I \to \mathbb{R}$ is a logarithmic mapping. Thus using this in (3.9) the solution (α_1)(i) holds. Furthermore, equations (3.1) and (α_1)(i) along with the facts that

$\alpha = \beta$ and $f(0) = 0$ yields the solution (α_1)(ii) by defining an additive mapping $\bar{a}_0 : \mathbb{R} \to \mathbb{R}$ as $\bar{a}_0(p) = a_0(p) - a(p)$ with $\bar{a}_0(1) = -nh(0)$.

Case 2. $\alpha \neq \beta$.

In this case, we may assume that $n \geq m$.

Choosing $p_{m+1} = \cdots = p_n = 0$ in (3.4) and using $f(0) = 0$, we have

$$\sum_{i=1}^{m}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{m} p_i^{\alpha} \sum_{j=1}^{m} f(q_j) + \sum_{j=1}^{m} q_j^{\beta} \sum_{i=1}^{m} f(p_i) - f(1) \sum_{i=1}^{m} p_i^{\alpha} \sum_{j=1}^{m} q_j^{\beta} \tag{3.10}$$

for all $(p_1, \ldots, p_m) \in \Gamma_m$ and $(q_1, \ldots, q_m) \in \Gamma_m$. Clearly, left hand side of (3.10) does not change if we interchange p_i and q_j, $i = 1, \ldots, m$; $j = 1, \ldots, m$. Consequently the right hand side must remain same on interchanging p_i and q_j, $i = 1, \ldots, m$; $j = 1, \ldots, m$. As a result, we get

$$\sum_{i=1}^{m} f(p_i) \left(\sum_{j=1}^{m} q_j^{\alpha} - \sum_{j=1}^{m} q_j^{\beta} \right) - \sum_{j=1}^{m} f(q_j) \left(\sum_{i=1}^{m} p_i^{\alpha} - \sum_{i=1}^{m} p_i^{\beta} \right)$$
$$+ f(1) \left(\sum_{i=1}^{m} p_i^{\alpha} \sum_{j=1}^{m} q_j^{\beta} - \sum_{j=1}^{m} q_j^{\alpha} \sum_{i=1}^{m} p_i^{\beta} \right) = 0.$$

Now we show that for $\alpha \neq \beta$, $\sum_{j=1}^{m} q_j^{\alpha} - \sum_{j=1}^{m} q_j^{\beta}$ does not vanish identically on Γ_m. On the contrary we assume that $\sum_{j=1}^{m} q_j^{\alpha} - \sum_{j=1}^{m} q_j^{\beta} = 0$ for all $(q_1, \ldots, q_m) \in \Gamma_m$. Then, if we choose $q_1 = \frac{1}{2}$, $q_2 = \frac{1}{2}$, $q_3 = \cdots = q_m = 0$, it follows that $\left(\frac{1}{2}\right)^{\alpha} = \left(\frac{1}{2}\right)^{\beta}$ which is possible only when $\alpha = \beta$. Thus our assumption is wrong. Hence there exists a probability distribution $(q_1^*, \ldots, q_m^*) \in \Gamma_m$ so that $\sum_{j=1}^{m} q_j^{*\alpha} - \sum_{j=1}^{m} q_j^{*\beta} \neq 0$. So from the above equation, it follows that

$$\sum_{i=1}^{m} f(p_i) = c_1 \left(\sum_{i=1}^{m} p_i^{\alpha} - \sum_{i=1}^{m} p_i^{\beta} \right) - c_2 \sum_{i=1}^{m} p_i^{\alpha} + c_3 \sum_{i=1}^{m} p_i^{\beta} \tag{3.11}$$

where $c_1, c_2, c_3 \in \mathbb{R}$ and $(p_1, \ldots, p_m) \in \Gamma_m$. Now, define a mapping $g : I \to \mathbb{R}$ as

