On soft closed graph and its characterizations

Sandeep Kaur*
Department of Mathematics and Statistics
Central University of Punjab
Bathinda-151001, Punjab
India
sandybrar86@gmail.com

Navpreet Singh Noorie
Department of Mathematics
Punjabi University
Patiala-147002, Punjab
India
noorie@pbi.ac.in

Abstract. In this paper, we first introduce soft graph and soft closed graph of a soft mapping. Among other results, we obtain characterization of soft closed graph of identity soft mapping and characterization of soft-closedness of soft graphs using soft nets. We also give characterization of soft closure of a soft set in terms of convergence of soft net using soft points.

Keywords: soft topology, soft closed sets, soft graph, soft points, soft nets.

1. Introduction

In 1999, Molodtsov [7] introduced the theory of soft sets as a new mathematical tool for dealing with uncertainties. Topological structures of soft sets has been studied by some authors [11],[1],[10]. In 2011, Shabir and Naz [10] initiated the study of soft topological space. Consequently, they defined some basic notions of soft topological space such as soft open and closed sets, soft closure, soft separation axioms and established their several properties. Kharal and Ahmad [5], defined the notion of a soft mapping on soft classes and studied some properties of images and preimages of soft sets under soft mapping. Aygünoğlu and Aygün [1], introduced soft continuity of soft mappings and soft product topology. Nazmul and Samanta [8], studied the new concept of soft points and neighbourhood properties in a soft topological space. The purpose of this paper is to obtain new and significant results on soft closed graphs and soft continuity to further our understanding of this new and important mathematical discipline.

In section 2, we recall the basic concept related to soft topological space. In section 3, we define soft closed graph of soft mapping and study some charac-

∗. Corresponding author

terizations of soft closed graph. We give characterization of soft closure of a set in terms of convergence of soft net by using notion of soft points.

2. Preliminaries

Zorlutuna, Min and Atmaca studied soft topological spaces in [11] and introduced the following important notions which we shall use in our results below. Let X be an initial universe set, $P(X)$ the power set of X and A a set of parameters. A pair F_A is called a *soft set* over X, where F is a mapping given by $F : A \to P(X)$. In other words, a soft set over X is a parameterized family of subsets of the universe X. The family of all soft sets F_A over X is denoted by $S(X, A)$. The complement of a soft set F_A is denoted by F_A^c and is defined by $(F_A)^c = (F^c, A)$ where $F^c : A \to P(X)$ is a mapping given by $F^c(a) = X - F(a)$, for all $a \in A$. F^c is said to be the *soft complement* function of F. $(F^c)^c$ is the same as F and $((F_A^c)^c = F_A$. A soft set F_A over X is said to be a *null soft set*, denoted by Φ_A , if for all $a \in A$, $F(a) = \phi$. A soft set F_A over X is said to be an *absolute soft set*, denoted by $\widetilde{X}$, if for all $a \in A$, $F(a) = X$. For two soft sets F_A and G_A over a common universe X, F_A is a *soft subset* of G_A if $\forall a \in A$, $F(a) \subseteq G(a)$ for all $a \in A$ and is denoted by $F_A \widetilde{\subseteq} G_A$.$F_A$ is said that to be a *soft superset* of G_A, if G_A is a soft subset of F_A and is denoted by $F_A \widetilde{\supseteq} G_A$. For two soft sets F_A and G_A over a common universe X, *union* of two soft sets of F_A and G_A is the soft set H_A, where $H(a) = F(a) \cup G(a)$ for all $a \in A$ and is denoted by $F_A \widetilde{\cup} G_A = H_A$. For two soft sets F_A and G_A over a common universe X, *intersection* of two soft sets of F_A and G_A is the soft set H_A, where $H(a) = F(a) \cap G(a)$ for all $a \in A$ and is denoted by $F_A \widetilde{\cap} G_A = H_A$.

Definition 2.1 ([11]). *Let τ be the collection of soft sets over X with the fixed set of parameters A. Then τ is said to be a soft topology on X, if:*

(i) Φ_A, $\widetilde{X}$ belong to τ,

(ii) the union of any number of soft sets in τ belongs to τ,

(iii) the intersection of any two soft sets in τ belongs to τ.

The triplet (X, τ, A) is called a soft topological space over X. The members of τ are called soft open sets. The soft complement of a soft open set is called a soft closed set in (X, τ, A). The family of all soft closed sets is denoted by τ^c.

