Some results for best coapproximation on Banach lattices

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Abstract. In this paper we introduce the concept of best coapproximation on Banach lattices with a strong unit. We study the existence problem of best coapproximation in these spaces. Also, we develop the theory of best coapproximation in quotient of Banach lattice spaces and discuss about the relationship between the coproximinal elements of a given space and its quotient space. Finally, we show that every lattice isomorphism is an coapproximation preserving operator.

Keywords: best coapproximation, Banach lattice spaces, downward set, quotient spaces.

1. Introduction and preliminaries

The theory of best approximation is an important topic in functional analysis. It is very extensive field which has many applications in mathematics and some other sciences. An interesting question in this field is: under what conditions on the set G does the point $g \in G$ nearest from point x in the spaces exist and this is unique? One of an important conditions on G is convexity. However, convexity is sometimes a very restrictive assumption, so there is a clear need to study the best approximation by not necessarily convex sets. Downward sets are nonconvex sets which play an important role in some parts of mathematical economics and game theory. Recently, many authors have been studied the notions of best approximation, best simultaneous approximation and farthest points for downward sets in Banach lattice spaces ([3],[5],[6],[9],[10],[11]). Another kind of approximation, called best coapproximation was introduced in normed linear spaces by C. Franchetti and M. Furi [4]. The theory of best coapproximation is much less developed as compared to the theory of best approximation in abstract spaces. Some results on best coapproximation theory in metric spaces and linear normed spaces have obtained by P.L. Papini, T.D. Narang, and others ([1],[7],[8],[12],[13],[14],[15],[16]).

In this parer, we develop the theory of best coapproximation by elements of closed downward sets, in a Banach lattice with a strong unit. Also, we introduce the notion of best coapproximation in quotient Banach lattice spaces. We shall determine some conditions which coproximinality can be transmitted to quotient spaces and vice versa. Finally, we obtain some conditions for maps under which preserve best coapproximation by downward subsets of Banach lattices.

A real vector space X is said to be an ordered vector space whenever it is equipped with an order relation $\leq$ (i.e., $\leq$ is a reflexive, antisymmetric, and transitive binary relation on X). A vector lattice space (or a Riesz space) is an ordered vector space X with the additional property that for each pair of vectors $x, y \in X$, the $\sup\{x, y\}$ and the $\inf\{x, y\}$ both exist in X. As usual, $\sup\{x, y\}$ is denoted by $x \vee y$ and $\inf\{x, y\}$ by $x \wedge y$. Recall that a vector subspace W of a vector lattice space X is said to be a vector sublattice, whenever W is closed under the lattice operations of X, i.e., whenever for each pair $x, y \in W$ the vector $x \vee y$ and $x \wedge y$ (taken in X) belongs to W. A subset A of a vector lattice space is called solid whenever $|x| \leq |y|$ and $y \in A$ imply $x \in A$. A solid vector subspace of a vector lattice space is referred to as an ideal. Note that from the lattice identity $x \vee y = \frac{1}{2}(x + y + |x - y|)$, it follows immediately that every ideal is a vector lattice subspace.

For any vector x in a vector lattice space define $x^+ := x \vee 0$, $x^- := x \wedge 0$ and $|x| := x \vee (-x)$. The element x^+ is called the positive part, x^- is called the negative part, and $|x|$ is called the absolute value of x. If X is an ordered vector space, then the set $X^+ = \{x \in X \ : \ x \geq 0\}$ is called a positive cone of X, and its members are called the positive elements of X. In a vector lattice space, two elements x and y are said to be disjoint (in symbols $x \perp y$) whenever $|x| \wedge |y| = 0$ holds. For each $x \in X$ we define $x^\perp = \{y \in X : |x| \wedge |y| = 0\}$. It is easy to show that $x^\perp$ is a closed vector sublattice of X.