$$g(x) = f(x) - c_1 x^{\alpha} + c_2 x^{\alpha} - c_3 x^{\beta} \tag{3.12}$$

for all $x \in I$. Clearly $g(0) = 0$. From equations (3.4), (3.11), (3.12) and performing necessary calculations, functional equation (A) follows. Hence by Result 2.2, $g(p) = g(1)p^{\beta} + a_1(p)$ for all $p \in I$, where $a_1 : \mathbb{R} \to \mathbb{R}$ is an additive mapping with $a_1(1) = 0$. Thus using this in equation (3.12), solution (α_2)(i) follows by assuming $c_1' = c_1 - c_2$ and $c_2' = c_3 - c_1$ as arbitrary real constants. Substituting (α_2)(i) in (3.1) and using the fact $f(0) = 0$, we obtain solution (α_2)(ii) by defining an additive mapping $\bar{a}_1 : \mathbb{R} \to \mathbb{R}$ as $\bar{a}_1(p) = a_1(p) - a(p)$ with $\bar{a}_1(1) = -nh(0)$. □

4. The stability of functional equation (1.5)

In this section, we prove:

Theorem 4.1. *Let $n \geq 3$, $m \geq 3$ be fixed integers; ε be a nonnegative real constant and $f : I \to \mathbb{R}$, $h : I \to \mathbb{R}$ be mappings which satisfy the functional inequality* (1.6) *for all $(p_1, \ldots, p_n) \in \Gamma_n$ and $(q_1, \ldots, q_m) \in \Gamma_m$; α and β be fixed positive real powers such that $0^\alpha := 0$, $0^\beta := 0$, $1^\alpha := 1$, $1^\beta := 1$, $\alpha \neq 1$, $\beta \neq 1$. Then, for all $p \in I$, either*

$$\text{(a)}\quad \left.\begin{aligned} &\text{(i)}\quad f(p) = f(1)p^\beta + p^\beta \ell(p) + a_0^*(p) + b_0(p), \quad a_0^*(1) = 0 \\ &\text{(ii)}\quad h(p) = p^\beta \ell(p) + \bar{a}_0^*(p) + \bar{b}_0(p) \end{aligned}\right\} \text{ if } \alpha = \beta$$

or

$$\text{(b)}\quad \left.\begin{aligned} &\text{(i)}\quad f(p) = c(p^\alpha - p^\beta) + a_1^*(p) + b_1(p) \\ &\text{(ii)}\quad h(p) = c(p^\alpha - p^\beta) + \bar{a}_1^*(p) + \bar{b}_1(p) \end{aligned}\right\} \text{ if } \alpha \neq \beta$$

where $a_0^* : \mathbb{R} \to \mathbb{R}$, $\bar{a}_0^* : \mathbb{R} \to \mathbb{R}$, $a_1^* : \mathbb{R} \to \mathbb{R}$ and $\bar{a}_1^* : \mathbb{R} \to \mathbb{R}$ are additive mappings; $b_0 : \mathbb{R} \to \mathbb{R}$, $\bar{b}_0 : \mathbb{R} \to \mathbb{R}$, $b_1 : \mathbb{R} \to \mathbb{R}$ and $\bar{b}_1 : \mathbb{R} \to \mathbb{R}$ are bounded mappings; $\ell : I \to \mathbb{R}$ is a logarithmic mapping and c is an arbitrary real constant.

Proof. Let us put $q_1 = 1$, $q_2 = \cdots = q_m = 0$ in (1.6). We obtain

$$\left| \sum_{i=1}^n f(p_i) + n(m-1)f(0) \sum_{i=1}^n p_i - (f(1) + (m-1)f(0)) \sum_{i=1}^n p_i^\alpha - \sum_{i=1}^n h(p_i) \right| \leq \varepsilon,$$

for all $(p_1, \ldots, p_n) \in \Gamma_n$. By Result 2.4, there exists an additive mapping $\bar{A}_1 : \mathbb{R} \to \mathbb{R}$ and a bounded mapping $\bar{B}_1 : \mathbb{R} \to \mathbb{R}$ with $\bar{B}_1(0) = 0$, such that

$$\begin{aligned} &f(p) + n(m-1)f(0)p - (f(1) + (m-1)f(0))p^\alpha - h(p) - f(0) + h(0) \\ &\quad = \bar{A}_1(p) + \bar{B}_1(p) \end{aligned} \tag{4.1}$$