Definition 2.2 ([11]). *Let (X, τ, A) be a soft topological space over X and (F, A) a soft set over X. Then the soft closure of F_A, denoted by $\overline{F_A}$, is the intersection of all soft closed supersets of F_A and $\overline{F_A}$ is the smallest soft closed set in (X, τ, A) which contains F_A.*

We shall also make use of the fundamental notion of soft points introduced in [8]. A soft set P_A over X is said to be a soft point, denoted by P_a^x, if for the element $a \in A$, $P(a) = \{x\}$ and $P(a^{'}) = \phi$, for all $a^{'} \in A - \{a\}$. The soft point P_a^x is said to be in the soft set G_A, denoted by $P_a^x \widetilde{\in} G_A$, if $x \in G(a)$. Two soft points $P_{a_1}^{x_1}$, $P_{a_2}^{x_2}$ are said to be equal if $a_1 = a_2$ and $x_1 = x_2$. Thus, $P_{a_1}^{x_1} \neq P_{a_2}^{x_2}$ $\Leftrightarrow a_1 \neq a_2$ or $x_1 \neq x_2$.

From now on, the family of all soft points over X will be denoted by SP(X,A).

Proposition 2.1 ([8]). *Let F_A,$G_A \in S(X,A)$ and $P_a^x \in SP(X,A)$ then we have,*

1. $P_a^x \widetilde{\in} G_A$ *if and only if* $P_a^x \widetilde{\notin} G_A^c$.
2. $P_a^x \widetilde{\in} F_A \widetilde{\cup} G_A$ *if and only if* $P_a^x \widetilde{\in} F_A$ *or* $P_a^x \widetilde{\in} G_A$.
3. $P_a^x \widetilde{\in} F_A \widetilde{\cap} G_A$ *if and only if* $P_a^x \widetilde{\in} F_A$ *and* $P_a^x \widetilde{\in} G_A$.
4. $F_A \widetilde{\subseteq} G_A$ *if and only if* $P_a^x \widetilde{\in} F_A$ *implies* $P_a^x \widetilde{\in} G_A$.

Definition 2.3 ([8]). *A soft set F_A in a soft topological space (X, τ, A) is called a soft neighbourhood of the soft point P_a^x if there exists a soft open set G_B such that $P_a^x \widetilde{\in} G_B \widetilde{\subseteq} F_A$.*

The soft neighbourhood system of soft point P_a^x, denoted by $\mathcal{N}_\tau(P_a^x)$, is the family of all its soft neighbourhoods.

Proposition 2.2 ([6]). *Let (X, τ, A) be a soft topological space. A point $P_a^x \widetilde{\in} \overline{F_A}$ if and only if each soft neighbourhood of P_a^x intersects F_A.*

We shall also make use of following definitions and propositions

Definition 2.4 ([5]). *Let $S(X,A)$ and $S(Y,B)$ be the families of all soft sets over X and Y respectively. Let $\varphi : X \to Y$ and $e : A \to B$ be two mappings. Then, the mapping (φ, e) is called a soft mapping X to Y, denoted $(\varphi, e) : S(X,A) \to S(Y,B)$, for which:*

(i) Let $F_A \in S(X,A)$. Then $(\varphi, e)(F_A)$ is the soft set over Y defined as follows:

$$(\varphi, e)(F_A)(b) = \begin{cases} \bigcup\limits_{a \in e^{-1}(b) \cap A} \varphi(F(a)), & \text{if } e^{-1}(b) \cap A \neq \phi, \\ \phi, & \text{otherwise,} \end{cases}$$

for all $b \in B$. $(\varphi, e)(F_A)$ is called soft image of a soft set F_A.

(ii) Let $G_A \in S(Y,B)$. Then $(\varphi, e)^{-1}(G_B)$ is the soft set over X defined as follows:

$$(\varphi, e)^{-1}(G_B)(a) = \begin{cases} \varphi(G(e((a))), & \text{if } e(a) \in B, \\ \phi, & \text{otherwise,} \end{cases}$$

for all $a \in A$. $(\varphi, e)^{-1}(G_B)$ is called soft inverse image of a soft set G_B.

Definition 2.5 ([3]). *Let $F_A \in S(X,A)$ and $G_B \in S(Y,B)$. The cartesian product $F_A \times G_B$ is defined by $H_{A \times B}$, where $H : A \times B \to P(X \times Y)$ and $H(a,b) = F(a) \times G(a)$ for all $(a,b) \in A \times B$.*

Definition 2.6 ([3]). *Let (X, τ_1, A) and (Y, τ_2, B) be two soft topological spaces. The soft topological space $(X \times Y, \tau_1 \times \tau_2, A \times B)$, where $\tau_1 \times \tau_2$ is the collection of all soft unions of elements of $\{F_A \times G_B \mid F_A \in \tau_1, G_B \in \tau_2\}$, is called soft product topological space over $X \times Y$.*

Definition 2.7 ([9]).