An element $\mathbf{1} \in X$ is called a strong unit if for each $x \in X$ there exists $0 < \lambda \in \mathbb{R}$ such that $x \leq \lambda \mathbf{1}$. Then for each $x \in X$ there exists $0 < \lambda \in \mathbb{R}$ such that $|x| \leq \lambda \mathbf{1}$. Using $\mathbf{1}$ we can define a norm on X by

$$\|x\| = \inf\{\lambda > 0 : \ |x| \leq \lambda \mathbf{1}\}. \tag{1}$$

Recall that a norm $\|.\|$ on a vector lattice space is said to be a lattice norm whenever $|x| \leq |y|$ implies $\|x\| \leq \|y\|$. A vector lattice space equipped with a lattice norm is known as a normed vector lattice space. If a normed vector lattice space is also norm complete, then it is referred to as a Banach lattice. It is well known that X equipped with the norm (1) is a Banach lattice which is called a Banach lattice with strong unit $\mathbf{1}$.

The closed ball with center at x and radius r defined on Banach lattice X as follows:

$$B(x, r) = \{y \in X \ : \ \|y - x\| \leq r\} = \{y \in X \ : \ x - r\mathbf{1} \leq y \leq x + r\mathbf{1}\}. \tag{2}$$

Let A be an ideal in Banach lattice space X. We recall that the equivalence class determined by x in $\frac{X}{A}$ will be denoted by $\dot{x} = x + A$. In $\frac{X}{A}$ we introduce a relation $\dot{\leq}$ by letting $\dot{x} \dot{\leq} \dot{y}$ whenever there exist $x_1 \in \dot{x}$ (i.e., $x_1 - x \in A$) and $y_1 \in \dot{y}$ with $x_1 \leq y_1$. Clearly, $\frac{X}{A}$ under the relation $\dot{\leq}$ is an ordered vector space and it is easy to show that $\frac{X}{A}$ is a vector lattice space.

Let A be a closed ideal of a Banach lattice space X. Then the vector lattice space $\frac{X}{A}$ under the quotient norm

$$\|\dot{x}\| = \inf\{\|y\| : y \in \dot{x}\}, \tag{3}$$

is a Banach lattice space. In fact, the quotient vector space $\frac{X}{A}$ is itself a Banach lattice space.

A linear operator $T : X \longrightarrow Y$ between two lattice vector spaces X and Y is said to be a lattice (or Riesz) homomorphism whenever $T(x \vee y) = T(x) \vee T(y)$ holds for all $x, y \in X$. Note that every lattice homomorphism is a positive operator, i.e., it carries positive vectors to positive vectors. This is equivalent to saying that every lattice homomorphism is order preserving, i.e., $x \leq y$ implies that $T(x) \leq T(y)$, and also it is equivalent to

$$|T(x)| = T(|x|) \tag{4}$$

for all $x \in X$. It is important to note that the range of a lattice homomorphism is a lattice subspace. A lattice homomorphism which is in addition one-to-one is referred to as a lattice isomorphism. Clearly, the map $x \to \dot{x}$, from X to $\frac{X}{A}$, is a linear operator called the canonical projection of X onto $\frac{X}{A}$. The lattice homomorphisms are closely related to ideals. For every ideal A of a Banach lattice space X, the canonical projection of X onto the Banach lattice space $\frac{X}{A}$ is a lattice homomorphism (for more details see [2]).

Let X be a normed vector space. For a nonempty subset G of X and $x \in X$, define

$$d(x, G) = \inf_{g \in G} \|x - g\|.$$

A point $g_0 \in G$ is called a best approximation for $x \in X$ if

$$\|x - g_0\| = d(x, G).$$

If each $x \in X$ has at least one best approximation $g_0 \in G$, then G is called a proximinal subset of X. Let $G \subset X$. For $x \in X$, the set of all best approximations of x in G,is denote by $P_G(x)$; that is

$$P_G(x) = \{g \in G : \|x - g\| = d(x, G)\}.$$

A counterpart to best approximation in normed linear spaces is best coapproximation which is defined as follows:

Let W be a nonempty subset of a normed linear space X. An element $w_0 \in W$ is called a best coapproximation to $x \in X$ from W if for every $w \in W$,

$$\|w - w_0\| \leq \|x - w\|.$$

The set of all elements of best coapproximation to $x \in X$ from W is denoted by $R_W(x)$. If each $x \in X$ has at least one best coapproximation $w_0 \in W$, then W is called a coproximinal subset of X.

2. Best coapproximation by downward sets

In this section we show that a closed downward set is coproximinal and obtain some results on best coapproximation elements.