for all $p \in I$. From equation (4.1), we can easily obtain the expression

$$h(p) = f(p) + A_1(p) + B_1(p) - (f(1) + (m-1)f(0))p^\alpha \tag{4.2}$$

where $A_1 : \mathbb{R} \to \mathbb{R}$ is an additive mapping defined as $A_1(p) = n(m-1)f(0)p - \bar{A}_1(p)$ and $B_1 : \mathbb{R} \to \mathbb{R}$ is a bounded mapping defined as $B_1(p) = -f(0) + h(0) - \bar{B}_1(p)$. From equations (1.6) and (4.2), we have

$$\begin{aligned} \sum_{j=1}^m \Bigg| &\sum_{i=1}^n f(p_i q_j) - \sum_{i=1}^n p_i^\alpha f(q_j) - q_j^\beta \\ &\times \left[\sum_{i=1}^n f(p_i) + A_1(1) + \sum_{i=1}^n B_1(p_i) - (f(1) + (m-1)f(0)) \sum_{i=1}^n p_i^\alpha \right] \Bigg| \leq \varepsilon. \end{aligned}$$

By Result 2.4, there exists a mapping $A_2 : \Gamma_n \times \mathbb{R} \to \mathbb{R}$, additive in the second variable and a bounded mapping $B_2 : \Gamma_n \times \mathbb{R} \to \mathbb{R}$ with $B_2(p_1, \ldots, p_n; 0) = 0$, such that

$$\begin{aligned}&\sum_{i=1}^{n} f(p_i q) - \sum_{i=1}^{n} p_i^{\alpha} f(q) - q^{\beta} \\ &\times \left[\sum_{i=1}^{n} f(p_i) + A_1(1) + \sum_{i=1}^{n} B_1(p_i) - (f(1) + (m-1)f(0)) \sum_{i=1}^{n} p_i^{\alpha}\right] \\ &- nf(0) + \sum_{i=1}^{n} p_i^{\alpha} f(0) = A_2(p_1, \ldots, p_n; q) + B_2(p_1, \ldots, p_n; q).\end{aligned} \tag{4.3}$$

Let $x \in I$ and $(r_1, \ldots, r_n) \in \Gamma_n$. Let us put $q = r_t x$, $t = 1, \ldots, n$ consecutively in (4.3); summing the resulting n equations so obtained and then substituting the value of the expression $\sum_{t=1}^{n} f(r_t x)$ calculated from (4.3). We get

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{t=1}^{n} f(p_i r_t x) - \big[f(x) - f(0) - x^{\beta}(f(1) \\ &+ (m-1)f(0))\big] \sum_{i=1}^{n} p_i^{\alpha} \sum_{t=1}^{n} r_t^{\alpha} - n^2 f(0) \\ &= x^{\beta}\Bigg\{ \left[\sum_{t=1}^{n} f(r_t) + A_1(1) + \sum_{t=1}^{n} B_1(r_t)\right] \sum_{i=1}^{n} p_i^{\alpha} \\ &+ \left[\sum_{i=1}^{n} f(p_i) + A_1(1) + \sum_{i=1}^{n} B_1(p_i)\right] \\ &\times \sum_{t=1}^{n} r_t^{\beta} - (f(1) + (m-1)f(0)) \sum_{i=1}^{n} p_i^{\alpha} \sum_{t=1}^{n} r_t^{\beta}\Bigg\} + A_2(p_1, \ldots, p_n; x) \\ &+ \sum_{t=1}^{n} B_2(p_1, \ldots, p_n; r_t x) + A_2(r_1, \ldots, r_n; x) \sum_{i=1}^{n} p_i^{\alpha} \\ &+ B_2(r_1, \ldots, r_n; x) \sum_{i=1}^{n} p_i^{\alpha}.\end{aligned} \tag{4.4}$$