1. *Let F_A be a soft set over X and Y be a nonempty subset of X. Then the sub-soft set of F_A over Y denoted by $({}^Y F_A)$ is defined as, ${}^Y F(a) = Y \cap F(a)$, for each $a \in A$.*
2. *Let (X, τ, A) be a soft topological space over X and Y is a nonempty subset of X. Then $\tau_Y = \{({}^Y F_A) | F_A \in \tau\}$ is said to be soft relative topology on Y and (Y, τ_Y, A) is called a soft subspace of (X, τ, A).*

Proposition 2.3 ([2]). *Let $(\varphi, e) : S(X, A) \to S(Y, B)$ be a soft mapping and $P^x_a \in SP(X, A)$. Then $(\varphi, e)(P^x_a) = P^{\varphi(x)}_{e(a)} \in SP(Y, B)$.*

Proposition 2.4 ([2]). *Let $(\varphi, e) : S(X, A) \to S(Y, B)$ be a soft mapping and $P^y_b \in SP(Y, B)$. If (φ, e) is bijective then $(\varphi, e)^{-1}(P^y_b) = P^{\varphi^{-1}(x)}_{e^{-1}(b)} \in SP(X, A)$.*

Proposition 2.5 ([10]). *Let (X, τ, A) be a soft topological space over X. Then the collection $\tau_a = \{F(a) : F_A \in \tau\}$ for each $a \in A$, defines a topology on X.*

3. Results

It is well known that the graph of a function $f : X \to Y$ is the collection of all ordered pair $(x, f(x))$ which is subset of $X \times Y$. In this section, we define soft graph of a soft mapping using the notion of soft points. Firstly, it must be a soft set in $X \times Y$ whose image of parameter (a, b) will be $\bigcup_x(x, \bigcup_a(\varphi, e)(P^x_a)(b))$. By Proposition 2.3, it can be easily seen that $(\varphi, e)(P^x_a)(b) = \varphi(x)$ when $b = e(a)$. Also, as every soft set is soft union of its soft points we get the following appropriate and simplified definition

Definition 3.1. *Let $S(X, A)$ and $S(Y, A)$ be the families of all soft sets over X and Y respectively and $(\varphi, e) : S(X, A) \to S(Y, B)$ be a soft mapping. Then Soft graph of (φ, e) is a soft set $\mathbb{G}(\varphi, e)_{A\times B}$, where $\mathbb{G}(\varphi, e) : A \times B \to \mathbf{P}(X \times Y)$ is defined by*

$$\mathbb{G}(\varphi, e)(a, b) = \begin{cases} \mathbb{G}(\varphi), & \text{if } b = e(a), \\ \phi, & \text{if } b \neq e(a). \end{cases}$$

Here $\mathbb{G}(\varphi)$ is usual graph of the function φ.

Definition 3.2. *Let (X, τ_1, A) and (Y, τ_2, B) be two soft topological spaces. If $\mathbb{G}(\varphi, e)_{A\times B}$ is soft closed in the soft product topological space $(X \times Y, \tau_1 \times \tau_2, A \times B)$, then we say that (φ, e) is a soft mapping with soft closed graph.*

Remark 3.1. It is straight forward to see that for soft topological spaces (X, τ_1, A) and (Y, τ_2, B) if the soft mapping $(\varphi, e) : S(X, A) \to S(Y, B)$ has a soft closed graph in soft product topological space $(X \times Y, \tau_1 \times \tau_2, A \times B)$ then the graph of $\varphi : X \to Y$ is also closed in the product topology on $X \times Y$. However, the converse is not true as shown in the example below

Example 3.1. Let $X = \{x, y, z\}$, $A = \{0, 1\}$ and $\tau = \{\Phi, \widetilde{X}, (F_1, A), (F_2, A), (F_3, A), (F_4, A), (F_5, A), (F_6, A)\}$ where

$$F_1(0) = \{x\}, F_1(1) = \{z\};$$
$$F_2(0) = \{y\}, F_2(1) = \{x\};$$
$$F_3(0) = \{z\}, F_3(1) = \{y\};$$
$$F_4(0) = \{x, y\}, F_4(1) = \{x, z\};$$
$$F_5(0) = \{x, z\}, F_5(1) = \{y, z\};$$
$$F_6(0) = \{y, z\}, F_6(1) = \{x, y\}.$$

Let $\varphi = 1_X$ and $e = 1_A$ and for each $a \in A$, (X, τ_a) is a Hausdorff space. Then, $\Delta(\varphi) = \{(x, x) | x \in X\}$ is closed subset of $X \times X$. Now $(\varphi, e) : S(X, A) \to S(Y, B)$ and soft graph of (φ, e), $\mathbb{G}(\varphi, e) : A \times A \to \mathbf{P}(X \times X)$ is defined by

$$\mathbb{G}(\varphi, e)(a, b) = \begin{cases} \Delta(\varphi), & \text{if } a = b \\ \phi, & \text{if } a \neq b \end{cases}$$

is not soft closed in $X \times X$.

