The structure of a class of inverse residuated lattices

Wei Chen
School of Mathematics and Statistics
Minnan Normal University
Zhangzhou, Fujian 363000
P.R. China
chenwei6808467@126.com

Abstract. In this paper, we study some special inverse residuated lattices, namely, E-unitary inverse residuated chains. After giving some properties of such residuated lattices, we obtain a structure theorem for E-unitary inverse residuated chains.
Keywords: residuated lattice, $E-$unitary inverse semigroup, chain.

1. Introduction

Let $(\mathfrak{P}, \leq)$ be a poset. A (binary) operation $\circ$ is called *residuated* if there exist (binary) operations $\backslash$ and $/$ on $\mathfrak{P}$ such that

$$(\forall x, y \in \mathfrak{P})\; x \circ y \leq z \Longleftrightarrow y \leq x\backslash z \Longleftrightarrow x \leq z/y.$$

In this case, the operation pair $(\backslash, /)$ is called a *residuals* of the operation $\circ$. It is well known that an operation $\circ$ on the poset $(\mathfrak{P}, \leq)$ is residuated if and only if $\circ$ is order preserving in each argument and such that for all $a, b \in \mathfrak{P}$, both $\{p \in \mathfrak{P} \mid a \circ p \leq b\}$ and $\{q \in \mathfrak{P} \mid q \circ a \leq b\}$ contain the greatest element(denote by $a\backslash b$ and b/a, respectively). A *residuated lattice* is defined as an algebra $(\mathfrak{L}, \wedge, \vee, \circ, \backslash, /, e)$ satisfying the following conditions:

$(RL1)$ $(\mathfrak{L}, \wedge, \vee)$ is a lattice;
$(RL2)$ $(\mathfrak{L}, \circ)$ is a monoid with identity e; and
$(RL3)$ the operation pair $(\backslash, /)$ is a residuals of the operation $\circ$.

In this case, we call the lattice $(\mathfrak{L}, \wedge, \vee)$ and the semigroup $(\mathfrak{L}, \circ)$ the *lattice reduct* and the *semigroup reduct* of the residuated lattice $(\mathfrak{L}, \wedge, \vee, \circ, \backslash, /, e)$, respectively. Some time, residuated lattices are also called *residuated lattice-ordered monoids*. A residuated lattice is called an *inverse residuated lattice* if its semigroup reduct is an inverse semigroup. An inverse residuated lattice is called an *$E-$unitary inverse residuated lattice* if its semigroup reduct is an $E-$unitary inverse semigroup. An $E-$unitary inverse residuated lattice is called an *$E-$unitary inverse residuated chain* if its lattice reduct is a chain.

Motivated by the ideal lattices of rings with identity, residuated lattices are initially introduced by Ward and Dilworth in the late of 1930's [21]. Since that time, there has been substantial research regarding some specific classes of residuated structures(see[4-16]). Recently, some authors investigated various types of residuated lattices from different perspectives . For example, Kühr in [10] considered the $\{\backslash, /, e\}$-subreducts of integral residuated lattices as Pseudo-BCK-algebras and used Pseudo-BCK-algebra theory to investigate integral residuated lattices. Residuated lattices can be considered as a class of partial ordered semigroups. We investigate idempotent residuated lattices from semigroup perspectives and obtain structure theorems and decomposition theorem for two class of idempotent residuated lattices in [5,6,7], which generalize [9, Theorem 20].

In the present paper we will use algebra theory of partial ordered semigroups to investigate E-unitary inverse residuated chains. In algebra theory of semigroups, E-unitary inverse semigroups are of the important kinds which are situated between groups and general semigroups. McAlister in [3] investigated E-unitary inverse semigroups. He obtained a structure theorem for E-unitary inverse semigroups. Saitô studied totally ordered E-unitary inverse semigroups in [2]. He described a structure theorem for totally ordered E-unitary inverse semigroups. In [1], Gomes, Giraldes and McAlister simplified Saitô's structure theorem for totally ordered E-unitary inverse semigroups by using McAlister's structure theorem for E-unitary inverse semigroups. We consider E-unitary inverse residuated chains as a class of totally ordered E-unitary inverse semigroups. The following question naturally arises: How can we characterize the structure of E-unitary inverse residuated chains? The purpose of this paper is to solve this question.

We proceed as follows: Section 2 presents some necessary notation and known results. In Section 3, we give some fundamental properties of E-unitary inverse residuated chains. In Section 4, we use the above results to give a solution to the question. We establish a structure theorem for E-unitary inverse residuated chains.

2. Preliminaries

In what follows, we shall use the notion and notation in [1,7]. Other undefined terms can be found in [17,18]. First we recall some of the basic facts about totally ordered E-unitary inverse semigroups.