We observe that the left hand side of (4.4) is symmetric in $(p_1, \ldots, p_n) \in \Gamma_n$ and $(r_1, \ldots, r_n) \in \Gamma_n$. Thus the right hand side of (4.4) must remain unchanged on interchanging p_i and r_t, $i = 1, \ldots, n$; $t = 1, \ldots, n$. Consequently, we have

$$\begin{aligned}&A_2(p_1, \ldots, p_n; x) \left(1 - \sum_{t=1}^{n} r_t^{\alpha}\right) - A_2(r_1, \ldots, r_n; x) \left(1 - \sum_{i=1}^{n} p_i^{\alpha}\right) \\ &= \sum_{i=1}^{n} B_2(r_1, \ldots, r_n; p_i x) + B_2(p_1, \ldots, p_n; x) \sum_{t=1}^{n} r_t^{\alpha}\end{aligned}$$

$$
\begin{aligned}
&-\sum_{t=1}^{n} B_2(p_1,\dots,p_n;r_t x) - B_2(r_1,\dots,r_n;x)\sum_{i=1}^{n} p_i^{\alpha} \\
&+ x^{\beta}\left\{\left[\sum_{i=1}^{n} f(p_i) + A_1(1) + \sum_{i=1}^{n} B_1(p_i)\right]\left(\sum_{t=1}^{n} r_t^{\alpha} - \sum_{t=1}^{n} r_t^{\beta}\right)\right. \\
&- \left[\sum_{t=1}^{n} f(r_t) + A_1(1) + \sum_{t=1}^{n} B_1(r_t)\right]\left(\sum_{i=1}^{n} p_i^{\alpha} - \sum_{i=1}^{n} p_i^{\beta}\right) \\
&\left. -(f(1)+(m-1)f(0))\left[\sum_{i=1}^{n} p_i^{\beta}\sum_{t=1}^{n} r_t^{\alpha} - \sum_{t=1}^{n} r_t^{\beta}\sum_{i=1}^{n} p_i^{\alpha}\right]\right\}.
\end{aligned}
\tag{4.5}
$$

Case 1. $\alpha = \beta$.

In this case, equation (4.5) reduces to

$$
\begin{aligned}
&A_2(p_1,\dots,p_n;x)\left(1-\sum_{t=1}^{n} r_t^{\beta}\right) - A_2(r_1,\dots,r_n;x)\left(1-\sum_{i=1}^{n} p_i^{\beta}\right) \\
&= \sum_{i=1}^{n} B_2(r_1,\dots,r_n;p_i x) + B_2(p_1,\dots,p_n;x)\sum_{t=1}^{n} r_t^{\beta} \\
&\quad -\sum_{t=1}^{n} B_2(p_1,\dots,p_n;r_t x) - B_2(r_1,\dots,r_n;x)\sum_{i=1}^{n} p_i^{\beta}.
\end{aligned}
$$

For fixed $(p_1,\dots,p_n) \in \Gamma_n$ and $(r_1,\dots,r_n) \in \Gamma_n$, the left hand side of the above equation is additive in x but the right hand side is bounded on I. Thus the left hand side is linear (Aczél [1], Sahoo and Kannappan [14]). We obtain

$$
\begin{aligned}
&[A_2(p_1,\dots,p_n;x) - xA_2(p_1,\dots,p_n;1)]\left[1-\sum_{t=1}^{n} r_t^{\beta}\right] \\
&= [A_2(r_1,\dots,r_n;x) - xA_2(r_1,\dots,r_n;1)]\left[1-\sum_{i=1}^{n} p_i^{\beta}\right].
\end{aligned}
\tag{4.6}
$$