We begin by giving characterization of soft closed graph of identity soft mapping

Theorem 3.1. *Let (X, τ, A) be a soft topological space. Let $1_X : X \to X$ and $1_A : A \to A$ be identity mappings. Then the soft mapping $(1_X, 1_A) : S(X, A) \to S(X, A)$ has a soft closed graph if and only if (X, τ, A) is soft Hausdorff space.*

Proof of Theorem 3.1. Let (X, τ, A) is soft Hausdorff space and soft graph of $(1_X, 1_A)$ is a soft set $\mathbb{G}(1_X, 1_A) : A \times A \to \mathbf{P}(X \times X)$ defined by

$$\mathbb{G}(1_X, 1_A)(a, b) = \begin{cases} \{(x, x) | x \in X\}, & \text{if } a = b, \\ \phi, & \text{if } a \neq b. \end{cases}$$

Assume that $P_{(a,b)}^{(x,y)} \widetilde{\in} \mathbb{G}(1_X, 1_A)^c_{A \times A}$ implies $P_a^x \neq P_b^y$. Since (X, τ, A) is soft Hausdorff space, there exist soft open sets F_A and G_B such that $P_a^x \widetilde{\in} F_A$ and $P_b^y \widetilde{\in} G_A$ and $(F_A) \widetilde{\cap} (G_A) = \Phi$. Hence $P_{(a,b)}^{(x,y)} \widetilde{\in} F_A \times G_A \widetilde{\subseteq} G(1_X, 1_A)^c_{A \times A}$ where $F_A \times G_A$ is soft open set in $X \times X$ implies $\mathbb{G}(1_X, 1_A)^c$ is soft open and hence, $\mathbb{G}(1_X, 1_A)_{A \times A}$ is soft closed in $X \times X$.

Conversely, let P_a^x, $P_b^y \in$ SP(X,A) be any two distinct soft points such that $P_{(a,b)}^{(x,y)} \widetilde{\in} \mathbb{G}(1_X, 1_A)^c_{A \times A}$ and $\mathbb{G}(1_X, 1_A)^c_{A \times A}$ is soft open in $X \times X$ then, there exist soft open sets M_A and N_A in X such that $P_{(a,b)}^{(x,y)} \widetilde{\in} M_A \times N_A \widetilde{\subseteq} \mathbb{G}(1_X, 1_A)^c_{A \times A}$. Therefore, $P_a^x \widetilde{\in} M_A$, $P_b^y \widetilde{\in} N_A$ and $(M_A) \widetilde{\cap} (N_A) = \Phi$.

Next, we need the characterization of soft closure in terms of convergence of soft nets to study soft graphs further. For this, we need the following

Definition 3.3 ([2]). *Let X be a set and $(D, \leq)$ be a directed set [4] where a directed set is a set with a relation $\leq$ (Throughout the paper, by $b \geq a$ we mean $a \leq b$) which is reflexive, transitive and upwards directive (where a set D is directed if for each m, $n \in D$, there is some $p \in D$ such that $p \geq m, p \geq n$). The function $T : D \to SP(X, A)$ defined by $T(\alpha) = P_{a_\alpha}^{x_\alpha}$ is called a soft net in X. For $\alpha \in D$, $T(\alpha)$ is denoted by T_α and hence a soft net is denoted by $\{T_\alpha = P_{a_\alpha}^{x_\alpha} \mid \alpha \in D\}$.*

Example 3.2 ([2]). Since $(\mathcal{N}_\tau(P_a^x), \widetilde{\subseteq}^*)$ is a directed set (where the relation $\widetilde{\subseteq}^*$ is defined by $F_A \widetilde{\subseteq}^* G_B$ if and only if $G_B \widetilde{\subseteq} F_A$) the function $T : \mathcal{N}_\tau(P_a^x) \to SP(X, A)$ is a soft net $\{T_{F_A} \mid F_A \in \mathcal{N}_\tau(P_a^x)\}$ where $T_{F_A} = P_a^x \widetilde{\in} F_A$.

Remark 3.2. Let (X, τ_1, A) and (Y, τ_2, B) be two soft topological spaces and $(\varphi, e) : S(X, A) \to S(Y, B)$ be a soft mapping. Let $\{T_\alpha = P_{a_\alpha}^{x_\alpha} \mid \alpha \in D\}$ be a soft net in (X, τ_1, A) then $\{(\varphi, e)(T_\alpha) = (\varphi, e)(P_{a_\alpha}^{x_\alpha}) = P_{e(a_\alpha)}^{\varphi(x_\alpha)} \mid \alpha \in D\}$ is a soft net in (Y, τ_2, B) and $\{P_{(a_\alpha, e(a_\alpha))}^{(x_\alpha, \varphi(x_\alpha))} \mid \alpha \in D\}$ is a soft net in $(X \times Y, \tau_1 \times \tau_2, A \times B)$.