An element a of a semigroup S is called *regular* if there exists x in S such that $axa = a$. A semigroup S is called *regular* if all its elements are regular. A semigroup S is called *inverse* if S is regular and its idempotents commute. For brevity, we use E(S) to denote the set of idempotents of S. An inverse semigroup is called *E-unitary* if for all $e \in E(S)$ and $s \in S$, $es \in E(S)$ implies $s \in E(S)$. An inverse semigroup S is said to be a *partially ordered semigroup*, or to be *partially ordered*, if it admits a compatible partial ordering $\leq$; that is, $\leq$ is a partial order on the set S and for each $a, b \in S$, $a \leq b$ implies $xay \leq xby$ for all $x, y \in S^1$. A partially ordered inverse semigroup S is said to be *totally ordered* or a *totally ordered inverse semigroup* if the imposed partial order is a total order; that is, if a, b are distinct elements of S then either $a < b$ or $b < a$.

A partially ordered set is said to be a *chain* if it is a totally ordered set. A semilattice S is called *tree* if for each $x \in S$, the principal ideal generated by x is a chain.

Suppose now that S is an E-unitary inverse semigroup and let $G = S/\sigma$ be its maximal group homomorphic image. Then S is isomorphic to a $P-$semigroup $P(G, \mathscr{X}, \mathscr{Y})$, where $(G, \mathscr{X}, \mathscr{Y})$ is a *McAlister triple* in the sense that $\mathscr{X}$ is a down directed partially ordered set, $\mathscr{Y}$ is a subsemilattice and order ideal of $\mathscr{X}$, and G acts, on the left, on $\mathscr{X}$, by order automorphisms, in such a way that $\mathscr{X} = G\mathscr{Y}$ and for $g \in G$, $g\mathscr{Y} \cap \mathscr{Y} \neq \emptyset$. Moreover, $\mathscr{Y}$ is isomorphic to the semilattice of idempotents of S. The elements of $P(G, \mathscr{X}, \mathscr{Y})$ are the pairs $(a, g) \in \mathscr{Y} \times G$ for which $g^{-1}a \in \mathscr{Y}$, and multiplication is defined by $(a, g)(b, h) = (a + gb, gh)$ where $+$ denotes greatest lower bound (when it exists) in the partially ordered set $\mathscr{Y}$. We always use $+$ for this operation to avoid confusion with $\vee$ and $\wedge$ when the semigroup in question is a totally ordered semigroup and use $\leq^*$ to denote the partial order on the set $\mathscr{X}$ following our convention of reserving $\leq$ for the imposed partial order. Saitô [20] has shown that in totally ordered E-unitary inverse semigroups the semilattice of idempotents must be a totally ordered semilattice and such a semilattice must be a binary tree. That is, it is a semilattice in which each principal ideal is a chain and in which there do not exist distinct incomparable elements a, b, c with $a \wedge b = b \wedge c = c \wedge a$.

We shall write $x \prec y (x \prec^* y)$ for the fact that x is covered by y; i.e., $x < y (x <^* y)$ and for every z, if $x \leq z \leq y (x \leq^* z \leq^* y)$, then $z = x$ or $z = y$. By Theorem 5.9.2 of [18] and Corollary 2.8 of [1], we have:

Theorem 2.1 ([1])**.** *Let G be a totally ordered group, $\mathscr{X}$ a tree, and $\mathscr{Y}$ an ideal of $\mathscr{X}$ which is a totally ordered binary tree. Suppose, further, that G acts by order automorphisms on $\mathscr{X}$ in such a way that $\mathscr{X} = G\mathscr{Y}$, for $g \in G$, $g\mathscr{Y} \cap \mathscr{Y} \neq \emptyset$ and if $a, b \in \mathscr{Y}, g \in G$ with $ga, gb \in \mathscr{Y}$ then $ga \leq gb$ if and only if $a \leq b$. Then, under the lexicographic ordering,*

$$(a, g) \leq (b, h) \text{ if and only if } g < h \text{ or } g = h \text{ and } a \leq b$$

$P(G, \mathscr{X}, \mathscr{Y})$ is a totally ordered E-unitary inverse semigroup. Conversely, any totally ordered E-unitary inverse semigroup is isomorphic to one of this form.

By a *lattice-ordered monoid*, we mean an ordered monoid $(M, \circ, \leq)$ in which the ordered set $(M, \leq)$ is still a lattice. Moreover, a lattice-ordered monoid is called *idempotent* if the monoid $(M, \circ)$ is a band. For the detail information on ordered semigroups, the reader is referred to references [16] and [18].

For our purpose, we need the following lemma.