As described in the previous section 3 that $1-\sum_{t=1}^{n} r_t^{\beta} \not\equiv 0$ for $\beta \neq 1$, $0 < \beta \in \mathbb{R}$, there exists a probability distribution $(r_1^*,\dots,r_n^*) \in \Gamma_n$ such that $1-\sum_{t=1}^{n} r_t^{*\beta} \neq 0$. Using this fact in equation (4.6), we have

$$
A_2(p_1,\dots,p_n;x) = xA_2(p_1,\dots,p_n;1) + a_0^*(x)\left[1-\sum_{i=1}^{n} p_i^{\beta}\right]
\tag{4.7}
$$

where $a_0^* : \mathbb{R} \to \mathbb{R}$ defined as $a_0^*(x) = [A_2(r_1^*,\dots,r_n^*;x) - xA_2(r_1^*,\dots,r_n^*;1)] \times \left[1-\sum_{t=1}^{n} r_t^{*\beta}\right]^{-1}$ is an additive mapping with $a_0^*(1) = 0$. Making use of the fact that $\alpha = \beta$ and $1^\beta := 1$ in equation (4.3), we obtain the expression

$$\begin{aligned} A_2(p_1,\dots,p_n;1) &= -A_1(1) - \sum_{i=1}^n B_1(p_i) \\ &\quad + mf(0)\sum_{i=1}^n p_i^\beta - nf(0) - B_2(p_1,\dots,p_n;1). \end{aligned} \tag{4.8}$$

Equations (4.3), (4.7), (4.8) together with the fact that $\alpha = \beta$ and $a_0^*(1) = 0$, yield

$$\begin{aligned} &\sum_{i=1}^n H(p_i q) - \sum_{i=1}^n p_i^\beta H(q) - q^\beta \sum_{i=1}^n H(p_i) \\ &= q^\beta \left[nf(0) + A_1(1) + \sum_{i=1}^n B_1(p_i) \right] \\ &+ q\bigg[-A_1(1) - \sum_{i=1}^n B_1(p_i) + mf(0)\sum_{i=1}^n p_i^\beta \\ &- nf(0) - B_2(p_1,\dots,p_n;1) \bigg] + B_2(p_1,\dots,p_n;q), \end{aligned} \tag{4.9}$$

where $H : I \to \mathbb{R}$ is a mapping defined as

$$H(x) = f(x) - a_0^*(x) - f(0) - (f(1) + (m-1)f(0))x^\beta \tag{4.10}$$

for all $x \in I$. Clearly, from (4.10) $H(0) = 0$ follows. Since the right hand side of (4.9) is bounded, we apply Result 2.4 and using $H(0) = 0$, we have

$$H(pq) - p^\beta H(q) - q^\beta H(p) = A_3(p,q) + B_3(p,q) \tag{4.11}$$

for all $p \in I$ and $q \in I$; where $A_3 : \mathbb{R} \times I \to \mathbb{R}$ is a mapping additive in the first variable and $B_3 : \mathbb{R} \times I \to \mathbb{R}$ is a bounded mapping with $B_3(0,q) = 0$. Define a mapping $G : I \times I \to \mathbb{R}$ as

$$G(p,q) = H(pq) - p^\beta H(q) - q^\beta H(p) \tag{4.12}$$

for all $p \in I$ and $q \in I$. Making use of (4.12), we can easily verify that

$$G(r,pq) + r^\beta G(p,q) = G(rp,q) + q^\beta G(r,p). \tag{4.13}$$

From equations (4.11), (4.12) and (4.13), we have

$$\begin{aligned} &A_3(r,pq) - A_3(rp,q) - q^\beta A_3(r,p) \\ &= B_3(rp,q) + q^\beta B_3(r,p) - B_3(r,pq) - r^\beta A_3(p,q) - r^\beta B_3(p,q). \end{aligned} \tag{4.14}$$

The right hand side of (4.14) is bounded while its left hand side is an additive mapping in r so it must be linear (Aczél [1], Sahoo and Kannappan [14]). Hence

$$A_3(r,pq)-A_3(rp,q)-q^{\beta}A_3(r,p)=r[A_3(1,pq)-A_3(p,q)-q^{\beta}A_3(1,p)]. \tag{4.15}$$

Now equation (4.14) with $r=1$, results in

$$A_3(1,pq)-q^{\beta}A_3(1,p)=q^{\beta}B_3(1,p)-B_3(1,pq). \tag{4.16}$$

From equations (4.14), (4.15) and (4.16), it follows that

$$\begin{aligned}(r^{\beta}-r)A_3(p,q)=B_3(rp,q)+q^{\beta}B_3(r,p)-B_3(r,pq)-r^{\beta}B_3(p,q)\\ -rq^{\beta}B_3(1,p)+rB_3(1,pq).\end{aligned}$$