Definition 3.4 ([2]). *Let (X, τ, A) be a soft topological space, $\{T_\alpha = P_{a_\alpha}^{x_\alpha} \mid \alpha \in D\}$ be a soft net in X and $F_A \in S(X, A)$*

1. *The soft net $\{P_{a_\alpha}^{x_\alpha} \mid \alpha \in D\}$ is called in F_A if $P_{a_\alpha}^{x_\alpha} \widetilde{\in} F_A$, for all $\alpha \in D$.*
2. *The soft net $\{P_{a_\alpha}^{x_\alpha} \mid \alpha \in D\}$ is called eventually in F_A if there exists some $\alpha_0 \in D$ such that $P_{a_\alpha}^{x_\alpha} \widetilde{\in} F_A$ for all $\alpha \geq \alpha_0$.*

Definition 3.5 ([2]). *A soft net $\{P_{a_\alpha}^{x_\alpha} \mid \alpha \in D\}$ in a soft topological space (X, τ, A) is said to converge to P_a^x, if it is eventually in every soft neighbourhood of P_a^x.*

Theorem 3.2. *$P_{a_\alpha}^{x_\alpha} \longrightarrow P_a^x$ where $P_a^x \in SP(X, A)$ and $P_{e(a_\alpha)}^{\varphi(x_\alpha)} \longrightarrow P_b^y$ where $P_b^y \in SP(X, A)$ if and only if $P_{(a_\alpha, e(a_\alpha))}^{(x_\alpha, \varphi(x_\alpha))} \longrightarrow P_{(a,b)}^{(x,y)}$ where $P_{(a,b)}^{(x,y)} \in SP(X \times Y, A \times B)$.*

Proof of Theorem 3.2. Let $P_{a_\alpha}^{x_\alpha} \longrightarrow P_a^x$ and $P_{e(a_\alpha)}^{\varphi(x_\alpha)} \longrightarrow P_b^y$. Assume $G_{A\times B} \in \mathcal{N}_\tau(P_{(a,b)}^{(x,y)})$ then there exist soft open sets F_A and H_B in X and Y respectively such that $P_{(a,b)}^{(x,y)} \widetilde{\in} F_A \times H_B \widetilde{\subseteq} G_{A\times B}$ which implies $P_a^x \widetilde{\in} F_A$ and $P_b^y \widetilde{\in} H_B$. As $P_{a_\alpha}^{x_\alpha} \longrightarrow P_a^x$ and $P_{e(a_\alpha)}^{\varphi(x_\alpha)} \longrightarrow P_b^y$ there exists $\alpha' \in D$ and $\alpha'' \in D$ such that $P_{a_\alpha}^{x_\alpha} \widetilde{\in} F_A$ for all $\alpha \geq \alpha'$ and $P_{e(a_\alpha)}^{\varphi(x_\alpha)} \widetilde{\in} H_B$ for all $\alpha \geq \alpha''$. Let $\alpha^0 = max\{\alpha', \alpha''\}$. This implies $P_{(a_\alpha, e(a_\alpha))}^{(x_\alpha, \varphi(x_\alpha))} \widetilde{\in} F_A \times H_B \widetilde{\subseteq} G_{A\times B}$ and hence $P_{(a_\alpha, e(a_\alpha))}^{(x_\alpha, \varphi(x_\alpha))} \longrightarrow P_{(a,b)}^{(x,y)}$.

Conversely, let $P_{(a_\alpha, e(a_\alpha))}^{(x_\alpha, \varphi(x_\alpha))} \longrightarrow P_{(a,b)}^{(x,y)}$. Assume $F_A \in \mathcal{N}_{\tau_1}(P_a^x)$ and $G_B \in \mathcal{N}_{\tau_2}(P_b^y)$ then $F_A \times G_B \in \mathcal{N}_{\tau_1\times\tau_2}(P_{(a,b)}^{(x,y)})$. Therefore, by assumption there exists an index $\alpha' \in D$ such that $P^{(x_\alpha,\varphi(x_\alpha))}_{(a_\alpha,e(a_\alpha))} \widetilde{\in} F_A \times G_B$ for all $\alpha \geq \alpha'$. This implies $P^{x_\alpha}_{a_\alpha} \widetilde{\in} F_A$ and $P^{\varphi(x_\alpha)}_{e(a_\alpha)} \widetilde{\in} G_B$ for all $\alpha \geq \alpha'$ and hence, $P^{x_\alpha}_{a_\alpha} \longrightarrow P^x_a$ and $P^{\varphi(x_\alpha)}_{e(a_\alpha)} \longrightarrow P^y_b$.

