A new interior-point method for $P_*(\kappa)$ linear complementarity problems based on a parameterized kernel function

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Abstract. In this paper, we propose a primal-dual interior-point method (IPM) for $P_*(\kappa)$ linear complementarity problems (LCPs) based on a new parameterized kernel function which is a generalization of the one presented by Bai et al. [A primal-dual interior-point method for linear optimization based on a new proximity function, Optim. method softw. (2002)]. A simple analysis shows that the iteration bound for large-update method obtained by Cho based on the original kernel function [A new large-update interior point algorithm for $P_*(\kappa)$ linear complementarity problems, J. Comput. Appl. Math. (2008)] is improved from $O((1+2\kappa)n^{\frac{3}{4}}\log\frac{n}{\epsilon})$ to $O((1+2\kappa)\sqrt{n}\log n\log\frac{n}{\epsilon})$, and the small-update method has $O((1+2\kappa)\sqrt{n}\log\frac{n}{\epsilon})$ iteration complexity. These are the currently best known complexity results for such methods. Some numerical results have been provided.
Keywords: interior-point method, $P_*(\kappa)$ LCPs, large-update method, parameterized kernel function.

1. Introduction

In this paper, we consider the following LCP in the standard form:

$$(1) \qquad s = Mx + q, \quad xs = 0, \quad x \geq 0,\ s \geq 0.$$

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where $x, s, q \in R^n$, $M \in R^{n\times n}$, xs denotes the componentwise (Hadamard) product of vectors x and s.

Problem (1.1) is called a $P_*(\kappa)$ LCP if M is a $P_*(\kappa)$ matrix, i.e. there exists a nonnegative number κ such that

$$(1+4\kappa)\sum_{i\in J_+} x_i(Mx)_i + \sum_{i\in J_-} x_i(Mx)_i \geq 0,$$

where $J = \{1, 2, ..., n\}$, $J_+ = \{i \in J : x_i(Mx)_i \geq 0\}$ and $J_- = \{i \in J : x_i(Mx)_i < 0\}$. Furthermore, we define $P_* = \bigcup_{\kappa\geq 0} P_*(\kappa)$. Note that the class P_* includes the class PSD of positive semi-definite matrices, and the class of P-matrices with all the principal minors positive.

Since the groundbreaking paper of Karmarkar in 1984 [1], some scholars and researchers are committed to the study of IPMs and a slew of results have been proposed (see [2, 3]). $P_*(\kappa)$ LCP is the largest class of LCPs which can be solved by interior-point methods (IPMs) in polynomial time. Indeed, the Karush-Kuhn-Tucker (KKT) optimality conditions for linear optimization (LO) and convex quadratic optimization can be formulated as LCPs, and LCPs are also closed related to variational inequalities. The basic theory, algorithms, and applications about LCPs can be referred in [3, 4, 5].

For a long time, most of the IPMs are designed based on the classical logarithmic barrier function. However, there is a gap between the theoretical behavior and practical computational efficiency of the algorithm: the large-update IPMs have better practical performance than the small-update IMPs, but theoretical result is relatively weaker. This gap was significantly reduced by Peng et al. who introduced the so-call self-regular kernel function and obtained the currently best known iteration bound of the large-update method for LO [6]. Subsequently, Bai et al. presented a series of eligible kernel functions and gave a scheme to analyze the complexity bound of the primal-dual IPMs for LO. They also obtained the best known iteration bound for large-update method [7]. Moreover, El Ghami et al. [8] first introduced a trigonometric kernel function, it is shown that in the IPMs based on this kernel function for large-update methods, the iteration bound is $O(n^{\frac{3}{4}}\log\frac{n}{\epsilon})$..

With the development of IPMs, the methods of solving LO based on a kernel function have been extended to some complementarity problems (CPs), which includes LCPs and nonlinear complementarity problems (NCPs), as well as some other important optimization problems such as semidefinite and second order cone optimization problems. For instance, Ji et al. proposed a large-update primal-dual IPM for $P_*(\kappa)$ LCPs based a new class kernel function [9], which is neither classical logarithmic nor self-regular function. Cai et al. introduced a new parametried kernel function with trigonometric barrier term [10], which yields a class of large-udate and small-update IPMs for the Cartesian $P_*(\kappa)$-LCP over symmetric cones. Li designed a large-update IPM for solving the $P_*(\kappa)$ nonlinear complementarity problem (NCP) based on a new class of parameterized kernel functions [11], the result coincides with the currently best known iteration bounds for such methods. Achache et al. presented a full-Newton feasible step IPM for solving monotone horizontal linear complementarity problems [12]. For more studies about IPMs based on kernel functions please refer to [14, 13, 15, 16, 17, 18, 19].

The best known iteration bound for large-update IPMs was obtained so far based on some kernel functions in Table 1.

Table 1. Several known parameterized kernel functions

kernel function	parameter	iteration bound	References
$\frac{t^2-1}{2}+\frac{t^{1-q}-1}{q-1}, q>1$	$q=\frac{1}{2}\log n$	$O(\sqrt{n}\log n\log\frac{n}{\epsilon})$	[20],[21]
$\frac{t^2-1}{2}+\frac{t^{1-q}-1}{q(q-1)}-\frac{q-1}{q}(t-1), q>1$	$q=\frac{1}{2}\log n$	$O(\sqrt{n}\log n\log\frac{n}{\epsilon})$	[20],[21]
$\frac{t^2-1}{2}+\frac{1}{b}(e^{b(1-t)}-1),\ b\geq 0$	$b=O(\log n)$	$O(\sqrt{n}\log n\log\frac{n}{\epsilon})$	[22]
$\frac{t^2-1}{2}-\int_1^t\left(\frac{e-1}{e^x-1}\right)^p dx,\ p\geq 1,$	$p=O(\log n)$	$O(\sqrt{n}\log n\log\frac{n}{\epsilon})$	[17]
$\frac{t^2-1}{2}-\int_1^t e^{q(\frac{1}{\xi}-1)}d\xi,\ q\geq 1$	$q=O(\log n)$	$O(\sqrt{n}\log n\log\frac{n}{\epsilon})$	[23]
$\frac{t^2-1}{2}+\frac{1}{q^2}\left(\frac{q}{t}-1\right)e^{q(\frac{1}{t}-1)}-\frac{q-1}{q^2}, q\geq 1$	$q=O(\log n)$	$O(\sqrt{n}\log n\log\frac{n}{\epsilon})$	[24]
$\frac{t^2-1}{2}+\frac{q^{\frac{1}{t}-1}-1}{\log q},\ q>1$	$q=1+O(n)$	$O(\sqrt{n}\log n\log\frac{n}{\epsilon})$	[25]

Note that all kernel functions in Table 1 depend on a parameter. Moreover, the last three kernel functions in the above table are all generalized by the original kernel function when $q=1$, and the iteration bounds for large-update IPMs are reduced from $O(\sqrt{n}\log^2 n\log\frac{n}{\epsilon})$ to $O(\sqrt{n}\log n\log\frac{n}{\epsilon})$. In fact, as indicated in the recent studies about primal-dual IPMs based on kernel functions, the parameter not only represents a barrier degree of the kernel function, but also plays a role in improving the iteration complexity and practical computational efficiency of the algorithm.

Motivated by these work, in this paper we introduce a parameterized version of a kernel function that was previously presented by Bai et al. in 2002 [26]. This kernel function was also used to design a IPM for $P_*(\kappa)$ LCP proposed by cho [27], and the iteration bound for large-update method is $O((1+2\kappa)n^{\frac{3}{4}}\log\frac{n}{\epsilon})$. As far as we know, no one has put forward its parameterized version. The new parameterized kernel function is not self-regular, but it is eligible. Based on this parameterized kernel function, we design a new primal-dual IPMs for $P_*(\kappa)$ LCP and obtain the best known iteration bounds for large-update and small-update method respectively, among which the result for large-update method is better than that of Cho.

The paper is organized as follows. In Section 2, we recall the basic concepts of the central path and search direction. We also give a generic IPM for $P_*(\kappa)$ LCP in Fig.1. New kernel function and its properties are introduced in Section 3. Section 4 is devoted to analyzing the convergence of the algorithm and deducing the iteration bound of the algorithm. In section 5, we give some numerical results. Finally, concluding remarks are given in Section 6.

Some notations used throughout the paper are as below: R^n denotes the set of n-dimensional vectors, the set of n-dimensional nonnegative vectors and positive vectors are denoted as R^n_+ and R^n_{++}, respectively. For any vector $x \in R^n$, the ith component of vector x is denoted as x_i, x_{min} denotes the smallest ith component of vector x, $X := diag(x)$. Furthermore, e denotes n-dimensional vector of ones and E denotes identity matrix. $\|\cdot\|$ denotes the 2-norm of a vector. If $f : R_+ \to R_+$ and $x \in R^n_+$, then $f(x)$ denotes a vector in R^n_+ whose ith component is $f(x_i)$. The notation $f(x) = O(g(x))$ denotes $f(x) \leq cg(x)$ for some positive constant c, and $f(x) = \Theta(g(x))$ denotes $c_1 g(x) \leq f(x) \leq c_2 g(x)$ for positive constants c_1 and c_2. The functions like f, g have a local meaning in this paper.