In view of our assumption that $\beta\neq 1$, the above equation yields that the additive mapping $A_3(p,q)$ is bounded in p on I. Thus as argued earlier it follows that it must be linear (Aczél [1], Sahoo and Kannappan [14]). Hence $A_3(p,q)=pA_3(1,q)$ for all $p\in I$ and $q\in I$. However equation (4.16) with the replacement $p=1$ yields that the mapping $q\to A_3(1,q)$ is bounded on I. As a consequence the mapping $A_3(p,q)$ is bounded on $I\times I$. Apparently, follows same for the mapping G on $I\times I$ and thus by applying Result 2.5 on (4.11), there exists a logarithmic mapping $\ell: I\to\mathbb{R}$ and a bounded mapping $B_4:\mathbb{R}\to\mathbb{R}$ such that $H(p)=p^{\beta}\ell(p)+B_4(p)$ for all $p\in I$. Thus from (4.10), (a)(i) holds where bounded mapping $b_0:\mathbb{R}\to\mathbb{R}$ is defined as $b_0(p)=B_4(p)+f(0)+(m-1)f(0)p^{\beta}$. From (4.2), (a)(i) and $\alpha=\beta$, we obtain solution (a)(ii) where an additive mapping $\bar{a}_0^*:\mathbb{R}\to\mathbb{R}$ is defined as $\bar{a}_0^*(p)=a_0^*(p)+A_1(p)$; a bounded mapping $\bar{b}_0:\mathbb{R}\to\mathbb{R}$ is defined as $\bar{b}_0(p)=b_0(p)+B_1(p)-(m-1)f(0)p^{\beta}$.

Case 2. $\alpha\neq\beta$.

In this case, there is no loss of generality in assuming $n\geq m$.

Let us put $p_{m+1}=\cdots=p_n=0$ in (1.6) and making use of (4.2), we have

$$\begin{aligned}&\Bigg|\sum_{i=1}^{m}\sum_{j=1}^{m}f(p_iq_j)-\sum_{i=1}^{m}p_i^{\alpha}\sum_{j=1}^{m}f(q_j)-\sum_{j=1}^{m}q_j^{\beta}\sum_{i=1}^{m}f(p_i)\\ &+(f(1)+(m-1)f(0))\sum_{i=1}^{m}p_i^{\alpha}\sum_{j=1}^{m}q_j^{\beta}+m(n-m)f(0)\\ &-[(n-m)f(0)+A_1(1)+(n-m)B_1(0)]\sum_{j=1}^{m}q_j^{\beta}-\sum_{i=1}^{m}B_1(p_i)\sum_{j=1}^{m}q_j^{\beta}\Bigg|\leq\varepsilon,\end{aligned} \tag{4.17}$$

for all $(p_1, \ldots, p_m) \in \Gamma_m$ and $(q_1, \ldots, q_m) \in \Gamma_m$. Now on interchanging p_i and q_j, $i = 1, \ldots, m$; $j = 1, \ldots, m$ in functional inequality (4.17), we obtain

$$\begin{aligned}
&\Bigg|\sum_{i=1}^{m}\sum_{j=1}^{m} f(p_i q_j) - \sum_{j=1}^{m} q_j^{\alpha} \sum_{i=1}^{m} f(p_i) - \sum_{i=1}^{m} p_i^{\beta} \sum_{j=1}^{m} f(q_j) \\
&+ (f(1) + (m-1)f(0)) \sum_{j=1}^{m} q_j^{\alpha} \sum_{i=1}^{m} p_i^{\beta} + m(n-m)f(0) \qquad (4.18)\\
&-[(n-m)f(0) + A_1(1) + (n-m)B_1(0)] \sum_{i=1}^{m} p_i^{\beta} - \sum_{j=1}^{m} B_1(q_j) \sum_{i=1}^{m} p_i^{\beta}\Bigg| \le \varepsilon\,.
\end{aligned}$$