Lemma 2.2 ([4])**.** *Let $(M, \circ, \leq)$ be a lattice-ordered idempotent monoid with identity e, and $a, b \in M$:*

(1) $a \wedge b \leq ab \leq a \vee b$.
(2) *If* $a, b \geq e$, *then* $ab = a \vee b$.
(3) *If* $a, b \leq e$, *then* $ab = a \wedge b$.
(4) *If* $a \leq e \leq ab$, *then* $ab = b$.
(5) *If* $ab \leq e \leq a$, *then* $ab = b$.

3. Fundamental properties

In this section, we will give some fundamental properties of E-unitary inverse residuated chains. We assume $\mathfrak{L} = P(G, \mathscr{X}, \mathscr{Y})$ is an E-unitary inverse residuated chain and $(e, 1)$ is an identity of $\mathfrak{L}$. By the proof of [17, Theorem 5.9.2], e is an identity of $\mathscr{Y}$ and 1 is an identity of G. Since $\mathfrak{L}$ is a totally ordered E-unitary inverse semigroup, by Theorem 2.1, $\mathscr{Y}$ is a binary tree and so $\mathscr{Y}$ is a chain with greatest element e with respect to $\leq^*$ by noting that e is an identity of $\mathscr{Y}$. We define $\mathscr{Y}^- = \{a \in \mathscr{Y} | a \leq e\}$, $\widetilde{\mathscr{Y}} = \{a \in \mathscr{Y} | (\exists b \in \mathscr{Y}^-) b \leq^* a\}$ and $A(\mathfrak{L}) = \{(a, 1) \in \mathfrak{L} | a \in \widetilde{\mathscr{Y}}\}$. We always denote by $\top$ the maximum element of $\mathscr{Y}$ with respect to the imposed ordering $\leq$ if it exists. We have the following results:

Proposition 3.1 ([8]). *Let* $\mathfrak{L} = P(G, \mathscr{X}, \mathscr{Y})$ *be an* E*-unitary inverse residuated chain. Then the following properties hold:*

(1) $\boldsymbol{A}(\mathfrak{L})$ *is a subalgebra of* $\mathfrak{L}$*;*

(2) *If* $\mathscr{Y} \backslash \widetilde{\mathscr{Y}} \neq \emptyset$, *then for every* $g \in G$, *there exists* $\bar{g} \in G$ *such that* $\bar{g} \prec g$ *and* $\top$ *exists; moreover, in this case,* $\top \leq^* a$ *for all* $a \in \mathscr{Y}$, $g\top = \top$ *for all* $g \in G$ *and* h *is central in* G *where* $h = \bar{1} \prec 1$*;*

(3) *If* $a \in \widetilde{\mathscr{Y}}$ *such that* $a > e$, *then there exists a unique element* $b \in \mathscr{Y}$ *such that* $b <^* a, b < e$ *and if* $c \in \mathscr{Y}$ *such that* $b <^* c <^* a$, *then* $c > e$*;*

(4) *If* $b \in \mathscr{Y}$ *such that* $b < e$, *then there exists a unique element* $a \in \mathscr{Y}$ *such that* $b <^* a, a \geq e$ *and if* $c \in \mathscr{Y}$ *such that* $b <^* c <^* a$, *then* $c < e$*;*

(5) *If* $(a, g), (b, h) \in \mathfrak{L}$ *such that* $a \in \mathscr{Y} \backslash \widetilde{\mathscr{Y}}$ *and* $a > b$, $(c, k) = (a, g) \backslash (b, h)$, *then* $c = \top$ *and* $k \prec g^{-1}h$*;*

(6) *If* $(a, g), (b, h) \in \mathfrak{L}$ *such that* $a \in \widetilde{\mathscr{Y}}$ *or* $a \in \mathscr{Y} \backslash \widetilde{\mathscr{Y}}$ *and* $a \leq b$, $(c, k) = (a, g) \backslash (b, h)$, *then* $k = g^{-1}h$*;*

(7) *If* $(a, g), (b, h) \in \mathfrak{L}$ *such that* $hg^{-1}a \in \mathscr{Y} \backslash \widetilde{\mathscr{Y}}$ *and* $hg^{-1}a > b$ *or* $hg^{-1}a \notin \mathscr{Y}$, $gh^{-1}b \notin \mathscr{Y}$ *and* $hg^{-1}\mathscr{Y} \cap \widetilde{\mathscr{Y}} = \emptyset$, $(d, l) = (b, h)/(a, g)$, *then* $d = \top$ *and* $l \prec hg^{-1}$*;*

(8) *If* $(a, g), (b, h) \in \mathfrak{L}$ *satisfying one of the following conditions:*

(*a*) $hg^{-1}a \notin \mathscr{Y}$, $gh^{-1}b \notin \mathscr{Y}$ *and* $hg^{-1}\mathscr{Y} \cap \widetilde{\mathscr{Y}} \neq \emptyset$;

(*b*) $hg^{-1}a \notin \mathscr{Y}$ *and* $gh^{-1}b \in \mathscr{Y}$;

(*c*) $hg^{-1}a \in \widetilde{\mathscr{Y}}$;

(*d*) $hg^{-1}a \in \mathscr{Y} \backslash \widetilde{\mathscr{Y}}$ *and* $hg^{-1}a \leq b$*; and* $(d, l) = (b, h)/(a, g)$, *then* $l = hg^{-1}$.