Definition 2.1. *A nonempty subset W of an ordered vector space X is called downward if*

$$(w \in W\ ,\ x \leq w) \Longrightarrow x \in W.$$

A simple example of a downward set is a set of the form $\{y \in X\ :\ y \leq g\}$, where $g \in X$. For another example, let $f : X \longrightarrow \mathbb{R}$ be an increasing function, then its lower level sets $S_c(f) = \{x \in X : f(x) \leq c\}$ for all $c \in \mathbb{R}$, are downward.

Proposition 2.1. *Let W be a downward and coproximinal subset of Banach lattice space X and $w_1, w_2 \in R_W(x)$. Then $w_1 \wedge w_2 \in R_W(x)$.*

Proof. Since W is a downward subset of X, it follows that $w_1 \wedge w_2 \in W$. Due to the relations $|x \wedge y| \leq |x| \vee |y|$ and $x + y \wedge z = (x + y) \wedge (x + z)$, for each $w \in W$ we have

$$\begin{aligned} |w - w_1 \wedge w_2| &= |(w - w_1) \wedge (w - w_2)| \\ &\leq |w - w_1| \vee |w - w_2| \\ &\leq |x - w| \vee |x - w| = |x - w|. \end{aligned}$$

This implies that $\|w - w_1 \wedge w_2\| \leq \|x - w\|$ for all $w \in W$. Hence $w_1 \wedge w_2 \in R_W(x)$ and the proof is complete. □

Corollary 2.1. *Let W be a vector sublattice and $R_W(x)$ be a subspace of Banach lattice space X, then $R_W(x)$ is a vector sublattice of X.*

Proof. Similar to the proof of Proposition 2.1, we can prove that $w_1 \wedge w_2 \in R_W(x)$ and $w_1 \vee w_2 \in R_W(x)$. This yields that $R_W(x)$ is a vector sublattice of X. □

Theorem 2.1. *Let W be a nonempty subset of Banach lattice space X and $x \in X \setminus W$. Then $w_0 \in R_W(x)$ if and only if for every $w \in W$,*

$$w - r_w \mathbf{1} \leq w_0 \leq w + r_w \mathbf{1}$$

where $r_w = \|x - w\|$.

Proof. Due to (2) for all $w \in W$, we have $w_0 \in R_W(x) \Longleftrightarrow \|w_0 - w\| \leq \|x - w\| = r_w \Longleftrightarrow -r_w\mathbf{1} \leq w_0 - w \leq r_w\mathbf{1} \Longleftrightarrow w - r_w\mathbf{1} \leq w_0 \leq w + r_w\mathbf{1}$. □

Theorem 2.2. *Let W be a closed downward subset of Banach lattice X. Then for each $x \in X$ such that $x \geq W$, W is a coproximinal subset of X.*

Proof. Let $x_0 \in X$ be arbitrary such that $x_0 \geq W$ and $r := d(x_0, W) = \inf_{w \in W} \|x_0 - w\| > 0$. Then, for each $\varepsilon > 0$ there exists $w_\varepsilon \in W$ such that $\|x_0 - w_\varepsilon\| < r + \varepsilon$. Now, by (1), we have

$$w_\varepsilon \geq x_0 - (r + \varepsilon)\mathbf{1}.$$

Set $w_0 = x_0 - r\mathbf{1}$. So we have $w_\varepsilon \geq x_0 - r\mathbf{1} - \varepsilon\mathbf{1} = w_0 - \varepsilon\mathbf{1}$. We claim that $w_0 \in R_W(x_0)$. Since W is a downward set and $w_\varepsilon \in W$, it follows that $w_0 - \varepsilon\mathbf{1} \in W$ for all $\varepsilon > 0$. Closedness of W implies that $w_0 \in W$. Now, suppose that $w \in W$ be arbitrary. In view of $\|x_0 - w\| = r_w \leq r + r_w$, $r \leq r_w$ and (1.2), we have

$$(r - r_w)\mathbf{1} \leq 0 \leq x_0 - w \leq (r + r_w)\mathbf{1}.$$

This implies that for each $w \in W$,

$$w - r_w\mathbf{1} \leq w_0 = x_0 - r\mathbf{1} \leq w + r_w\mathbf{1}.$$

Thus, by Theorem 2.1, $w_0 \in R_W(x_0)$ and so W is coproximinal. □

Example 2.3. Let $X = \mathbb{R}$ and $W = (-\infty, 0]$. Then W is a closed downward set in X and so is coproximinal and we have $R_W(1) = [-1, 0]$.