Now, applying triangle inequality to inequalities (4.17) and (4.18), we get

$$\begin{aligned}
&\Bigg|\left(\sum_{j=1}^{m} q_j^{\alpha} - \sum_{j=1}^{m} q_j^{\beta}\right) \sum_{i=1}^{m} f(p_i) - \left(\sum_{i=1}^{m} p_i^{\alpha} - \sum_{i=1}^{m} p_i^{\beta}\right) \sum_{j=1}^{m} f(q_j) \\
&+ (f(1) + (m-1)f(0)) \left(\sum_{i=1}^{m} p_i^{\alpha} \sum_{j=1}^{m} q_j^{\beta} - \sum_{i=1}^{m} p_i^{\beta} \sum_{j=1}^{m} q_j^{\alpha}\right) \\
&+ [(n-m)f(0) + A_1(1) + (n-m)B_1(0)] \left(\sum_{i=1}^{m} p_i^{\beta} - \sum_{j=1}^{m} q_j^{\beta}\right) \\
&- \sum_{i=1}^{m} B_1(p_i) \sum_{j=1}^{m} q_j^{\beta} + \sum_{j=1}^{m} B_1(q_j) \sum_{i=1}^{m} p_i^{\beta}\Bigg| \le 2\varepsilon.
\end{aligned}$$

Now, as explained in the previous section 3 that for $\alpha \neq \beta$, $\sum_{j=1}^{m} q_j^{\alpha} - \sum_{j=1}^{m} q_j^{\beta}$ does not vanish identically on Γ_m. Hence we choose a probability distribution $(q_1^*, \ldots, q_m^*) \in \Gamma_m$ such that $\sum_{j=1}^{m} q_j^{*\alpha} - \sum_{j=1}^{m} q_j^{*\beta} \neq 0$ and $\varepsilon' = 2\varepsilon \left[\sum_{j=1}^{m} q_j^{*\alpha} - \sum_{j=1}^{m} q_j^{*\beta}\right]^{-1}$. Hence the above inequality reduces to

$$\left|\sum_{i=1}^{m} f(p_i) - c_1 \sum_{i=1}^{m} p_i^{\alpha} + c_2 \sum_{i=1}^{m} p_i^{\beta} - c_3 - c_4 \sum_{i=1}^{m} B_1(p_i)\right| \le \varepsilon',$$

where $c_1, c_2, c_3, c_4 \in \mathbb{R}$ and $(p_1, \ldots, p_m) \in \Gamma_m$. By Result 2.4, there exists an additive mapping $A_4 : \mathbb{R} \to \mathbb{R}$ and a bounded mapping $B_5 : \mathbb{R} \to \mathbb{R}$ with $B_5(0) = 0$ satisfying

$$f(p) - c_1 p^{\alpha} + c_2 p^{\beta} - c_3 p - c_4 B_1(p) - f(0) + c_4 B_1(0) = A_4(p) + B_5(p)$$

for all $p \in I$. Thus (b)(i) holds where $a_1^* : \mathbb{R} \to \mathbb{R}$ defined as $a_1^*(p) = A_4(p) + c_3 p$, for all $p \in \mathbb{R}$ is an additive mapping; $b_1 : \mathbb{R} \to \mathbb{R}$ defined as $b_1(p) = B_5(p) +$

$c_4B_1(p) - c_4B_1(0) + f(0) + (c_1 - c_2)p^\beta$, for all $p \in \mathbb{R}$ is a bounded mapping and $c = c_1 \in \mathbb{R}$. From equations (b)(i) and (4.2), (b)(ii) follows where $\bar{a}_1^* : \mathbb{R} \to \mathbb{R}$ defined as $\bar{a}_1^*(p) = a_1^*(p) + A_1(p)$, for all $p \in \mathbb{R}$ is an additive mapping and $\bar{b}_1 : \mathbb{R} \to \mathbb{R}$ defined as $\bar{b}_1(p) = b_1(p) + B_1(p) - (f(1) + (m-1)f(0))p^\alpha$, for all $p \in \mathbb{R}$ is a bounded mapping. □

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