Let $\mathfrak{L}$ be an inverse semigroup. The natural ordering $\leq_n$ on $\mathfrak{L}$ is given by for all $a, b \in \mathfrak{L}$, $a \leq_n b$ if and only if there exists $e \in E(\mathfrak{L})$ such that $a = eb$. Let

σ be the minimum group congruence on $\mathfrak{L}$. $\mathfrak{L}$ is called *$F-$inverse semigroup* if every $\sigma-$class contains a greatest element for $\leq_n$. By Exercise 5.11.36 of [18], an $F-$inverse semigroup contains an identity element and is $E-$unitary.

Proposition 3.2. *Let $\mathfrak{L}$ be a totally ordered inverse monoid. Then $\mathfrak{L}$ is an E-unitary semigroup if and only if $\mathfrak{L}$ is an F-inverse semigroup.*

Proof. We need only to verify the sufficiency because the necessity is obvious. Now assume $\mathfrak{L} = P(G, \mathscr{X}, \mathscr{Y})$ is a totally ordered E-unitary semigroup with an identity. Then by Theorem 2.1, $\mathscr{Y}$ is a principal ideal of $\mathscr{X}$ and $\mathscr{X}$ is a tree, hence by Exercise 5.38 of [18], $\mathfrak{L}$ is an F-inverse semigroup. □

Proposition 3.3. *Let $\mathfrak{L} = P(G, \mathscr{X}, \mathscr{Y})$ be a totally ordered E-unitary semigroup with an identity $(e, 1)$.*

(1) *For all $g \in G$, $g\mathscr{Y} \cap \mathscr{Y}$ is a principal ideal of $\mathscr{Y}$.*

(2) *If $\mathfrak{L}$ is a residuated lattice and $g \in G$, $g\mathscr{Y} \cap \mathscr{Y}^- \neq \emptyset$, then $g\mathscr{Y} \cap \mathscr{Y}^-$ is a principal ideal of $\mathscr{Y}^-$.*

Proof. (1) Since $\mathscr{X}$ is a tree and $\mathscr{Y}$ is a principal ideal of $\mathfrak{L}$, $e+g^{-1}e, ge+e \in \mathscr{Y}$. Then $ge+e = g(e+g^{-1}e) \in g\mathscr{Y} \cap \mathscr{Y}$. Let $b \in g\mathscr{Y} \cap \mathscr{Y}$. Then there exists $c \in \mathscr{Y}$ such that $b = gc \in \mathscr{Y}$. Since $c \leq^* e$, $b = gc \leq^* ge$. Hence $b \leq^* ge + e$. Let $d \leq^* e + ge$. Since $e + g^{-1}e \in \mathscr{Y}$, $g^{-1}d \in \mathscr{Y}$. Then $d = g(g^{-1}d) \in g\mathscr{Y}$ and so $d \in g\mathscr{Y} \cap \mathscr{Y}$. Thus $g\mathscr{Y} \cap \mathscr{Y}$ is a principal ideal of $\mathscr{Y}$.

(2) Suppose that $\mathfrak{L}$ is a residuated lattice and $g \in G$, $g\mathscr{Y} \cap \mathscr{Y}^- \neq \emptyset$. If $e + ge \leq e$, then by (1), $g\mathscr{Y} \cap \mathscr{Y}$ is a principal ideal of $\mathscr{Y}$ with the greatest element $e + ge$ with respect to $\leq^*$, hence $g\mathscr{Y} \cap \mathscr{Y}^-$ is a principal ideal of $\mathscr{Y}^-$. If $e + ge > e$, we can claim that $e + ge \in \widetilde{\mathscr{Y}}$. Otherwise, if $e + ge \in \mathscr{Y} \backslash \widetilde{\mathscr{Y}}$, then for $b \in \mathscr{Y}$ such that $b <^* e + ge, b > e$, which implies $g\mathscr{Y} \cap \mathscr{Y}^- = \emptyset$, contrary to the hypothesis. Consequently, $e + ge \in \widetilde{\mathscr{Y}}$. By Proposition 3.1 (3), there exists a unique element $b \in \mathscr{Y}^-$ such that b is the maximum element in $g\mathscr{Y} \cap \mathscr{Y}^-$ for $\leq^*$. This means that $g\mathscr{Y} \cap \mathscr{Y}^-$ is a principal ideal of $\mathscr{Y}^-$. □

4. Construction

In this section, we will show how to construct E-unitary inverse residuated chains from some given specific ingredients.