2. Preliminaries

It is well known that the basic idea of the primal-dual IPM is to replace the complementarity condition $xs = 0$ in (1) by the parameterized equation $xs = \mu e (\mu > 0)$, thus we have

$$s = Mx + q, \quad xs = \mu e, \quad x > 0, \ s > 0. \tag{2}$$

A solution of (2) exists if and only if LCP satisfies the *interior-point condition* (IPC), i.e., there exists (x^0, s^0) satisfying $s^0 = Mx^0 + q$, $x^0 > 0$, $s^0 > 0$. Without loss of generality, we can assume that the IPC is satisfied. Since M is a $P_*(\kappa)$ matrix and the IPC holds, then for each $\mu > 0$, a unique solution for parameterized system (2) exists. The solution of (2) is denoted as $(x(\mu), s(\mu))$, where $(x(\mu), s(\mu))$ is called the μ-center of the LCP. The set of μ-center gives a homotopy path, which is called the *central path* of the LCP. Under the above assumptions, if $\mu \to 0$, then the limit of the central path yields optimal solutions for LCP [28].

Let't define

$$d_x := \frac{\upsilon \Delta x}{x}, \qquad d_s := \frac{\upsilon \Delta s}{s}, \qquad \text{where } \ \upsilon := \sqrt{\frac{xs}{\mu}}. \tag{3}$$

The search direction of the algorithm is determined by Newton's method, and this yields the following scaled system:

$$\begin{cases} -\overline{M} d_x + d_s = 0, \\ d_x + d_s = \upsilon^{-1} - \upsilon, \end{cases} \tag{4}$$

where $\overline{M} := X^{\frac{1}{2}} S^{-\frac{1}{2}} M X^{\frac{1}{2}} S^{-\frac{1}{2}}$, $X = diag(x)$, $S = diag(s)$. A crucial observation is that the second equation in (4) is called the *scaled centering equation*, where $\upsilon - \upsilon^{-1}$ is exactly the negative gradient of the classical logarithmic barrier function, which is given by $\Psi_c(\upsilon) := \sum_{i=1}^n \left(\frac{\upsilon_i^2 - 1}{2} - log \ \upsilon_i \right)$.

In this paper, we replace the classical logarithmic barrier function $\Psi_c(v)$ by a new barrier function $\Psi(v)$. Therefore, the second equation in system (4) can be rewritten as $d_x + d_s = -\bigtriangledown \Psi(v)$. By using this result and (3), we have the following modified Newton system:

$$\begin{cases} -M\Delta x + \Delta s = 0, \\ X\Delta s + S\Delta x = -\mu v \bigtriangledown \Psi(v). \end{cases} \tag{5}$$

The barrier function $\Psi(v)$ is determined by its kernel function $\psi(t)$ as below:

$$\Psi(v) := \sum_{i=1}^{n} \psi(v_i), \tag{6}$$

where a twice differentiable function $\psi(t) : R_{++} \to R_+$ is called a kernel function if $\psi(t)$ satisfies the following conditions:

$$\psi(1) = \psi'(1) = 0; \qquad \psi''(t) > 0, \ \ \forall t > 0; \qquad \lim_{t\downarrow 0} \psi(t) = \lim_{t\to\infty} \psi(t) = \infty.$$

Hence $\psi(t)$ can be completely determined by its second derivative as follows:

$$\psi(t) = \int_1^t \int_1^\xi \psi''(\zeta) d\zeta d\xi. \tag{7}$$

The system (5) uniquely defines a centering search direction $(\Delta x, \Delta s)$ [28]. By taking a step along the search direction, with the step size α defined by some line search rules. The new point (x_+, s_+) is then computed by

$$x_+ = x + \alpha\Delta x; \qquad s_+ = s + \alpha\Delta s. \tag{8}$$

A generic primal-dual algorithm for LCP is given below:

Generic Primal-Dual Algorithm for LCP

Input:
  A threshold parameter $\tau > 0$;
  an accuracy parameter $\varepsilon > 0$;
  a fixed barrier update parameter θ, $0 < \theta < 1$;
begin
  $x := x^0 > 0$;$s := s^0 > 0$; $x^0 s^0 = \mu^0 e$; $\mu = \mu^0$;
  while $n\mu \geq \epsilon$ **do**
  begin
    $\mu := (1-\theta)\mu$;
    while $\Psi(v) > \tau$ **do**
    begin
      solve system (5) for Δx and Δs;
      determine a step size α;
      $x := x + \alpha\Delta x$; $s := s + \alpha\Delta s$;
    end
  end
end

Fig.1 Generic Algorithm

Remark 2.1. Usually, if $\tau = O(n)$ and $\theta = \Theta(1)$, the algorithm is called a *large-update* method. If $\tau = O(1)$ and $\theta = \Theta(\frac{1}{\sqrt{n}})$, then we call the algorithm a *small-update* method.

3. New kernel function and its properties

Bai et al. introduced the following kernel function in [26]:

$$\psi_A(t) = \frac{t^2 - 1}{2} + \frac{(e-1)^2}{e(e^t - 1)} - \frac{e-1}{e}, \; t > 0. \tag{9}$$

In this paper, we parameterize it and obtain a new kernel functions as follows:

$$\psi(t) = \frac{t^2 - 1}{2} + \frac{(e-1)^{q+1}}{qe(e^t - 1)^q} - \frac{e-1}{qe}, \; t > 0, \; q \geq 1. \tag{10}$$

For simplicity of presentation, we define $M = \frac{(e-1)^{q+1}}{e}$. Some straightforward computations yield the first three derivatives of $\psi(t)$ with respect to t as below:

$$\psi'(t) = t - \frac{Me^t}{(e^t - 1)^{q+1}}, \tag{11a}$$

$$\psi''(t) = 1 + \frac{M(qe^{2t} + e^t)}{(e^t - 1)^{q+2}}, \tag{11b}$$

$$\psi'''(t) = -\frac{M[q^2 e^{3t} + (3q+1)e^{2t} + e^t]}{(e^t - 1)^{q+3}}. \tag{11c}$$

In the following convergence analysis of the method we use the norm-based proximity measure $\delta(v)$ defined by

$$\delta(v) := \frac{1}{2}\|\bigtriangledown\Psi(v)\| = \frac{1}{2}\|d_x + d_s\|. \tag{12}$$

It is known from the definition of the barrier function that the barrier function $\Psi(v)$ is strictly convex and the minimum value is obtained at $v = e$, thus we have $\Psi(v) = 0 \Leftrightarrow \delta(v) = 0 \Leftrightarrow v = e$.

Bai et al. gave five more conditions on the kernel function in the literature [7], namely,

$$t\psi''(t) + \psi'(t) > 0, \quad t < 1; \tag{13a}$$

$$t\psi''(t) - \psi'(t) > 0, \quad t > 1; \tag{13b}$$

$$\psi'''(t) < 0, \quad t > 0; \tag{13c}$$

$$2\psi''(t)^2 - \psi'(t)\psi'''(t) > 0, \quad t < 1; \tag{13d}$$

$$\psi''(t)\psi'(\beta t) - \beta\psi'(t)\psi''(\beta t) > 0, \quad t > 1, \; \beta > 1. \tag{13e}$$

Obviously, the condition (13a) is satisfied if $t \geq 1$, because then $\psi'(t) \geq 0$ and $\psi''(t) > 0$. Thus the condition (13a) holds for $t > 0$. Similar cases also occur on conditions (13b) and (13d). Following [7], a kernel function $\psi(t)$ is called an eligible kernel function if $\psi(t)$ satisfies conditions (13a), (13c), (13d) and (13e). Moreover, If $\psi(t)$ satisfies (13b) and (13c), then $\psi(t)$ satisfies (13e). Now one easily checks that our kernel function $\psi(t)$ is eligible.

Theorem 3.1. *$\psi(t)$ is an eligible kernel function.*

Proof. Let $N = \frac{Me^t}{(e^t-1)^{q+1}},\ t > 0,\ q \geq 1$. To prove that the kernel function $\psi(t)$ is eligible, it is sufficient to prove that $\psi(t)$ satisfies (13a), (13b), (13c) and (13d).

(i) By using (11a) and (11b), we have

$$\begin{aligned} t\psi''(t) + \psi'(t) &= t\left[1 + \frac{N(qe^t+1)}{e^t-1}\right] + t - N \\ &= 2t + \frac{N[(qe^t+1)t - e^t + 1]}{e^t-1},\ t > 0,\ q \geq 1. \end{aligned}$$

Let $f(t) = (qe^t + 1)t - e^t + 1$. One has $f'(t) = (q-1)e^t + tqe^t + 1 > 0,\ t > 0,\ q \geq 1$, which implies $f(t)$ is strictly monotonically increasing. Using that $f(t)$ is continuous, it follows that if $t > 0$, then $f(t) > f(0) = 0$ holds. In conclusion, we have $t\psi''(t) + \psi'(t) = 2t + \frac{Nf(t)}{e^t-1} > 0,\ t > 0,\ q \geq 1$.