The following theorem gives characterization of soft closure of a soft set in terms of convergence of its soft net by using soft points. It is important to note that sufficient part of the following theorem is not true when we use concept of points belongs to a soft set, rather than soft points and therefore, by using the notion of soft points we get better results and are able to characterize soft closure below.

Theorem 3.3. *Let (X, τ, A) be a soft topological space. Let $G_A \in S(X, A)$ be a soft set and $P^x_a \in SP(X, A)$. Then $P^x_a \widetilde{\in} \overline{G_A}$ if and only if there exists a soft net $\{P^{x_\alpha}_{a_\alpha} | \alpha \in D\}$ in G_A i.e. $P^{x_\alpha}_{a_\alpha} \widetilde{\in} G_A$ for all $\alpha \in D$ such that $P^{x_\alpha}_{a_\alpha} \to P^x_a$.*

Proof of Theorem 3.3. Let $\{P^{x_\alpha}_{a_\alpha} | \alpha \in D\}$ be a soft net in G_A such that $P^{x_\alpha}_{a_\alpha} \longrightarrow P^x_a$. Let $F_A \in \mathcal{N}_\tau(P^x_a)$. Now as $P^{x_\alpha}_{a_\alpha} \longrightarrow P^x_a$, there exists an index $\alpha' \in D$ such that $P^{x_\alpha}_{a_\alpha} \widetilde{\in} F_A$ for all $\alpha \geq \alpha'$. Thus, $P^{x_{\alpha'}}_{a_{\alpha'}} \widetilde{\in} F_A \widetilde{\cap} G_A$ i.e. $x_{\alpha'} \in F_A(a_{\alpha'}) \cap G_A(a_{\alpha'})$ and therefore we have $F_A \widetilde{\cap} G_A \neq \Phi$. Hence $P^x_a \widetilde{\in} \overline{G_A}$.

Conversely, let $P^x_a \in \overline{G_A}$. This implies every soft neighborhood of P^x_a intersects G_A, that is $H_A \widetilde{\cap} G_A \neq \Phi$ for every $H_A \in \mathcal{N}_\tau(P^x_a)$. So there exists an $a \in A$ such that $H_A(a) \cap G_A(a) \neq \phi$. Let $y_{H_A} \in H_A(a) \cap G_A(a)$ and so $P^{y_{H_A}}_a \widetilde{\in} H_A \widetilde{\cap} G_A$. Now $(\mathcal{N}_\tau(P^x_a), \widetilde{\supseteq}^*)$ is a directed set (where the relation $\widetilde{\supseteq}^*$ is defined by $F_A \widetilde{\supseteq}^* G_B$ if and only if $F_A \widetilde{\subseteq} G_B$). Define a soft net, $T : \mathcal{N}_\tau(P^x_a) \to SP(X, A)$ as $T_{H_A} = P^{y_{H_A}}_a \widetilde{\in} H_A \widetilde{\cap} G_A$. Now $P^{y_{H_A}}_a \widetilde{\in} G_A$ for every $H_A \in \mathcal{N}_\tau(P^x_a)$. This implies $P^{y_{H_A}}_a$ is a soft net in G_A. For proving $P^{y_{H_A}}_a \to P^x_a$, assume $F_A \in \mathcal{N}_\tau(P^x_a)$ and $H_A \widetilde{\supseteq}^* F_A$ such that $H_A \widetilde{\subseteq} F_A$. Therefore, $T_{H_A} = P^{y_{H_A}}_a \widetilde{\in} H_A \widetilde{\subseteq} F_A$ which implies $P^{y_{H_A}}_a \widetilde{\in} F_A$, for every $H_A \widetilde{\supseteq}^* F_A$. Hence $P^{y_{H_A}}_a \longrightarrow P^x_a$.

We are now able to give the following another characterization of soft closed graph of a soft mapping.

Theorem 3.4. *Let (X, τ_1, A) and (Y, τ_2, B) be two soft topological spaces. A soft mapping $(\varphi, e) : S(X, A) \to S(Y, B)$ has a soft closed graph if and only if whenever $P^{x_\alpha}_{a_\alpha} \longrightarrow P^x_a$ in X and $(\varphi, e)(P^{x_\alpha}_{a_\alpha}) = P^{\varphi(x_\alpha)}_{e(a_\alpha)} \longrightarrow P^y_b$ in Y then $P^y_b = (\varphi, e)(P^x_a) = P^{\varphi(x)}_{e(a)}$.*