Let S be a vector sublattice of X, we define

$$\check{S} = \{x \in X : \|w\| \leq \|x - w\| \ \ \forall w \in S\} = R^{-1}(\{0\}).$$

Proposition 2.2. *Let S be a vector sublattice of Banach lattice X. Then we have $S \cap \check{S} = \{0\}$.*

Proof. Suppose that $x \in S \cap \check{S}$. Since $x \in \check{S}$ so by definition of $\check{S}$ for all $w \in S$ we have $\|w\| \leq \|x - w\|$. On the other hand, since $x \in S$, by replacing w by x we obtain $\|x\| \leq \|x - x\| = 0$. Therefore $x = 0$ and hence $S \cap \check{S} = \{0\}$. □

Proposition 2.3. *Let S be a vector sublattice of Banach lattice X. Then for all $w \in S$ we have $d(w, \check{S}) = \|w\|$.*

Proof. Suppose that $x \in \check{S}$. Then for all $w \in S$ we have $\|w\| \leq \|x - w\|$. Fixed $w \in S$ and so for all $x \in \check{S}$ we obtain

$$\|w\| \leq \inf_{x \in S \cap \check{S}} \|x - w\| = d(w, \check{S}) \leq \|w\|.$$

It follows that $d(w, \check{S}) = \|w\|$. □

Corollary 2.2. *Let S be a vector sublattice of Banach lattice X. Then for all $w \in S$ we have $0 \in P_{\check{S}}(w)$.*

Proposition 2.4. *Let X be a Banach lattice. Then for all $x \in X$ we have $d(x, x^\perp) = \|x\|$.*

Proof. Suppose that $x \in X$, then in view of the relation $|x| = |x - y + y| \leq |x - y| + |y|$ for all $y \in x^{\perp}$ we have

$$\begin{aligned} |x| = |x| \wedge |x| &\leq |x| \wedge (|x - y| + |y|) \\ &= (|x - y| \wedge |x|) + (|x| \wedge |y|) \\ &= |x - y| \wedge |x| \leq |x|. \end{aligned}$$

Hence $|x| = |x - y| \wedge |x|$ and so $|x| \leq |x - y|$. Therefore for all $y \in x^{\perp}$, we have $\|x\| \leq \|x - y\|$ which implies that $d(x, x^{\perp}) = \|x\|$. □

Corollary 2.3. *Let S be a vector sublattice of Banach lattice X. Then for all $x \in X$ we have $0 \in P_{x^{\perp}}(x)$.*

Theorem 2.4. *Let S be a coproximinal vector sublattice of Banach lattice X. If $\check{S}$ is a vector sublattice of X, then $\check{S}$ is proximinal.*

Proof. Suppose that S be coproximinal, $x \in X$ and $w_1 \in R_S(x)$. Then $w_2 = x - w_1 \in \check{S}$ and so $x = w_1 + w_2$ where $w_1 \in S$ and $w_2 \in \check{S}$. In view of Proposition 2.3, we have

$$\|x - w_2\| = \|w_1\| = d(w_1, \check{S}) = d(x - w_2, \check{S}) = d(x, \check{S}).$$

Thus, $w_2 \in P_{\check{S}}(x)$, and therefore $\check{S}$ is proximinal. □

3. Best coapproximation in quotient spaces

In this section we discuss about coproximality of sublattices in quotient of Banach lattice spaces.

Theorem 3.1. *Let A be an ideal in Banach lattice X. Then the following statements are equivalent:*

(i) A is coproximinal.

(ii) The canonical mapping $Q : X \longrightarrow \frac{X}{A}$ maps $\check{A}$ onto $\frac{X}{A}$, i.e. $Q(\check{A}) = \frac{X}{A}$.