(ii) From (11a) and (11b), we derive that

$$\begin{aligned} t\psi''(t) - \psi'(t) &= t\left[1 + \frac{N(qe^t+1)}{e^t-1}\right] - t + N \\ &= \frac{N(qe^t+1)t}{e^t-1} + N > 0,\ t > 0,\ q \geq 1. \end{aligned}$$

(iii) Due to (11c), we immediately get $\psi'''(t) < 0,\ t > 0,\ q \geq 1$.

(iv) Combining with (11a), (11b) and (11c), it is verified that

$$\begin{aligned} &2\psi''(t)^2 - \psi'(t)\psi'''(t) \\ &= 2\left[1 + \frac{N(qe^t+1)}{e^t-1}\right]^2 + (t-N)\frac{N\left[q^2e^{2t} + (3q+1)e^t + 1\right]}{(e^t-1)^2} \\ &\geq \frac{N^2[2(qe^t+1)^2 - (q^2e^{2t} + (3q+1)e^t + 1)]}{(e^t-1)^2} \\ &= \frac{N^2[q^2e^{2t} + (q-1)e^t + 1]}{(e^t-1)^2} > 0,\ t > 0, q \geq 1. \end{aligned}$$

This proves that $\psi(t)$ satisfies (13a), (13b), (13c) and (13d), the proof is completed. □

In the following lemma we list some formulas that are equivalent to (13a).

Lemma 3.2 ([21],Lemma 2.1.2)**.** *The following three formulas are equivalent if $\psi(t)$ is a twice differentiable function for $t>0$:*

(*i*) $\psi(\sqrt{t_1t_2}) \le \frac{1}{2}(\psi(t_1)+\psi(t_2)),\ t_1,\ t_2>0$;

(*ii*) $\psi'(t)+t\psi''(t)\ge 0,\ t>0$;

(*iii*) $\psi(e^{\xi})$ *is convex.*

In fact, a twice differentiable function is called *exponential convex* or *e-convex* if it satisfies the property described in Lemma 3.2, and this property has been proved to be essential in analyzing the convergence of the primal-dual IPMs based on kernel functions. It is clear that our kernel function $\psi(t)$ is e-convex by Theorem 3.1.

Define $\psi_b(t) := \frac{(e-1)^{q+1}}{qe(e^t-1)^q} - \frac{e-1}{qe},\ t>0,\ q\ge 1$. One may easily verify that $\psi_b(t)$ is monotonically decreasing and $\psi_b'(t)$ is monotonically increasing for $t\in(0,\infty)$. We also have $\psi_b(1)=0$ and $\psi_b'(1)=1$.

The following lemmas give several crucial properties which are important in the analysis of the algorithm.

Lemma 3.3. *Let $\psi(t)$ be as defined in (10), we have*

(*i*) $\frac{1}{2}(t-1)^2 \le \psi(t) \le \frac{1}{2}\psi'(t)^2,\ t>0,\ q\ge 1$;

(*ii*) $\psi(t)\le (q+1)(t-1)^2,\ t>1,\ q\ge 1$;

(*iii*) $\delta(v)\ge \sqrt{\frac{\Psi}{2}},\ t>0,\ q\ge 1.$

Proof. (i) By using (7) and $\psi''(t)\ge 1$, for $q\ge 1$ and $t>0$, we have

$$\psi(t)=\int_1^t\int_1^{\xi}\psi''(\zeta)d\zeta d\xi \ge \int_1^t\int_1^{\xi} 1 d\zeta d\xi = \frac{1}{2}(t-1)^2$$

and

$$\begin{aligned}\psi(t) &= \int_1^t\int_1^{\xi}\psi''(\zeta)d\zeta d\xi \le \int_1^t\int_1^{\xi}\psi''(\xi)\psi''(\zeta)d\zeta d\xi\\ &= \int_1^t \psi''(\xi)\psi'(\xi)d\xi = \int_1^t \psi'(\xi)d\psi'(\xi) = \frac{1}{2}\psi'(t)^2.\end{aligned}$$

(ii) Using Taylor's theorem, $\psi(1)=\psi'(1)=0$, $\psi''(1)=\frac{qe+1}{e-1}$, and $\psi'''(t)<0,\ t>0$, we have

$$\begin{aligned}\psi(t) &= \psi(1)+\psi'(1)(t-1)+\frac{1}{2}\psi''(1)(t-1)^2+\frac{1}{3!}\psi'''(\xi)(t-1)^3\\ &= \frac{1}{2}\psi''(1)(t-1)^2+\frac{1}{3!}\psi'''(\xi)(t-1)^3\\ &< \frac{qe+1}{2(e-1)}(t-1)^2 < (q+1)(t-1)^2,\ 1<\xi<t,\ q\ge 1.\end{aligned}$$

(iii) From the right inequality of (i), we obtain

$$\Psi(v)=\sum_{i=1}^n \psi(v_i) \le \frac{1}{2}\sum_{i=1}^n \psi'(v_i)^2 = \frac{1}{2}\|\bigtriangledown \Psi(v)\|^2 = 2\delta(v)^2,$$

which means $\delta(v) \geq \sqrt{\frac{\Psi}{2}}$. This completes the proof. □

Lemma 3.4. *Let $\rho : [0, \infty) \to (0, 1]$ denote the inverse function of the restriction of $-\frac{1}{2}\psi'(t)$ to the interval $(0, 1]$. Then*

$$\frac{1}{e^{\rho(s)} - 1} \leq \left(\frac{2s+1}{M}\right)^{\frac{1}{q+1}}, \ q \geq 1.$$

Proof. Putting $t = \rho(s)$, which implies $s = -\frac{1}{2}\psi'(t) = -\frac{1}{2}(t - \frac{Me^t}{(e^t-1)^{q+1}})$. Hence we may get

$$2s + 1 \geq 2s + t = \frac{Me^t}{(e^t - 1)^{q+1}} \geq \frac{M}{(e^t - 1)^{q+1}}, \ t \in (0, 1], \ q \geq 1.$$

This implies

$$\frac{1}{e^t - 1} = \frac{1}{e^{\rho(s)} - 1} \leq \left(\frac{2s+1}{M}\right)^{\frac{1}{q+1}}, \ t \in (0, 1], \ q \geq 1,$$

which proves the lemma. □

Lemma 3.5. *Let $\varrho : [0, \infty) \to [1, \infty)$ be the inverse function of $\psi(t)$ for $t \in [1, \infty)$. Then $\sqrt{1 + 2s} \leq \varrho \leq 1 + \sqrt{2s}$.*

Proof. Let $t = \varrho(s)$, then we have $s = \psi(t) = \frac{t^2-1}{2} + \psi_b(t), \ q \geq 1, \ t \in [1, \infty)$. Since $\psi_b(t)$ is decreasing for $t > 1$ and $\psi_b(1) = 0$, one has $s = \psi(t) \leq \frac{t^2-1}{2}, \ t \in [1, \infty)$. This gives $\sqrt{1 + 2s} \leq \varrho(s)$. For the second inequality we use the result (i) in Lemma 3.3, thus we obtain $s = \psi(t) \geq \frac{1}{2}(t-1)^2$, therefore $\varrho(s) = t \leq 1 + \sqrt{2s}$. This completes the proof of the lemma. □

4. Complexity analysis of the algorithm

Note that our algorithm consists of two parts: inner iteration and outer iteration. Every time before the outer iteration of the algorithm begins, just before the $\mu-$update with the factor $1 - \theta$, $0 < \theta < 1$, we have $\Psi(v) \leq \tau$. The vector v is divided by the factor $\sqrt{1 - \theta}$ in the outer iteration, which in general leads to an increase in the value of $\Psi(v)$. Then the algorithm starts executing the inner iterations if $\Psi(v) > \tau$, and the inner iterations will decrease the value of $\Psi(v)$. The algorithm returns to the outer iteration again when $\Psi(v) \leq \tau$. Repeat the above iteration process until μ is small enough, say, until $n\mu \leq \epsilon$, at this stage we have found an $\epsilon-$solution of the $P_*(\kappa)$ LCP.

4.1 Growth behavior

From the above analysis we conclude that the largest value of $\Psi(v)$ occur after the μ-update, just before the inner iteration begins. What we want is to find an upper bound of $\Psi(\upsilon)$ to research the amount of decrease of the barrier function during an inner iteration. Due to the fact that $\psi(t)$ is eligible, and also [7], we have the following lemma.