Let $(G, \mathscr{X}, \mathscr{Y})$ be a McAlister triple such that G is a totally ordered group, $\mathscr{X}$ is a tree and $\mathscr{Y}$ is a principal ideal of $\mathscr{X}$ which is a totally ordered semilattice with an identity e. put $\mathfrak{L} = \{(a, g) \in \mathscr{Y} \times G | g^{-1}a \in \mathscr{Y}\}$. Let $\mathscr{Y}^- = \{a \in \mathscr{Y} | a \leq e\}$, $\widetilde{\mathscr{Y}} = \{a \in \mathscr{Y} | (\exists b \in \mathscr{Y}^-) b \leq^* a\}$ and $\widetilde{\mathscr{Y}}^+ = \{a \in \widetilde{\mathscr{Y}} | a \geq e\}$. Given two mappings

$$\varphi : \widetilde{\mathscr{Y}}^+ \backslash \{e\} \to \mathscr{Y}^-; \quad a \mapsto \varphi(a)$$

and

$$\psi : \mathscr{Y}^- \backslash \{e\} \to \widetilde{\mathscr{Y}}^+; \quad a \mapsto \psi(a).$$

We call $(\mathfrak{L}; G, \mathscr{X}, \mathscr{Y}; \varphi, \psi)$ an *EIRC*-system if the following conditions hold:

$(E1)$ If $\mathscr{Y}\backslash\widetilde{\mathscr{Y}} \neq \emptyset$, then the greatest element $\top$ of $\mathscr{Y}$ exists with respect to $\leq$ and for every $g \in G$, there exists $\bar{g} \in G$ such that $\bar{g} \prec g$.

$(E2)$ For every $a \in Dom\varphi$, $\varphi(a) <^* a$.

$(E3)$ For every $a \in Dom\psi$, $a <^* \psi(a)$.

$(E4)$ If $a \in Dom\varphi, b \in \mathscr{Y}$ such that $\varphi(a) <^* b <^* a$, then $b > e$.

$(E5)$ If $a \in Dom\psi, b \in \mathscr{Y}$ such that $a <^* b <^* \psi(a)$, then $b < e$.

$(E6)$ If $a, b \in \mathscr{Y}, g \in G$ with $ga, gb \in \mathscr{Y}$, then $ga \leq gb$ if and only if $a \leq b$.

$(E7)$ If $g \in G$ such that $g\mathscr{Y} \cap \mathscr{Y} \neq \emptyset$, then $g\mathscr{Y} \cap \mathscr{Y}^-$ is a principal ideal of $\mathscr{Y}^-$.

Give an $EIRC-$system $(\mathfrak{L}; G, \mathscr{X}, \mathscr{Y}; \varphi, \psi)$. We define a multiplication on $\mathfrak{L}$ by the rule that

$$(a, g) \circ (b, h) = (a + gb, gh).$$

By the proof of Theorem 5.9.2 of [18], we have the following result:

Lemma 4.1. *If $(\mathfrak{L}; G, \mathscr{X}, \mathscr{Y}; \varphi, \psi)$ is an $EIRC-$system, then $(\mathfrak{L}, \circ)$ is an E-unitary inverse semigroup with an identity $(e, 1)$.*

We define the lexicographic ordering on $\mathfrak{L}$ by

$$(a, g) \leq (b, h) \text{ if and only if } g < h \text{ or } g = h \text{ and } a \leq b.$$

By Theorem 2.1, we have the following lemma:

Lemma 4.2. *If $(\mathfrak{L}; G, \mathscr{X}, \mathscr{Y}; \varphi, \psi)$ is an $EIRC-$system, then $(\mathfrak{L}, \circ, \leq)$ is a totally ordered E-unitary inverse semigroup.*

Give an $EIRC-$system $(\mathfrak{L}; G, \mathscr{X}, \mathscr{Y}; \varphi, \psi)$. Let $g \in G$. We always denote $\bar{g}$ the element k of G satisfying $k \prec g$, if it exists. If $g\mathscr{Y} \cap \mathscr{Y}^- \neq \emptyset$, then by (E7), $g\mathscr{Y} \cap \mathscr{Y}^-$ is a principal ideal of $\mathscr{Y}^-$; in this case, we denote the largest element of $g\mathscr{Y} \cap \mathscr{Y}^-$ for $\leq^*$ by t_g. We define two division operations $/$ and $\backslash$ on $\mathfrak{L}$ as follows: for $(a, g), (b, h) \in \mathfrak{L}$,