Proof. (i)$\Longrightarrow$ (ii). Let $x + A \in \frac{X}{A}$ be an arbitrary element and $w_0 \in R_A(x)$. Then $x - w_0 \in \check{A}$ and we have

$$Q(x - w_0) = (x - w_0) + A = x + A.$$

(ii)$\Longrightarrow$ (i). Suppose that $x \in X$.Then $x + A \in \frac{X}{A} = Q(\check{A})$, i.e., $x + A = Q(y)$ where $y \in \check{A} = R_A^{-1}(0)$. So $x + A = y + A$ and $0 \in R_A(y)$. Thus, $x - y = w_0 \in A$ and $0 \in R_A(x - w_0)$. This implies that $x - w_0 \in \check{A}$ and therefore $w_0 \in R_A(x)$. Hence A is coproximinal. □

Theorem 3.2. *A subset W of a vector lattice space X is downward in X if and only if $\frac{W}{A}$ is downward in $\frac{X}{A}$.*

Proof. Suppose that W is downward set. Let $\dot{x}, \dot{y} \in \frac{W}{A}$ such that $\dot{x} \dot{\leq} \dot{y}$ and $\dot{y} \in \frac{W}{A}$. Since $\dot{x} \dot{\leq} \dot{y}$, it follows that there exist $x_1 \in \dot{x}$ and $y_1 \in \dot{y}$ such that $x_1 \leq y_1$. In view of $\dot{y_1} = \dot{y} \in \frac{W}{A}$ we have $y_1 \in W$. Since W is downward set so $x_1 \in W$.Therefore $\dot{x} = \dot{x_1} \in \frac{W}{A}$and hence $\frac{W}{A}$ is a downward set in $\frac{X}{A}$.

Conversely, let $\frac{W}{A}$ be a downward set. Assume that $x, y \in W$such that $x \leq y$ and $y \in W$, so $\dot{y} \in \frac{W}{A}$. Since the canonical map is order preserving we have $\dot{x} \dot{\leq} \dot{y}$. Now, since $\frac{W}{A}$ is a downward set, it follows that $\dot{x} = x + A \in \frac{W}{A}$ and hence $x \in W$. Therefore W is downward set. □

Corollary 3.1. *Let W be a closed and downward subset and A be an ideal of X. Then $\frac{W}{A}$is coproximinal in $\frac{X}{A}$.*

Proof. In view of Theorem 3.2, $\frac{W}{A}$ is a downward subset of $\frac{X}{A}$. By continuity of the canonical map $Q : X \longrightarrow \frac{X}{A}$ we have $\frac{W}{A}$ is closed. Therefore, due to Theorem 2.2, $\frac{W}{A}$is coproximinal in $\frac{X}{A}$. □

Theorem 3.3. *Let Sbe a sublattice and A be an closed ideal in Banach lattice X such that $A \subseteq S$. If $S$is coproximinal in X, then $\frac{S}{A}$ is a coproximinal sublattice of $\frac{X}{A}$.*

Proof. In view of (3), $\frac{S}{A}$ is a Banach sublattice of $\frac{X}{A}$. Now we assume that $w_0 \in R_W(x)$and $w \in S$. Then for each $a \in A$ we have

$$\begin{aligned} \|\dot{w_0} - \dot{w}\| &= \|(w_0 - w) + A\| \\ &\leq \|w_0 - (w - a)\| \\ &\leq \|x - (w - a)\| = \|(x - w) + a\|. \end{aligned}$$

So for each $\dot{w} \in \frac{S}{A}$ we have $\|\dot{w_0} - \dot{w}\| \leq \|\dot{x} - \dot{w}\|$and therefore $\dot{w_0} \in R_{\frac{S}{A}}(\dot{x})$. Hence $\frac{S}{A}$ is coproximinalin $\frac{X}{A}$. □

Theorem 3.4. *Let S be a sublattice and A be an closed and proximinal ideal in Banach lattice X such that $A \subseteq S$. If $\frac{S}{A}$is coproximinal in $\frac{X}{A}$, then S is a coproximinal sublattice of X.*