Lemma 4.1. *Let ϱ be as defined in Lemma 3.5, and $\upsilon \in R^n_{++}$, $\beta \geq 1$, then*

$$\Psi(\beta\upsilon) \leq n\psi\left(\beta\varrho\left(\frac{\Psi(\upsilon)}{n}\right)\right).$$

Define

$$L_\psi(n,\theta,\tau) := n\psi\left(\frac{1}{\sqrt{1-\theta}}\varrho\left(\frac{\tau}{n}\right)\right). \tag{14}$$

Obviously, if $\Psi(\upsilon) \leq \tau$ and $\beta = \frac{1}{\sqrt{1-\theta}}$, then $L_\psi(n,\theta,\tau)$ is an upper bound for $\Psi\left(\frac{\upsilon}{\sqrt{1-\theta}}\right)$ by using Lemma 4.1.

Lemma 4.2. *Using the notations of (14), if $\Psi(\upsilon) \leq \tau$, then we have*

(i) $L_\psi(n,\theta,\tau) \leq \frac{n\theta+2\sqrt{2n\tau}+2\tau}{2(1-\theta)}$;

(ii) $L_\psi(n,\theta,\tau) \leq \frac{(q+1)(\sqrt{n}\theta+\sqrt{2\tau})^2}{1-\theta}$.

Proof. (i) Since $\psi_b(t)$ is monotonically decreasing for $t \geq 1$, and $\psi_b(1) = 0$, we get

$$\psi(t) = \frac{t^2-1}{2} + \psi_b(t) \leq \frac{t^2-1}{2},\ t \geq 1,\ q \geq 1.$$

Since $\frac{1}{\sqrt{1-\theta}} > 1$ for $0 < \theta < 1$ and $\varrho(\frac{\tau}{n}) > 1$, we have $\frac{\varrho(\frac{\tau}{n})}{\sqrt{1-\theta}} > 1$. Using above results, Lemma 3.5 and $\psi(t)$ is monotonically increasing for $t \in [1,\infty)$, we have

$$\begin{aligned} L_\psi(n,\theta,\tau) &= n\psi\left(\frac{\varrho(\frac{\tau}{n})}{\sqrt{1-\theta}}\right) \leq n\psi\left(\frac{1+\sqrt{\frac{2\tau}{n}}}{\sqrt{1-\theta}}\right) \\ &\leq n\frac{\left(\frac{1+\sqrt{\frac{2\tau}{n}}}{\sqrt{1-\theta}}\right)^2-1}{2} = \frac{n\theta+2\sqrt{2n\tau}+2\tau}{2(1-\theta)}. \end{aligned}$$

(ii) From the result (ii) in Lemma 3.3 and Lemma 3.5, we deduce that

$$\begin{aligned} L_\psi(n,\theta,\tau) &= n\psi\left(\frac{\varrho(\frac{\tau}{n})}{\sqrt{1-\theta}}\right) \leq n\psi\left(\frac{1+\sqrt{\frac{2\tau}{n}}}{\sqrt{1-\theta}}\right) \\ &\leq n(q+1)\left(\frac{1+\sqrt{\frac{2\tau}{n}}}{\sqrt{1-\theta}}-1\right)^2 \\ &= n(q+1)\frac{\left(1+\sqrt{\frac{2\tau}{n}}-\sqrt{1-\theta}\right)^2}{1-\theta} \leq \frac{(q+1)\left(\sqrt{n}\theta+\sqrt{2\tau}\right)^2}{1-\theta}, \end{aligned}$$

where the last inequality is obtained by $1-\sqrt{1-\theta} = \frac{\theta}{1+\sqrt{1-\theta}} \leq \theta$. Hence the lemma is proved. □

Defining

$$L_{\psi 1} := L_{\psi 1}(n,\theta,\tau) = \frac{n\theta + 2\sqrt{2n\tau} + 2\tau}{2(1-\theta)};$$

$$L_{\psi 2} := L_{\psi 2}(n,\theta,\tau) = \frac{(q+1)\left(\sqrt{n}\theta + \sqrt{2\tau}\right)^2}{1-\theta}.$$

We will use $L_{\psi 1}$ and $L_{\psi 2}$ for the upper bounds of $\Psi(v)$ for large-update and small-update methods, respectively. From Remark 2.1, if we set $\tau = O(n)$ and $\theta = \Theta(1)$, then $L_{\psi 1} = O(n)$. If we choose $\tau = O(1)$ and $\theta = \Theta(\frac{1}{\sqrt{n}})$, then $L_{\psi 2} = O(q)$.

4.2 Decrease behavior and the choice of step size

This section serves to analyze the decreasing behavior of the barrier function and the choice of step size during an inner iteration. After a damped step we have

$$x_+ = x + \alpha\Delta x = x(e + \alpha\frac{\Delta x}{x}) = x(e + \frac{d_x}{v}) = \frac{x}{v}(v + \alpha d_x);$$
$$s_+ = s + \alpha\Delta s = s(e + \alpha\frac{\Delta s}{s}) = s(e + \frac{d_s}{v}) = \frac{s}{v}(v + \alpha d_s).$$

Thus we obtain

$$v_+ = \sqrt{\frac{x_+ s_+}{\mu}} = \sqrt{(v + \alpha d_x)(v + \alpha d_s)}. \tag{15}$$

Since ψ is e-convex, then one can get

$$\Psi(v_+) = \Psi(\sqrt{(v + \alpha d_x)(v + \alpha d_s)} \leq \frac{1}{2}\big(\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)\big). \tag{16}$$

Define

$$f(\alpha) := \Psi(v_+) - \Psi(v),$$

which denotes the amount of decrease of the barrier function after an inner iteration. An immediate consequence following from (16) is

$$f(\alpha) \leq f_1(\alpha) := \frac{1}{2}(\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)) - \Psi(v).$$

The first derivative and the second derivative of $f_1(\alpha)$ are given as follows:

$$f_1'(\alpha) = \frac{1}{2}\sum_{i=1}^{n}(\psi'(v_i + \alpha d_{x_i})d_{x_i} + \psi'(v_i + \alpha d_{s_i})d_{s_i}); \tag{17}$$

$$f_1''(\alpha) = \frac{1}{2}\sum_{i=1}^{n}(\psi''(v_i + \alpha d_{x_i})d_{x_i}^2 + \psi''(v_i + \alpha d_{s_i})d_{s_i}^2). \tag{18}$$

We thus have

$$f(0) = f_1(0) = 0; \quad f_1'(0) = \frac{1}{2}\nabla\Psi(\upsilon)^T(d_x + d_s) = -\frac{1}{2}\nabla\Psi(\upsilon)^T\nabla\Psi(\upsilon) = -2\delta(\upsilon)^2.$$

Since M is a $P_*(\kappa)$ matrix and $M\Delta x = \Delta s$ from (5), for $\Delta x, \Delta s \in R^n$ we have

$$(1 + 4\kappa)\sum_{i\in J_+}\Delta x_i\Delta s_i + \sum_{i\in J_-}\Delta x_i\Delta s_i \geq 0.$$

where $J_+ = \{i \in J : \Delta x_i\Delta s_i \geq 0\}$ and $J_- = J - J_+$. Note that $d_x d_s = \frac{\upsilon^2\Delta x\Delta s}{xs} = \frac{\Delta x\Delta s}{\mu}$, $\mu > 0$, we obtain

$$(1 + 4\kappa)\sum_{i\in J_+} d_{x_i} d_{s_i} + \sum_{i\in J_-} d_{x_i} d_{s_i} \geq 0.$$

With $\delta(\upsilon)$ as defined in (12), then we use the following notations:

$$\upsilon_1 := \upsilon_{\min}, \qquad \delta := \delta(\upsilon).$$

The following lemmas give the bound for $\|d_x\|$ and $\|d_s\|$.

Lemma 4.3 ([27], Lemma 4.2)**.** *$\sum_{i=1}^n(d_{x_i}^2 + d_{s_i}^2) \leq 4(1+2\kappa)\delta^2$, $\|d_x\| \leq 2\sqrt{1+2\kappa}\delta$, and $\|d_s\| \leq 2\sqrt{1+2\kappa}\delta$.*

Lemma 4.4. *$f_1''(\alpha) \leq 2(1 + 2\kappa)\delta^2\psi''\big(\upsilon_1 - 2\alpha\delta\sqrt{1+2\kappa}\big)$.*

Proof. Using Lemma 4.3, we have the following inequalities:

$$\upsilon_i + \alpha d_{x_i} \geq \upsilon_1 - 2\alpha\delta\sqrt{1+2\kappa}, \quad \upsilon_i + \alpha d_{s_i} \geq \upsilon_1 - 2\alpha\delta\sqrt{1+2\kappa},$$

where $1 \leq i \leq n$. From the above result, Lemma 4.3, (18) and $\psi''(t)$ is monotonically decreasing, we deduce

$$\begin{aligned} f_1''(\alpha) &= \frac{1}{2}\sum_{i=1}^n(\psi''(\upsilon_i + \alpha d_{x_i})d_{x_i}^2 + \psi''(\upsilon_i + \alpha d_{s_i})d_{s_i}^2) \\ &\leq \frac{1}{2}\sum_{i=1}^n\Big(\psi''\big(\upsilon_1 - 2\alpha\delta\sqrt{1+2\kappa}\big)d_{x_i}^2 + \psi''\big(\upsilon_1 - 2\alpha\delta\sqrt{1+2\kappa}\big)d_{s_i}^2\Big) \\ &= \frac{1}{2}\psi''\big(\upsilon_1 - 2\alpha\delta\sqrt{1+2\kappa}\big)\sum_{i=1}^n(d_{x_i}^2 + d_{s_i}^2) \\ &\leq \frac{1}{2}\psi''\big(\upsilon_1 - 2\alpha\delta\sqrt{1+2\kappa}\big)\big(\|d_x\|^2 + \|d_s\|^2\big) \\ &\leq 2(1 + 2\kappa)\delta^2\psi''\big(\upsilon_1 - 2\alpha\delta\sqrt{1+2\kappa}\big). \end{aligned}$$

We get the desired result. □

Lemma 4.5. *If α satisfies the inequality*

$$-\psi'\big(\upsilon_1 - 2\alpha\delta\sqrt{1+2\kappa}\big) + \psi'(\upsilon_1) \leq \frac{2\delta}{\sqrt{1+2\kappa}}, \tag{19}$$

then $f_1'(\alpha) \leq 0$.