$$(b,h)/(a,g)$$

$$(4.1) \quad = \begin{cases} (t_{hg^{-1}}, hg^{-1}) & \text{if } h^{-1}ga \notin \mathscr{Y}, g^{-1}hb \notin \mathscr{Y}, hg^{-1}\mathscr{Y} \cap \widetilde{\mathscr{Y}} \neq \emptyset \\ & \text{or } hg^{-1}a \leq^* b, hg^{-1}a < e, gh^{-1}\psi(hg^{-1}a) \notin \mathscr{Y}; \\ (b, hg^{-1}) & \text{if } h^{-1}ga \notin \mathscr{Y}, g^{-1}hb \in \mathscr{Y} \\ & \text{or } b <^* h^{-1}ga \in \mathscr{Y} \\ & \text{or } hg^{-1}a = b \geq e; \\ (\psi(hg^{-1}a), hg^{-1}) & \text{if } hg^{-1}a \leq^* b, hg^{-1}a < e, gh^{-1}\psi(hg^{-1}a) \in \mathscr{Y}; \\ (\varphi(hg^{-1}a), hg^{-1}) & \text{if } hg^{-1}a \in \widetilde{\mathscr{Y}}, hg^{-1}a <^* b, hg^{-1}a > e; \\ (\top, \overline{hg^{-1}}) & \text{if } hg^{-1}a \in \mathscr{Y}\backslash\widetilde{\mathscr{Y}}, hg^{-1}a <^* b \\ & \text{or } h^{-1}ga \notin \mathscr{Y}, g^{-1}hb \notin \mathscr{Y}, hg^{-1}\mathscr{Y} \cap \widetilde{\mathscr{Y}} = \emptyset. \end{cases}$$

$$(a,g)\backslash(b,h)$$

$$(4.2)\quad = \begin{cases} (t_{g^{-1}h}, g^{-1}h) & \text{if } a \leq^* b, a \leq e, g^{-1}a < e, h^{-1}g\psi(g^{-1}a) \notin \mathscr{Y}; \\ & \text{or } a = b, a > e, g^{-1}a < e, h^{-1}g\psi(g^{-1}a) \notin \mathscr{Y}; \\ (\psi(g^{-1}a), g^{-1}h) & \text{if } a \leq^* b, a \leq e, g^{-1}a < e, h^{-1}g\psi(g^{-1}a) \in \mathscr{Y} \\ & \text{or } a = b, a > e, g^{-1}a < e, h^{-1}g\psi(g^{-1}a) \in \mathscr{Y}; \\ (g^{-1}a, g^{-1}h) & \text{if } a \leq^* b, a \leq e, g^{-1}a \geq e \\ & \text{or } a = b, a > e, g^{-1}a \geq e; \\ (g^{-1}\varphi(a), g^{-1}h) & \text{if } a \in \widetilde{\mathscr{Y}}, a <^* b, a > e, \\ (\top, \overline{g^{-1}h}) & \text{if } a \in \mathscr{Y}\backslash\widetilde{\mathscr{Y}}, a <^* b; \\ (g^{-1}b, g^{-1}h) & \text{if } b <^* a. \end{cases}$$

We denote by $EIRC(\mathfrak{L}; G, \mathscr{X}, \mathscr{Y}; \varphi, \psi)$ the above $(\mathfrak{L}, \wedge, \vee, \circ, \backslash, /, e)$.

Theorem 4.3. *If* $(\mathfrak{L}; G, \mathscr{X}, \mathscr{Y}; \varphi, \psi)$ *is an EIRC−system, then* $EIRC(\mathfrak{L}; G, \mathscr{X}, \mathscr{Y}; \varphi, \psi)$ *is an E-unitary inverse residuated chain.*

Proof. We only need to prove that $(b,h)/(a,g) = \max\{(c,k) \in \mathfrak{L} | (c,k)(a,g) \leq (b,h)\}$ and $(a,g)\backslash(b,h) = \max\{(c,k) \in \mathfrak{L} | (a,g)(c,k) \leq (b,h)\}$ for all $(a,g),(b,h) \in \mathfrak{L}$. To show that $(b,h)/(a,g) = \max\{(c,k) \in \mathfrak{L} | (c,k)(a,g) \leq (b,h)\}$ for all $(a,g),(b,h) \in \mathfrak{L}$, we need to consider the following cases:

Case 1. $hg^{-1}a \notin \mathscr{Y}, gh^{-1}b \notin \mathscr{Y}$ and $hg^{-1}\mathscr{Y} \cap \widetilde{\mathscr{Y}} \neq \emptyset$ or $hg^{-1}a \leq^* b, hg^{-1}a < e$ and $gh^{-1}\psi(hg^{-1}a) \notin \mathscr{Y}$. We need only to check the following subcases:

If $hg^{-1}a \notin \mathscr{Y}, gh^{-1}b \notin \mathscr{Y}$ and $hg^{-1}\mathscr{Y} \cap \widetilde{\mathscr{Y}} \neq \emptyset$, then since $\mathscr{Y}$ is a totally ordered semilattice with an identity e, by Proposition 3.3, $hg^{-1}\mathscr{Y} \cap \mathscr{Y}^-$ is a principal ideal of $\mathscr{Y}^-$. Since $t_{hg^{-1}} \in hg^{-1}\mathscr{Y} \cap \mathscr{Y}^-$, $gh^{-1}t_{hg^{-1}} \in \mathscr{Y}$, which implies that $(t_{hg^{-1}}, hg^{-1}) \in \mathfrak{L}$. Furthermore, we have $t_{hg^{-1}} <^* b$. Otherwise, if $b \leq^* t_{hg^{-1}}$, then $gh^{-1}b \leq^* gh^{-1}t_{hg^{-1}} \in \mathscr{Y}$ by noting that $t_{hg^{-1}} \in hg^{-1}\mathscr{Y} \cap \mathscr{Y}$, so $gh^{-1}b \in \mathscr{Y}$, contrary to $gh^{-1}b \notin \mathscr{Y}$. Thus $t_{hg^{-1}} <^* b$. Similarly, $gh^{-1}t_{hg^{-1}} <^* a$. We can claim that $t_{hg^{-1}} < b$; whereas, if $t_{hg^{-1}} \geq b$, then $b \leq e$, hence by Lemma 2.2(3), $t_{hg^{-1}} = t_{hg^{-1}} + b = t_{hg^{-1}} \wedge b = b$, contrary to $t_{hg^{-1}} <^* b$. Consequently, $t_{hg^{-1}} < b$. By equation (1), $(b,h)/(a,g) = (t_{hg^{-1}}, hg^{-1})$ and by the definition of $\circ$, $(t_{hg^{-1}}, hg^{-1}) \circ (a,g) = (t_{hg^{-1}} + hg^{-1}a, h) = (hg^{-1}(gh^{-1}t_{hg^{-1}} + a), h) = (t_{hg^{-1}}, h) \leq (b,h)$. Let $(c,k) \in \mathfrak{L}$ such that $(c,k)(a,g) \leq (b,h)$. Then $kg < h$ or $kg = h$ and $c + ka \leq b$. We shall prove that $(c,k) \leq (t_{hg^{-1}}, hg^{-1})$. If $kg < h$, then $k < hg^{-1}$, hence $(c,k) \leq (t_{hg^{-1}}, hg^{-1})$. If $kg = h$ and $c + ka \leq b$, then $k = hg^{-1}$, $gh^{-1}c = k^{-1}c \in \mathscr{Y}$ and $c + hg^{-1}a \leq b$. Furthermore, we have $gh^{-1}c <^* a$. Otherwise, if $a \leq^* gh^{-1}c$, then $hg^{-1}a \leq^* hg^{-1}gh^{-1}c = c \in \mathscr{Y}$, hence $hg^{-1}a \in \mathscr{Y}$, contrary to $hg^{-1}a \notin \mathscr{Y}$. Thus $gh^{-1}c <^* a$, which implies that $c = hg^{-1}gh^{-1}c = hg^{-1}(gh^{-1}c + a) = c + hg^{-1}a \leq b$. Note that $gh^{-1}c = k^{-1}c \in \mathscr{Y}$ and $gh^{-1}b \notin \mathscr{Y}$, by similar arguments as in the proof of $gh^{-1}c <^* a$, we may prove $c <^* b$. Furthermore, $c < e$; whereas, if $c \geq e$, then

$b \geq c \geq e$, hence by Lemma 2.2(2), $c = c + b = c \vee b = b$, contrary to $c <^* b$. Consequently, $c < e$. So $c \in hg^{-1}\mathscr{Y} \cap \mathscr{Y}^-$ and $c \leq^* t_{hg^{-1}}$. Then by Lemma 2.2(3), $c = c + t_{hg^{-1}} = c \wedge t_{hg^{-1}} \leq t_{hg^{-1}}$. Thus $(c,k) = (c, hg^{-1}) \leq (t_{hg^{-1}}, hg^{-1})$.