Proof. Let $\dot{x} \in \frac{X}{A}$and $\dot{w_0} \in R_{\frac{S}{A}}(\dot{x})$. Then for each $\dot{w} \in \frac{S}{A}$we have

$$\|\dot{w_0} - \dot{w}\| \leq \|\dot{x} - \dot{w}\|$$

or in the other word

$$\|(w_0 - w) + A\| \leq \|(x - w) + A\|.$$

Now, since A is proximinal in X, we conclude that there exist $a_0 \in A$ such that

$$\|(w_0 - w) + a_0\| = \|(w_0 - w) + A\| \leq \|(x - w) + A\| \leq \|x - w\|.$$

Then for each $w \in S$ we have

$$\|(w_0 + a_0) - w\| \leq \|x - w\|.$$

So $w_0 + a_0 \in R_S(x)$ and therefore S is coproximinal in X. □

Theorem 3.5. *Let S be a sublattice and A be an closed ideal in Banach lattice X such that $A \subseteq S$. If S is coproximinal in X, then we have*

$$Q(R_S(x)) \subseteq R_{\frac{S}{A}}(Q(x)),$$

In particular if A is proximinal in X, then

$$Q(R_S(x)) = R_{\frac{S}{A}}(Q(x)).$$

Proof. Due to Theorem 3.3, it is obvious that $Q(R_S(x)) \subseteq R_{\frac{S}{A}}(Q(x))$. Now, let A is proximinal, $\dot{x} \in \frac{X}{A}$ and $\dot{w_0} \in R_{\frac{S}{A}}(\dot{x})$.In view of the Theorem 3.4, there exist $a_0 \in A$ such that $w_0 + a_0 \in R_S(x)$. Now, we have $\dot{w_0} = Q(w_0 + a_0) \in Q(R_S(x))$. Hence $R_{\frac{S}{A}}(Q(x)) \subseteq Q(R_S(x))$ and the proof is complete. □

4. Coapproximation preserving operators

In this section we shall obtain characterization of coapproximation preserving maps on Banach lattices.

The following lemma characterizes the maps which preserve downwardness of downward subsets of vector lattices.

Lemma 4.1 ([9])**.** (1) *Let X and Y be two vector lattices and $T : X \longrightarrow Y$ be an injective positive operator, such that T^{-1} is a positive operator. Then W is a downward subset of X if and only if $T(W)$ is a downward subset of Y.*

(2) *If $T : X \longrightarrow X$ is a positive operator and $f : X \longrightarrow \mathbb{R}$ is an increasing function, then $S_c(f \circ T) = \{x \in X : f \circ T(x) \leq c\}$ for all $c \in \mathbb{R}$ are downward.*

The next theorem which has been proved in [2], page 94, described necessary and sufficient conditions for an operator between two vector lattice spaces, that is a lattice isomorphism.

Theorem 4.1. *Assume that an operator $T : X \longrightarrow Y$ between two vector lattice spaces is one-to-one and onto. Then T is a lattice isomorphism if and only if T and T^{-1} are both positive operators.*

The following proposition which is a consequence of Theorem 4.1 and relations (1), (4), plays a key role in proving our results.

Proposition 4.1. *Let X and Y be two Banach lattices with strong units $\mathbf{1}_X$ and $\mathbf{1}_Y$, respectively and $T : X \longrightarrow Y$ be an injective positive operator such that T^{-1} is positive and $T(\mathbf{1}_X) = \mathbf{1}_Y$. Then, T is a norm isometry, i.e. $\|T(x)\| = \|x\|$ for all $x \in X$.*

Definition 4.1. *Let X and Y be Banach lattices with strong units $\mathbf{1}_X$ and $\mathbf{1}_Y$, respectively. A linear operator $T : X \longrightarrow Y$ is called a coapproximation preserving operator if for all downward sets W in X and all $x \in X$:*

(*i*) *W is a downward subset of X if and only if $T(W)$ is a downward subset of Y.*

(*ii*) *$T(R_W(x)) = R_{T(W)}(T(x))$.*

Theorem 4.2. *Let X and Y be Banach lattices with strong units $\mathbf{1}_X$ and $\mathbf{1}_Y$, respectively. Let $T : X \longrightarrow Y$ be an injective positive operator which T^{-1} is a positive operator and $T(\mathbf{1}_X) = \mathbf{1}_Y$. Then T is a coapproximation persevering operator.*