Proof of Theorem 3.4. Suppose (φ, e) has a soft closed graph and $P^{x_\alpha}_{a_\alpha} \to P^x_a$ in X and $(\varphi, e)(P^{x_\alpha}_{a_\alpha}) = P^{\varphi(x_\alpha)}_{e(a_\alpha)} \to P^y_b$ in Y. Then by Proposition 3.2, $P^{x_\alpha}_{a_\alpha} \to P^x_a$ in X and $P^{(x_\alpha,\varphi(x_\alpha))}_{(a_\alpha,e(a_\alpha))} \to P^{(x,y)}_{(a,b)}$ where $P^{(x_\alpha,\varphi(x_\alpha))}_{(a_\alpha,e(a_\alpha))} \widetilde{\in} \mathbb{G}(\varphi, e)_{A\times B}$, soft graph of (φ, e) which implies $P^{(x_\alpha,\varphi(x_\alpha))}_{(a_\alpha,e(a_\alpha))}$ is a soft net in $\mathbb{G}(\varphi, e)_{A\times B}$ converging to $P^{(x,y)}_{(a,b)}$. Therefore by Theorem 3.3, $P^{(x,y)}_{(a,b)} \widetilde{\in} \overline{\mathbb{G}(\varphi, e)}_{A\times B}$. Since $\mathbb{G}(\varphi, e)_{A\times B}$ is soft closed, then $P^{(x,y)}_{(a,b)} \widetilde{\in} \mathbb{G}(\varphi, e)_{A\times B}$. Hence $b = e(a)$ and $y = \varphi(x)$. Therefore, $P^y_b = P^{\varphi(x)}_{e(a)} = (\varphi, e)(P^x_a)$.

Conversely, suppose $P^{x_\alpha}_{a_\alpha} \to P^x_a$ and $(\varphi, e)(P^{x_\alpha}_{a_\alpha}) = P^{\varphi(x_\alpha)}_{e(a_\alpha)} \to P^y_b$ implies $P^y_b = (\varphi, e)(P^x_a) = P^{\varphi(x)}_{e(a)}$ and let $P^{(x,y)}_{(a,b)} \widetilde{\in} \overline{\mathbb{G}(\varphi, e)}_{A\times B}$. Then by Theorem 3.3, there exists a soft net $P^{(x_\alpha,\varphi(x_\alpha))}_{(a_\alpha,e(a_\alpha))}$ in $\mathbb{G}(\varphi, e)_{A\times B}$ converging to $P^{(x,y)}_{(a,b)}$ which implies $P^{x_\alpha}_{a_\alpha} \to P^x_a$ and $(\varphi, e)(P^{x_\alpha}_{a_\alpha}) = P^{\varphi(x_\alpha)}_{e(a_\alpha)} \to P^y_b$. Thus by assumption $P^y_b = P^{\varphi(x)}_{e(a)} = (\varphi, e)(P^x_a)$ implies $b = e(a)$ and $y = \varphi(x)$. Therefore, $P^{(x,y)}_{(a,b)} \widetilde{\in} \mathbb{G}(\varphi, e)_{A\times B}$ and hence $\mathbb{G}(\varphi, e)_{A\times B}$ is soft closed.

Remark 3.3. We note that the soft graph of the restriction of soft mapping with soft closed graph is also soft closed where restriction of soft mapping is defined in [12].

Theorem 3.5. *Let the soft mapping $(\varphi, e) : S(X, A) \to S(Y, B)$ has a soft closed graph and $M \subset X$, $N \subset Y$ such that $\varphi(M) \subset N$. Then $(\varphi, e)|_{S(M,A)} : S(M, A) \to S(Y, B)$ also has a soft closed graph.*

Proof of Theorem 3.5. Let $\{P^{x_\alpha}_{a_\alpha} | \alpha \in D\}$ be a soft net in M such that $P^{x_\alpha}_{a_\alpha} \longrightarrow P^x_a$ where $P^x_a \in SP(M, A)$ and $\{(\varphi, e)(P^{x_\alpha}_{a_\alpha}) = P^{\varphi(x_\alpha)}_{e(a_\alpha)} \mid \alpha \in D\}$ be a soft net in N such that $P^{\varphi(x_\alpha)}_{e(a_\alpha)} \longrightarrow P^y_b$ where $P^y_b \in SP(N, B)$. Therefore, $\{P^{x_\alpha}_{a_\alpha} | \alpha \in D\}$ is a soft net in X and $\{P^{\varphi(x_\alpha)}_{e(a_\alpha)} \mid \alpha \in D\}$ is a soft net in Y. Since (φ, e) has a soft closed graph then by Theorem 3.4, $P^y_b = (\varphi, e)(P^x_a) = P^{\varphi(x)}_{e(a)} = (\varphi, e)|_{S(M,A)}(P^x_a)$.

Finally, in this section we give another characterization of soft closed graph using soft open sets.