Proof. By Lemma 4.4, we obtain

$$\begin{aligned}
f_1'(\alpha) &= f_1'(0) + \int_0^\alpha f_1''(\xi)d\xi \\
&\leq -2\delta^2 + 2\delta^2(1+2\kappa)\int_0^\alpha \psi''\big(\upsilon_1 - 2\xi\delta\sqrt{1+2\kappa}\big)d\xi \\
&= -2\delta^2 - \delta\sqrt{1+2\kappa}\int_0^\alpha \psi''\big(\upsilon_1 - 2\xi\delta\sqrt{1+2\kappa}\big)d\big(\upsilon_1 - 2\xi\delta\sqrt{1+2\kappa}\big) \\
&= -2\delta^2 - \sqrt{1+2\kappa}\delta\Big(\psi'\big(\upsilon_1 - 2\xi\delta\sqrt{1+2\kappa}\big) - \psi'\big(\upsilon_1\big)\Big) \\
&\leq -2\delta^2 + \sqrt{1+2\kappa}\delta\frac{2\delta}{\sqrt{1+2\kappa}} \\
&= 0.
\end{aligned}$$

The lemma is proved. □

Lemma 4.6. *Using the notions of Lemma 3.4, if step size α satisfies (19), then the largest step size α is given by*

$$\bar{\alpha} := \frac{1}{2\delta\sqrt{1+2\kappa}}\Big(\rho(\delta) - \rho\Big(\big(1 + \frac{1}{\sqrt{1+2\kappa}}\big)\delta\Big)\Big). \tag{20}$$

Proof. Since $\psi''(t)$ is monotonically decreasing, the derivative of the left hand side in (19) with respect to υ_1 satisfies

$$-\psi''\big(\upsilon_1 - 2\alpha\delta\sqrt{1+2\kappa}\big) + \psi''(\upsilon_1) < 0.$$

This implies that the left hand side in (19) is monotonically decreasing in υ_1. Therefore, for fixed δ, the smaller the value of υ_1, the smaller the value of α. By the definition of δ and $\Psi(\upsilon)$, we have

$$\delta = \frac{1}{2}\|\nabla\Psi(\upsilon)\| = \frac{1}{2}\sqrt{\sum_{i=1}^{n}(\Psi(\upsilon))^2} \geq \frac{1}{2}|\psi'(\upsilon_1)| \geq -\frac{1}{2}\psi'(\upsilon_1).$$

Note that equality holds if and only if υ_1 is the only coordinate in υ which differs from 1 and $\upsilon_1 \leq 1$, i.e., $\psi'(\upsilon_1) \leq 0$. Hence the smallest step size α occurs when υ_1 satisfies

$$-\frac{1}{2}\psi'(\upsilon_1) = \delta. \tag{21}$$

In this case, we have

$$v_1 = \rho(\delta). \tag{22}$$

Since $\psi''(t) > 0$, the derivative with respect to α of the left hand side in (19) satisfies

$$2\delta\sqrt{1+2\kappa}\psi''\big(v_1 - 2\alpha\delta\sqrt{1+2\kappa}\big) \geq 0,$$

which means the left hand side in (19) is monotone increasing with respect to α. Therefore, the largest step size satisfying (19) occurs when

$$-\psi'\big(v_1 - 2\alpha\delta\sqrt{1+2\kappa}\big) + \psi'(v_1) = \frac{2\delta}{\sqrt{1+2\kappa}}. \tag{23}$$

From (21), (23) can be written as

$$-\frac{1}{2}\psi'\big(v_1 - 2\alpha\delta\sqrt{1+2\kappa}\big) = \delta\left(1 + \frac{1}{\sqrt{1+2\kappa}}\right).$$

By above equation and the definition of ρ, we obtain

$$v_1 - 2\alpha\delta\sqrt{1+2\kappa} = \rho\left(\delta\Big(1 + \frac{1}{\sqrt{1+2\kappa}}\Big)\right).$$

Hence, by (22), the largest step size α follows that

$$\bar{\alpha} = \frac{1}{2\delta\sqrt{1+2\kappa}}\left(\rho(\delta) - \rho\Big(\Big(1 + \frac{1}{\sqrt{1+2\kappa}}\Big)\delta\Big)\right),$$

and the lemma is proved. □

Lemma 4.7. *Let ρ and $\bar{\alpha}$ be defined as in Lemma 4.6, then*

$$\bar{\alpha} \geq \tilde{\alpha} := \frac{1}{1+2\kappa}\frac{1}{\psi''(\rho((1+\frac{1}{\sqrt{1+2\kappa}})\delta))}, \tag{24}$$

and we will use $\tilde{\alpha}$ as the default step size.

Proof. By the definition of ρ, we have

$$-\frac{1}{2}\psi'(\rho(\delta)) = \delta$$

Taking the derivative with respect to δ, we obtain

$$-\frac{1}{2}\psi''(\rho(\delta))\rho'(\delta) = 1.$$

Since $\psi''(\rho(\delta)) > 0$, we have

$$\rho'(\delta) = -\frac{2}{\psi''(\rho(\delta))} < 0,$$

which means that $\rho(\delta)$ is monotonically decreasing in δ. Using the above result, (20) and the fundamental theorem of calculus, we get

$$\begin{aligned}\bar{\alpha} &= \frac{1}{2\delta\sqrt{1+2\kappa}}\left(\rho(\delta) - \rho\Big(\big(1+\frac{1}{\sqrt{1+2\kappa}}\big)\delta\Big)\right)\\ &= \frac{1}{2\delta\sqrt{1+2\kappa}}\int_{(1+\frac{1}{\sqrt{1+2\kappa}})\delta}^{\delta} \rho'(\xi)d\xi\\ &= \frac{1}{\delta\sqrt{1+2\kappa}}\int_{\delta}^{(1+\frac{1}{\sqrt{1+2\kappa}})\delta} \frac{1}{\psi''(\rho(\xi))}d\xi\end{aligned}$$

We want to obtain a lower bound for $\bar{\alpha}$ by replacing the argument of the last integral with its minimal value. Hence we would like to know when the $\psi''(\rho(\xi))$ attains the maximum value for $\xi \in [\delta, (1+\frac{1}{\sqrt{1+2\kappa}})\delta]$. Since ψ'' and ρ are monotonically decreasing, we have

$$\rho(\xi) \le \rho\Big(\big(1+\frac{1}{\sqrt{1+2\kappa}}\big)\delta\Big)$$

and

$$\psi''(\rho(\xi)) \le \psi''\Big(\rho\Big(\big(1+\frac{1}{\sqrt{1+2\kappa}}\big)\delta\Big)\Big).$$

As a consequence, we obtain

$$\frac{1}{\psi''(\rho(\xi))} \le \frac{1}{\psi''\Big(\rho\Big(\big(1+\frac{1}{\sqrt{1+2\kappa}}\big)\delta\Big)\Big)}.$$

Therefore, we have

$$\begin{aligned}\bar{\alpha} &= \frac{1}{\delta\sqrt{1+2\kappa}}\int_{\delta}^{(1+\frac{1}{\sqrt{1+2\kappa}})\delta} \frac{1}{\psi''(\rho(\xi))}d\xi\\ &\ge \frac{1}{\delta\sqrt{1+2\kappa}}\frac{1}{\psi''\Big(\rho\Big(\big(1+\frac{1}{\sqrt{1+2\kappa}}\big)\delta\Big)\Big)}\int_{\delta}^{(1+\frac{1}{\sqrt{1+2\kappa}})\delta} d\xi\\ &= \frac{1}{1+2\kappa}\frac{1}{\psi''\Big(\rho\Big(\big(1+\frac{1}{\sqrt{1+2\kappa}}\big)\delta\Big)\Big)}\end{aligned}$$