Proof. In view of the Proposition 4.1, we have $\|T(x)\| = \|x\|$, for all $x \in X$. Then for all $x \in X$ and all downward sets W of X and $w, w_0 \in W$,

$$\|w - w_0\| \leq \|x - w\| \Leftrightarrow \|T(w) - T(w_0)\| \leq \|T(x) - T(w)\|.$$

Therefore, $T(R_W(x)) = R_{T(W)}(T(x))$. Also by Lemma 4.1, W is downward subset of X if and only if $T(W)$ is a downward subset of Y. □

Corollary 4.1. *Let X and Y be two Banach lattices with strong units $\mathbf{1}_X$ and $\mathbf{1}_Y$, respectively. Every lattice isomorphism between X and Y is a coapproximation preserving operator.*

Proof. From Theorems 4.1 and 4.2, the result obtained. □

Example 4.3. Let X be a Banach lattice with strong unit $\mathbf{1}$ and $a \in X$. Define

$$T_a : X \longrightarrow X, \quad T_a(x) = a + x.$$

Then $T^{-1} = T_{-a}$. Suppose that W is a downward subset of X; then $T_a(W) = a+ W$ is a downward subset of X, and if $T(W)$ is downward, then $W = T_{-a}(T(W))$ is downward. Also for all $w, w_0 \in W$ and $x \in X$ we have

$$\|T_a(w_0) - T_a(w)\| \leq \|T_a(x) - T_a(w)\| \Longleftrightarrow \|w_0 - w\| \leq \|x - w\|.$$

Then $T(R_W(x)) = R_{T(W)}(T(x))$ and therefore T_a is a coapproximation preserving operator on X, for all $a \in X$.

Theorem 4.4. *Let X and Y be two Banach lattices with strong units $\mathbf{1}_X$ and $\mathbf{1}_Y$ respectively. If $T : X \longrightarrow Y$ is a coapproximation preserving operator, then W is a downward and coproximinal subset of X if and only if $T(W)$ is a downward and coproximinal subset of Y.*

Corollary 4.2. *The canonical mapping $Q : X \longrightarrow \frac{X}{A}$ is a coapproximation preserving operator.*

Proof. From Theorems 3.2 and 3.5, it is obvious. □

Definition 4.2. *A subset G of the cone X^+ is called normal if*

$$(g \in G, x \in X^+, x \leq g) \Longrightarrow x \in G.$$

For example, if f is an increasing function defined on X^+, then its lower level sets $S_c^+(f) = \{x \in X^+ : f(x) \leq c\}$ for all c are normal.

If $T : X \longrightarrow Y$ is a map between vector lattices X and Y, we assume that $T^+ = T|_{X^+}$. Then for all $x \in X^+$,

$$T(x) = T^+(x). \tag{5}$$

Lemma 4.2. (1) *Let X and Y be vector lattices and $T : X \longrightarrow Y$ be an injective positive operator such that T^{-1} is positive. Then G is closed normal subset of X^+ if and only if $T^+(G)$ is a closed normal subset of Y^+.*

(2) *Let $T : X \longrightarrow X$ be a positive operator; then $S_c^+(f \circ T^+)$ are normal for all $c \in \mathbb{R}$.*

Theorem 4.5. *Let X and Y be two Banach lattices with strong units $\mathbf{1}_X$ and $\mathbf{1}_Y$, respectively. Let $T : X \longrightarrow Y$ be an injective positive operator such that T^{-1} is positive and $T(\mathbf{1}_X) = \mathbf{1}_Y$. Then for all normal subset G of $X^+$and for all $x \in X^+$,*

$$T^+(R_G(x)) = R_{T^+(G)}(T^+(x)).$$

That is, T^+ preserves coapproximation.

Proof. It is clear from (5), Lemma 4.2 and Theorem 4.2. □

Conclusion. In this paper, we obtained some results on best coapproximation by element of downward sets and showed that every closed downward sets are coproximinal. Also, we developed of the theory of best coapproximation in quotient of Banach lattice spaces and give a necessary and sufficient condition about the coproximinal elements of downward sets of a given space and its quotient space. Finally, We have presented coapproximation preserving operators on Banach lattices and have shown that every lattice isomorphism is an approximation preserving operator. In particular, these operators preserve coproximinal downward subsets of X and normal subsets of X^+. For the future research, study on the uniqness of best coapproximation elements by downward sets is recommended.

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