Theorem 3.6. *The soft mapping $(\varphi, e) : S(X, A) \to S(Y, B)$ has a soft closed graph if and only if for every $P^x_a \in SP(X, A)$ and $P^y_b \in SP(Y, B)$ where $P^y_b \neq (\varphi, e)(P^x_a) = P^{\varphi(x)}_{e(a)}$, there exist soft open sets F_A in X where $P^x_a \widetilde{\in} F_A$ and G_B in Y where $P^y_b \widetilde{\in} G_B$ such that $(\varphi, e)(F_A) \widetilde{\cap} G_B = \Phi$.*

Proof of Theorem 3.6. Suppose the given condition holds. To prove sufficiency it is enough to show that $\overline{\mathbb{G}(\varphi, e)}_{A\times B} \widetilde{\subseteq} \mathbb{G}(\varphi, e)_{A\times B}$. Let us assume $P^{(x,y)}_{(a,b)} \widetilde{\in} \mathbb{G}(\varphi, e)^C_{A\times B}$ which implies either $b \neq e(a)$ or $y \neq \varphi(x)$. Then $P^y_b \neq (\varphi, e)(P^x_a) = P^{\varphi(x)}_{e(a)}$. Therefore by hypothesis, there exist soft open sets F_A in X where $P^x_a \widetilde{\in} F_A$ and G_B in Y where $P^y_b \widetilde{\in} G_B$ such that $(\varphi, e)(F_A) \widetilde{\cap} G_B = \Phi$, which implies $(F_A \times G_B) \widetilde{\cap} \mathbb{G}(\varphi, e)_{A\times B} = \Phi$, where $F_A \times G_B$ is soft open and $P^{(x,y)}_{(a,b)} \widetilde{\in} F_A \times G_B$. Hence $P^{(x,y)}_{(a,b)} \widetilde{\in} \overline{\mathbb{G}(\varphi, e)}^C_{A\times B}$ and so $\mathbb{G}(\varphi, e)_{A\times B}$ is soft closed.

Conversely, let $\mathbb{G}(\varphi, e)_{A\times B}$ be soft closed and $P^x_a \in SP(X, A)$ and $P^y_b \in SP(Y, B)$ where $P^y_b \neq P^{\varphi(x)}_{e(a)}$, Therefore either $b \neq e(a)$ or $y \neq \varphi(x)$, which implies $P^{(x,y)}_{(a,b)} \widetilde{\notin} \mathbb{G}(\varphi, e)_{A\times B}$. As $\mathbb{G}(\varphi, e)_{A\times B}$ is soft closed, $P^{(x,y)}_{(a,b)} \widetilde{\notin} \overline{\mathbb{G}(\varphi, e)}_{A\times B}$. Therefore there exist soft open sets F_A and G_B in X and Y respectively where $P^{(x,y)}_{(a,b)} \widetilde{\in} F_A \times G_B$ such that $(F_A \times G_B) \widetilde{\cap} \mathbb{G}(\varphi, e)_{A\times B} = \Phi$, which implies $(\varphi, e)(F_A) \widetilde{\cap} G_B = \Phi$.

Acknowledgments

The authors are very grateful to the reviewer's valuable inputs which were of great help to improve the quality of the paper.

References

[1] A. Aygünoğlu, H. Aygün, *Some notes on soft topological spaces*, Neural Comput. Appl., 21 (2012), 113-119.

[2] *İ*, Demir, O.B. Özbakir, *Soft Hausdorff spaces and their properties*, Annal. Fuzzy. Math. Inform., 8 (2014), 769-783.

[3] D. N. Georgiou, A. C. Megaritis, V. I. Petropoulos, *On soft topological spaces*, Appl. Math. Inf. Sci., 7(5) (2013), 1889-1901.

[4] J.L. Kelly, *General topology*, Springer-Verlag, New York, 1975.

[5] A. Kharal, B. Ahmad, *Mappings on soft classes*, New Math. Nat. Comput., 7 (2011), 471-481.

[6] F. Lin, *Soft connected spaces and soft paracompact spaces*, Int. J. Math. Comput. Sci. Eng., 7 (2013), 37-43.

[7] D.A. Molodtsov, *Soft set theory-first results*, Comput. Math. Appl., 37 (1999), 19-31.

[8] S. Nazmul, S.K. Samanta, *Neighbourhood properties of soft topological spaces*, Annal. Fuzzy. Math. Inform., 6 (2013), 1-15.

[9] E. Peyghan, B. Samadi, A. Tayebi, *Some results related to soft topological spaces*, Facta Universitatis Ser. Math. Inform., 29 (2014), 325-336.

[10] M. Shabir, M. Naz, *On soft topological spaces*, Comput. Math. Appl., 61 (2011), 1786-1799.

[11] I. Zorlutuna, M. Akdag, W.K. Min, S. Atmaca, *Remarks on soft topological spaces*, Annal. Fuzzy. Math. Inform., 3 (2012), 171- 185.

[12] I. Zorlutuna, H. Çakir, *On continuity of soft mappings*, Appl. Math. Inf. Sci., 9 (2015), 403-409.

Accepted: 20.03.2019