The corollary is proved. □

Lemma 4.8. *If $\bar{\alpha}$ is defined as in Lemma 4.6, then*

$$\bar{\alpha} \ge \frac{1}{4(1+2\kappa)(2q+1)(4\delta+1)^{\frac{q+2}{q+1}}}, \quad q \ge 1.$$

Proof. Let $r = \rho\Big(\big(1 + \frac{1}{\sqrt{1+2\kappa}}\big)\delta\Big)$. By using ρ defined in Lemma 3.4, we have

$$\frac{1}{e^{\rho\left((1+\frac{1}{\sqrt{1+2\kappa}})\delta\right)} - 1} = \frac{1}{e^r - 1} \le \left(\frac{2\delta(1 + \frac{1}{\sqrt{1+2\kappa}}) + 1}{M}\right)^{\frac{1}{q+1}}, \ q \ge 1.$$

Combining the above results and (11b), through the simple calculation, we derive that

$$\begin{aligned}
\psi''(r) &= 1 + \frac{M(qe^{2r} + e^r)}{(e^r - 1)^{q+2}} \\
&\le 1 + \left(\frac{2\delta(1 + \frac{1}{\sqrt{1+2\kappa}}) + 1}{M}\right)^{\frac{q+2}{q+1}} M(qe^{2r} + e^r) \\
&= 1 + M^{-\frac{1}{q+1}}\left(2\delta\big(1 + \frac{1}{\sqrt{1+2\kappa}}\big) + 1\right)^{\frac{q+2}{q+1}}(qe^{2r} + e^r) \\
&\le 1 + (4\delta + 1)^{\frac{q+2}{q+1}}(qe^{2r} + e^r) \\
&\le 4(2q + 1)(4\delta + 1)^{\frac{q+2}{q+1}}, \ q \ge 1.
\end{aligned}$$

From Lemma 4.6, we derive that

$$\begin{aligned}
\bar{\alpha} \ge \tilde{\alpha} := \frac{1}{1 + 2\kappa}\frac{1}{\psi''(\rho(1 + \frac{1}{\sqrt{1+2\kappa}}\delta))} \\
\ge \frac{1}{4(1 + 2\kappa)(2q + 1)(4\delta + 1)^{\frac{q+2}{q+1}}}, \ q \ge 1.
\end{aligned}$$

This proves the lemma. □

Lemma 4.9 ([6], Lemma 12)**.** *Suppose that $h(t)$ is a twice differentiable convex function with $h(0) = 0$, $h'(0) < 0$ and $h(t)$ attains its global minimum at $t^* > 0$. If $h''(t)$ is monotonically increasing for $t \in [0, t^*]$, then*

$$h(t) \le \frac{th'(0)}{2}, \ 0 \le t \le t^*.$$

Lemma 4.10. *If the step size α is such that $\alpha \le \bar{\alpha}$, then*

$$f(\alpha) \le -\alpha\delta^2. \tag{25}$$

Proof. Define the univariate function h as below:

$$\begin{aligned}
&h(0) = f_1(0) = 0, h'(0) = f_1'(0) = -2\delta^2, \\
&h''(\alpha) = 2(1 + 2\kappa)\delta^2\psi''(v_1 - 2\alpha\delta\sqrt{1 + 2\kappa}).
\end{aligned}$$

According to Lemma 4.4, we have $f_1''(\alpha) \le h''(\alpha)$, which implies $f_1'(\alpha) \le h'(\alpha)$ and $f_1(\alpha) \le h(\alpha)$. Since $\psi''(t) > 1$, $h''(\alpha) \ge 2(1 + \kappa)\delta^2$. Thus $h(\alpha)$ is strongly convex and $h(\alpha)$ attains its global minimum for some $\alpha^* > 0$. Taking $\alpha \le \bar{\alpha}$, with $\bar{\alpha}$ as defined by (20), using the fundamental theorem of calculus, and Lemma 4.5, we may get

$$\begin{aligned} h'(\alpha) &= h'(0) + \int_0^\alpha h''(\xi)d\xi \\ &= -2\delta^2 + 2\delta^2(1+2\kappa)\int_0^\alpha \psi''(\upsilon_1 - 2\xi\delta\sqrt{1+2\kappa}))d\xi \\ &= -2\delta^2 - \delta\sqrt{1+2\kappa}\Big(\psi'(\upsilon_1 - 2\alpha\delta\sqrt{1+2\kappa}) - \psi'(\upsilon_1)\Big) \\ &\le -2\delta^2 + \delta\sqrt{1+2\kappa}\frac{2\delta}{\sqrt{1+2\kappa}} = 0 \end{aligned}$$

Since $h'''(\alpha) = -4(1+2\kappa)^{\frac{3}{2}}\delta^3\psi'''\Big(\upsilon_1 - 2\xi\delta\sqrt{1+2\kappa}\Big)$ and $\psi'''(t) < 0$, $h''(\alpha)$ is increasing in α. From Lemma 4.9, we may write

$$f_1(\alpha) \le h(\alpha) \le \frac{1}{2}\alpha h'(0) = -\alpha\delta^2.$$

Due to $f(\alpha) \le f_1(\alpha)$, the proof is completed. □

Lemma 4.11. *One has*

$$(26)\quad f(\tilde{\alpha}) \le -\frac{1}{1+2\kappa}\frac{\delta^2}{\psi''(\rho(1+\frac{1}{\sqrt{1+2\kappa}}\delta))} \le -\frac{\delta^{\frac{q}{q+1}}}{4(1+2\kappa)(2q+1)\Big(4+\frac{1}{\delta}\Big)^{\frac{q+2}{q+1}}}.$$

Proof. According to Lemmas 4.7, 4.8 and 4.10, we have

$$\begin{aligned} f(\tilde{\alpha}) &\le -\tilde{\alpha}\delta^2 \le -\frac{1}{1+2\kappa}\frac{\delta^2}{\psi''(\rho(1+\frac{1}{\sqrt{1+2\kappa}}\delta))} \\ &\le -\frac{\delta^2}{4(1+2\kappa)(2q+1)(4\delta+1)^{\frac{q+2}{q+1}}} \\ &\le -\frac{\delta^{\frac{q}{q+1}}}{4(1+2\kappa)(2q+1)\Big(4+\frac{1}{\delta}\Big)^{\frac{q+2}{q+1}}}. \end{aligned}$$

Hence, the result of this lemma holds. □

4.3 Iteration complexity

Our aim in this section is to analyze the convergence of the algorithm. The question now is to count how many inner iterations are required to return to the situation where $\Psi(\upsilon) \le \tau$. To investigate this, we define the value of $\Psi(\upsilon)$ after each $\mu-$update as Ψ_0, and the subsequent values during inner iterations as

$\Psi_\kappa,\ \kappa = 1, 2, \ldots, K$. Thus K is the number of iterations in the inner iteration after once $\mu-$update.

By using Lemmas 4.1, 4.2 and the definition of Ψ_0, we have $L_\Psi \geq \Psi_0 \geq \Psi \geq \tau$. In what follows we assume $L_\psi \geq \Psi_0 \geq \Psi \geq \tau \geq 2$. From Lemma 3.5, we deduce that $\delta(\upsilon) \geq \sqrt{\frac{\Psi}{2}} \geq 1$. Substitution into (26) gives

$$f(\tilde{\alpha}) \leq -\frac{\delta^{\frac{q}{q+1}}}{4(1+2\kappa)(2q+1)\left(4+\frac{1}{\delta}\right)^{\frac{q+2}{q+1}}} \leq -\frac{\Psi^{\frac{q}{2(q+1)}}}{56(1+2\kappa)(2q+1)}, \tag{27}$$

which implies

$$\Psi_{\kappa+1} \leq \Psi_\kappa - \frac{\Psi_\kappa^{\frac{q}{2(q+1)}}}{56(1+2\kappa)(2q+1)},\ \kappa = 0, 1, 2, \ldots, K-1. \tag{28}$$

To derive an upper bound for the total number of inner iterations in an outer iteration, we recall the following technical lemma, and its elementary proof please refer to [6].

Lemma 4.12. *Let $t_0, t_1, \ldots, t_K$ be a sequence of positive numbers such that*

$$t_{\kappa+1} \leq t_\kappa - \beta t_\kappa^{1-\gamma},\ \kappa = 0, 1, 2, \ldots, K-1,$$

where $\beta > 0$ and $0 < \gamma \leq 1$. Then $K \leq \lfloor \frac{t_0^\gamma}{\beta\gamma} \rfloor$.

Lemma 4.13. *The following inequality holds:*

$$K \leq 112(1+2\kappa)(2q+1)\Psi_0^{\frac{q+2}{2(q+1)}},\ q \geq 1.$$

Proof. Let $t_\kappa = \Psi_\kappa,\ \beta = \frac{1}{56(2q+1)},\ \gamma = \frac{q+2}{2(q+1)}$. Using Lemma 4.12 and substitution gives, we have

$$\begin{aligned} K &\leq \frac{\Psi_0^\gamma}{\beta\gamma} = \frac{112(1+2\kappa)(2q+1)(q+1)\Psi_0^{\frac{q+2}{2(q+1)}}}{q+2} \\ &\leq 112(1+2\kappa)(2q+1)\Psi_0^{\frac{q+2}{2(q+1)}},\ q \geq 1. \end{aligned}$$

This completes the proof of the lemma. □

Lemma 4.13 gives an upper bound for the number of iterations in the inner iteration after once $\mu-$update. Multiplication the number K by the number of barrier parameter updates yields an upper bound for the total number of iterations [2], where the number of barrier parameter updates is bounded by $\frac{1}{\theta}\log\frac{n}{\epsilon}$. Thus we obtain an upper bound for the total number of iterations as follows:

$$K\frac{1}{\theta}\log\frac{n}{\epsilon} \leq \frac{\Psi_0^\gamma}{\theta\beta\gamma}\log\frac{n}{\epsilon}.$$

Recall that $L_\psi \geq \Psi_0$, and L_ψ is bounded by $L_{\psi 1}$ and $L_{\psi 2}$ in Lemma 4.2. Using $L_{\psi 1}$ and $L_{\psi 2}$ for the upper bounds of Ψ_0 for large-update and small-update methods, respectively, we immediately obtain the following theorem:

Theorem 4.14. *The total number of iterations required by the large-update and small-update methods are at most*

$$112\frac{(1+2\kappa)(2q+1)(n\theta+2\sqrt{2n\tau}+2\tau)^{\frac{q+2}{2(q+1)}}}{\theta[2(1-\theta)]^{\frac{q+2}{2(q+1)}}}\log\frac{n}{\epsilon}=O(q(1+2\kappa)n^{\frac{q+2}{2(q+1)}}\log\frac{n}{\epsilon})$$

and

$$112\frac{(1+2\kappa)(2q+1)(q+1)^{\frac{q+2}{2(q+1)}}(\sqrt{n}\theta+\sqrt{2\tau})^{\frac{2(q+2)}{2(q+1)}}}{\theta(1-\theta)^{\frac{q+2}{2(q+1)}}}\log\frac{n}{\epsilon}$$
$$=O(q^{\frac{3q+4}{2(q+1)}}(1+2\kappa)\sqrt{n}\log\frac{n}{\epsilon}),$$

respectively.

By choosing $q = O(\log n)$, we obtain the iteration bound for large-update method is $O((1+2\kappa)\sqrt{n}\log n \log\frac{n}{\epsilon})$. If we choose $q = 1$, then small-update method have $O((1+2\kappa)\sqrt{n}\log\frac{n}{\epsilon})$ iteration complexities. These are curently the best known bound for such methods. It's worth noting that $\psi(t)$ is precisely the kernel function proposed by Bai et al. in [26] for $q = 1$, and the large-update method for $P_*(\kappa)$ LCP has $O((1+2\kappa)n^{\frac{3}{4}}\log\frac{n}{\epsilon})$ iteration complexities from Cho, which means the iteration bound for large-update method is improved based on our parameterized kernel function.

5. Numerical results

In this section, we give some numerical results. We present three test problems in subsection 5.1 to verify the effectiveness of our algorithm. Subsection 5.2 gives numerical results of large-update IPMs based on our kernel function and some other kernel functions, including classical logarithmic kernel function, trigonometric kernel function, parameterized kernel function with integral types, etc. The numerical result of the Mizuno-Todd-Ye (MTY) predictor-corrector algorithm are given in subsection 5.3.

In the implementation, we set threshold parameter $\tau = 2.5$, accuracy parameter $\epsilon = 10^{-6}$. The numerical results are obtained by using MATLAB 2012b. All numerical results include iteration numbers (iter), the duality gap (Gap) and the CPU time (CPU).

5.1 Test problem

Consider the following three $P_*(\kappa)$ LCPs. In fact, there has not been a polynomial algorithm for which we can calculate the value of the parameter κ for a given $P_*(\kappa)$ matrix, the problem 5.1 is the only $P_*(\kappa)$ LCPs with $\kappa \neq 0$ ($\kappa = \frac{1}{4}$) we currently known. Therefore, we also test Problem 5.2 and Problem 5.3 with $\kappa = 0$ (monotone).

Problem 5.1 (Lee et al.'s example in [29]).

$$M = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad x^0 = \begin{pmatrix} 0.4 \\ 0.45 \end{pmatrix}.$$

Problem 5.2 (Harker et al.'s example in [30]).

$$M = \begin{pmatrix} 1 & 2 & 2 & \cdots & 2 \\ 2 & 5 & 6 & \cdots & 6 \\ 2 & 6 & 9 & \cdots & 10 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 2 & 6 & 10 & \cdots & 4n-3 \end{pmatrix}, \qquad q = \begin{pmatrix} -1 \\ -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}, \qquad x^0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Problem 5.3 (Geiger et al.'s example in [32]).

$$M = \begin{pmatrix} 4 & -1 & 0 & \cdots & 0 \\ -1 & 4 & -1 & \cdots & 0 \\ 0 & -1 & 4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & 4 \end{pmatrix}, \qquad q = \begin{pmatrix} -1 \\ -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}, \qquad x^0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

5.2 Numerical results for large-update IPMs

The best iteration numbers for Problem 5.1 based on our kernel function with different values of the parameters q and θ are stated in Table 2, where $q \in \{1, 2, 3, 5, 10\}$ and $\theta \in \{0.2, 0.4, 0.6, 0.8, 0.99\}$. We also give a series of numerical results in Table 3 to compare the practical computational efficiency of the methods based on different kernel functions and θ for Problem 5.1. Some kernel functions we used in our experiments are as follows:

$$\psi_{new}(t) = \frac{t^2-1}{2} + \frac{(e-1)^{q+1}}{qe(e^t-1)^q} - \frac{e-1}{qe}, \; q \geq 1,$$

$$\psi_1(t) = \frac{t^2-1}{2} - \log t, \qquad \psi_2(t) = \frac{t^2-1}{2} + \frac{t^{-1}-t}{2},$$

$$\psi_3(t) = \frac{t^2-1}{2} + \frac{6}{\pi}\tan(h(t)), \quad h(t) = \frac{\pi(1-t)}{4t+2},$$

$$\psi_4(t) = \frac{t^2-1}{2} - \int_1^t \Big(\frac{e-1}{e^x-1}\Big)^p dx, \; p \geq 1,$$

$$\psi_5(t) = \frac{t^2-1}{2} + \frac{1}{q^2}\big(\frac{q}{t} - 1\big)e^{q(\frac{1}{t}-1)} - \frac{q-1}{q^2}, \; q \geq 1.$$

The first kernel function $\psi_{new}(t)$ is proposed in this paper, and it is exactly the kernel function presented by Bai et al. in [26] for $q = 1$. $\psi_1(t)$ is the classical logarithmic kernel function. $\psi_2(t)$ is a self-regular kernel function presented by Peng et al. in [31]. $\psi_3(t)$ is a trigonometric kernel function [8]. $\psi_4(t)$ [17] and $\psi_5(t)$ [24] are parameterized kernel functions, which obtain the best iteration complexity for $q = O(\log n)$.

Furthermore, the best iteration numbers for Problems 5.2 and 5.3 based on different kernel functions are stated in Table 4 and Table 5 when the scale of the problems changes. For the parameterized kernel functions $\psi_4(t)$, $\psi_5(t)$ and $\psi_{new}(t)$, we take the numerical results of the parameterized p or $q = 1, 2, 3$ respectively.

Table 2. Numerical results based on ψ_{new} for Problem 5.1

q		θ 0.2	0.4	0.6	0.8	0.99	Average
1	Iter	71	35	22	16	9	30.6
	Gap	9.89e-07	7.26e-07	8.46e-07	2.02e-07	1.97e-08	
	CPU	0.0130	0.0119	0.0106	0.0105	0.0093	0.0111
2	Iter	71	34	22	16	9	30.4
	Gap	9.89e-07	7.26e-07	8.46e-07	2.02e-07	1.97e-08	
	CPU	0.0131	0.0113	0.0110	0.0100	0.0091	0.0109
3	Iter	72	35	22	17	9	31
	Gap	9.89e-07	7.26e-07	8.46e-07	2.02e-07	1.97e-08	
	CPU	0.0134	0.0121	0.0106	0.0102	0.0101	0.0113
5	Iter	73	37	24	19	12	33
	Gap	9.89e-07	7.26e-07	8.46e-07	2.02e-07	1.97e-08	
	CPU	0.0127	0.0118	0.0104	0.0103	0.0102	0.0111
10	Iter	74	38	25	20	12	33.8
	Gap	9.89e-07	7.26e-07	8.46e-07	2.02e-07	1.97e-08	
	CPU	0.0131	0.0120	0.0115	0.0115	0.0102	0.0117

From the numerical results of Tables 2, 3, 4, and 5, we can draw the following conclusions:

- The results in Tables 2 and 3 show that the larger θ is, the better iteration numbers will be.
- Note that the number of iteration results using $\psi_3(t)$ is efficient, but it consumes significantly more CPU time than other kernel functions, probably because its integral form leads to more complex operations.
- As the scale of the problems 5.2 and 5.3 increases, there is little or no increase in the number of iteration steps, which is precisely what is hoped for IPMs.
- Compared with the original kernel function, that is, when the parameter $q = 1$, the numerical result is obviously improved.
- From Tables 2-5, our kernel function has better numerical results than the other kernel functions.

Table 3. Numerical results based on different kernel functions for Problem 5.1

ψ_i			θ 0.2	0.4	0.6	0.8	0.99	Average
ψ_1		Iter	78	43	26	20	10	35.4
		Gap	9.89e-07	7.26e-07	8.46e-07	2.02e-07	1.97e-08	
		CPU	0.0141	0.0106	0.0097	0.0099	0.0089	0.0106
ψ_2		Iter	75	38	24	19	12	33.6
		Gap	9.89e-07	7.26e-07	8.46e-07	2.02e-07	1.97e-08	
		CPU	0.0127	0.0097	0.0095	0.0090	0.0089	0.0100
ψ_3		Iter	80	42	31	20	12	37
		Gap	9.89e-07	7.26e-07	8.46e-07	2.02e-07	1.97e-08	
		CPU	0.0137	0.0111	0.0105	0.0097	0.0087	0.0119
ψ_4	p=1	Iter	71	35	22	16	9	30.6
		Gap	9.89e-07	7.26e-07	8.46e-07	2.02e-07	1.97e-08	
		CPU	0.9113	0.4736	0.3568	0.2980	0.2820	0.4643
	p=2	Iter	72	36	22	17	9	31.2
		Gap	9.89e-07	7.26e-07	8.46e-07	2.02e-07	1.97e-08	
		CPU	0.8660	0.5065	0.3823	0.3163	0.2807	0.4704
	p=3	Iter	72	36	23	18	11	32
		Gap	9.89e-07	7.26e-07	8.46e-07	2.02e-07	1.97e-08	
		CPU	0.9036	0.5200	0.4019	0.3366	0.2817	0.4888
ψ_5	q=1	Iter	71	35	22	16	9	30.6
		Gap	9.89e-07	7.26e-07	8.46e-07	2.02e-07	1.97e-08	
		CPU	0.0121	0.0106	0.0102	0.0095	0.0091	0.0103
	q=2	Iter	72	36	22	17	9	31.2
		Gap	9.89e-07	7.26e-07	8.46e-07	2.02e-07	1.97e-08	
		CPU	0.0120	0.0111	0.0104	0.0100	0.0091	0.0105
	q=3	Iter	72	36	23	18	12	32.2
		Gap	9.89e-07	7.26e-07	8.46e-07	2.02e-07	1.97e-08	
		CPU	0.0123	0.0106	0.0103	0.0102	0.0093	0.0105

Table 4. Numerical results based on different kernel functions for Problem 5.2

$\psi_i(\theta = 0.99)$		n=10			n=50			n=100			Average	
		Iter	Gap	CPU	Iter	Gap	CPU	Iter	Gap	CPU	Iter	CPU
ψ_1		22	1.32e-07	0.0112	26	1.67e-07	0.0607	28	1.33e-08	0.0379	25.3	0.0366
ψ_2		25	1.32e-07	0.0138	28	1.67e-07	0.0450	30	1.33e-08	0.0352	27.7	0.0313
ψ_3		27	1.32e-07	0.0156	26	1.67e-07	0.0680	33	1.33e-08	0.0493	28.7	0.0443
ψ_4	p=1	19	1.32e-07	1.1234	21	1.67e-07	5.2673	22	1.33e-08	11.3652	20.7	5.9186
	p=2	19	1.32e-07	1.1746	20	1.67e-07	5.5205	22	1.33e-08	11.7512	20.3	6.1488
	p=3	19	1.32e-07	1.1638	23	1.67e-07	5.8152	26	1.33e-08	12.5233	22.7	6.5008
ψ_5	q=1	18	1.32e-07	0.0131	22	1.67e-07	0.0293	24	1.33e-08	0.0419	21.3	0.0281
	q=2	19	1.32e-07	0.0138	23	1.67e-07	0.0356	26	1.33e-08	0.0461	22.7	0.0318
	q=3	19	1.32e-07	0.0132	22	1.67e-07	0.0214	25	1.33e-08	0.0388	22	0.0245
ψ_{new}	q=1	19	1.32e-07	0.0107	22	1.67e-07	0.0393	24	1.33e-08	0.0469	21.7	0.0323
	q=2	18	1.32e-07	0.0127	20	1.67e-07	0.0287	22	1.33e-08	0.0272	20	0.0229
	q=3	19	1.32e-07	0.0133	23	1.67e-07	0.0330	26	1.33e-08	0.0346	22.7	0.0270

Table 5. Numerical results based on different kernel functions for Problem 5.3

$\psi_i(\theta = 0.99)$		n=10			n=50			n=100			Average	
		Iter	Gap	CPU	Iter	Gap	CPU	Iter	Gap	CPU	Iter	CPU
ψ_1		21	1.20e-07	0.0110	25	5.20e-07	0.0401	27	1.02e-08	0.0374	24.3	0.0295
ψ_2		19	1.20e-07	0.0129	22	5.20e-07	0.0399	30	1.02e-08	0.0332	23.7	0.0287
ψ_3		22	1.20e-07	0.0147	25	5.20e-07	0.0592	33	1.02e-08	0.0455	26.3	0.0398
ψ_4	p=1	17	1.20e-07	1.1023	18	5.20e-07	4.6845	22	1.02e-08	9.5526	19	5.1131
	p=2	16	1.20e-07	1.1324	17	5.20e-07	4.5205	21	1.02e-08	9.8843	18	5.1791
	p=3	17	1.20e-07	1.1478	17	5.20e-07	5.0012	21	1.02e-08	10.1011	18.3	5.4167
ψ_5	q=1	17	1.20e-07	0.0118	17	5.20e-07	0.0205	22	1.02e-08	0.0336	18.7	0.0220
	q=2	16	1.20e-07	0.0110	17	5.20e-07	0.0201	22	1.02e-08	0.0321	18.3	0.0211
	q=3	17	1.20e-07	0.0122	17	5.20e-07	0.0194	21	1.02e-08	0.0277	18.3	0.0197
ψ_{new}	q=1	17	1.20e-07	0.0105	17	5.20e-07	0.0293	22	1.02e-08	0.0315	18.7	0.0238
	q=2	16	1.20e-07	0.0100	17	5.20e-07	0.0225	21	1.02e-08	0.0252	18	0.0192
	q=3	17	1.20e-07	0.0115	17	5.20e-07	0.0230	21	1.02e-08	0.0319	18.3	0.0221

5.3 Numerical results for MTY predictor-corrector method

In the previous subsection, the number results of the method based on the classical logarithmic kernel function have been given, that is, the so-called primal-dual path-following algorithm. The numerical results of our algorithm are better. Our goal in this subsection is to make comparison between our algorithm and MTY predictor-corrector algorithm in [33].

Table 6. Numerical results for MTY predictor-corrector method

Problem		our (q=2,θ=0.99)			MTY		
		Iter	Gap	CPU	Iter	Gap	CPU
5.1		9	1.97e-08	0.0093	49	9.22e-07	0.7532
5.2	n=10	18	1.32e-07	0.0127	21	9.81-09	0.6823
	n=50	20	1.67e-07	0.0287	24	4.08e-08	1.2595
	n=100	22	1.33e-08	0.0272	50	8.09e-07	7.4915
5.3	n=10	16	1.20e-07	0.0100	24	6.94e-07	0.6511
	n=50	17	5.20e-07	0.0225	37	5.45e-07	1.7485
	n=100	21	1.02e-08	0.0346	36	6.74e-07	5.4926

- Table 6 show that the iteration numbers and CPU time of our algorithm are significantly better than the MTY predictor-corrector method

These results imply that our kernel function is quite efficient and promising.

6. Concluding remarks

In this paper, a new parameterized kernel function is proposed, which is a kind of generalization of the one used in [26]. We apply it in the designing of a primal-dual IPM for $P_*(\kappa)$ LCP. The properties of the kernel function have been analyzed. As a result, the iteration bound in [27] is improved from $O((1+2\kappa)n^{\frac{3}{4}}\log\frac{n}{\epsilon})$ to $O((1+2\kappa)\sqrt{n}\log n\log\frac{n}{\epsilon})$ for large-update method, the latter is currently the best known iteration bound for such methods, and we also obtain the best iteration bound for small-update method. The numerical results show that the algorithm depend on the value of kernel function is effective.

Further research may focus on finding a kernel function which makes the iteration bound of the large-update method based on this kernel function improve to $O((1+2\kappa)\sqrt{n}\log\frac{n}{\epsilon})$ (or even better), or prove that such a kernel function does not exist. Moreover, the extensions of the proposed kernel function in this paper to semidefinite optimization and second order cone optimization can be investigated.

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