If $hg^{-1}a \leq^* b, hg^{-1}a < e$ and $gh^{-1}\psi(hg^{-1}a) \notin \mathscr{Y}$, then since $\mathscr{Y}$ is a totally ordered semilattice with an identity e, by Proposition 3.3, $hg^{-1}\mathscr{Y} \cap \mathscr{Y}^-$ is a principal ideal of $\mathscr{Y}^-$ and so $hg^{-1}a \leq^* t_{hg^{-1}}$. Since $t_{hg^{-1}} \in hg^{-1}\mathscr{Y} \cap \mathscr{Y}^-$, $gh^{-1}t_{hg^{-1}} \in \mathscr{Y}$, which implies that $(t_{hg^{-1}}, hg^{-1}) \in \mathfrak{L}$. Furthermore, we have $hg^{-1}a \leq b$. Otherwise, if $hg^{-1}a > b$, then $b < hg^{-1}a < e$, hence by Lemma 2.2(3), $hg^{-1}a = hg^{-1}a + b = hg^{-1}a \wedge b = b$, contrary to $hg^{-1}a > b$. We conclude that $hg^{-1}a \leq b$. By equation (4.1), $(b,h)/(a,g) = (t_{hg^{-1}}, hg^{-1})$ and by the definition of $\circ$, $(t_{hg^{-1}}, hg^{-1}) \circ (a,g) = (t_{hg^{-1}} + hg^{-1}a, h) = (hg^{-1}a, h) \leq b$. Let $(c,k) \in \mathfrak{L}$ such that $(c,k)(a,g) \leq (b,h)$. Then $kg < h$ or $kg = h$ and $c + ka \leq b$. We shall prove that $(c,k) \leq (t_{hg^{-1}}, hg^{-1})$. If $kg < h$, then $k < hg^{-1}$, hence $(c,k) \leq (t_{hg^{-1}}, hg^{-1})$. If $kg = h$ and $c + ka \leq b$, then $k = hg^{-1}$. If $c <^* hg^{-1}a$, then $c <^* b$ and $c = c + hg^{-1}a \leq b$. Furthermore, we have $c < e$. Otherwise, if $c \geq e$, then $b \geq c \geq e$, hence by Lemma 2.2(2), $c = c + b = c \vee b = b$, contrary to $c <^* b$. Consequently, $c < e$, which, together with $c = hg^{-1}gh^{-1}c = hg^{-1}(k^{-1}c) \in hg^{-1}\mathscr{Y} \cap \mathscr{Y}^-$, derives $c \leq^* t_{hg^{-1}}$, so $c = c + t_{hg^{-1}} = c \wedge t_{hg^{-1}} \leq t_{hg^{-1}}$. Thus $(c,k) = (c, hg^{-1}) \leq (t_{hg^{-1}}, hg^{-1})$. If $hg^{-1}a \leq^* c$, then we can claim that $c <^* \psi(hg^{-1})$; whereas, if $\psi(hg^{-1}) \leq^* c$, then $gh^{-1}\psi(hg^{-1}a) \leq^* gh^{-1}c = k^{-1}c \in \mathscr{Y}$, hence $gh^{-1}\psi(hg^{-1}a) \in \mathscr{Y}$, contrary to $gh^{-1}\psi(hg^{-1}a) \notin \mathscr{Y}$. We conclude that $hg^{-1}a \leq^* c <^* \psi(hg^{-1})$. Hence by $(E5)$, $c < e$, which together with $c \in hg^{-1}\mathscr{Y} \cap \mathscr{Y}^-$, derives $c = c + t_{hg^{-1}} = c \wedge t_{hg^{-1}} \leq t_{hg^{-1}}$. This implies that $(c,k) = (c, hg^{-1}) \leq (t_{hg^{-1}}, hg^{-1})$.

Case 2. $hg^{-1}a \notin \mathscr{Y}$ and $gh^{-1}b \in \mathscr{Y}$ or $b <^* hg^{-1}a \in \mathscr{Y}$ or $hg^{-1}a = b \geq e$. We need only to check the following subcases:

If $hg^{-1}a \notin \mathscr{Y}$ and $gh^{-1}b \in \mathscr{Y}$, then $(b, hg^{-1}) \in \mathfrak{L}$. We can claim that $gh^{-1}b <^* a$; whereas, if $a \leq^* gh^{-1}b$, then $hg^{-1}a \leq^* b \in \mathscr{Y}$, which implies that $hg^{-1}a \in \mathscr{Y}$, contrary to $hg^{-1}a \notin \mathscr{Y}$. Thus $gh^{-1}b <^* a$. By equation (4.1), $(b,h)/(a,g) = (b, hg^{-1})$ and by the definition of $\circ$, $(b, hg^{-1}) \circ (a,g) = (b + hg^{-1}a, h) = (b,h) \leq (b,h)$. Let $(c,k) \in \mathfrak{L}$ such that $(c,k)(a,g) \leq (b,h)$. Then $kg < h$ or $kg = h$ and $c + ka \leq b$. We shall prove that $(c,k) \leq (b, hg^{-1})$. If $kg < h$, then $k < hg^{-1}$, hence $(c,k) \leq (b, hg^{-1})$. If $kg = h$ and $c + ka \leq b$, then $k = hg^{-1}$. Furthermore, we have $k^{-1}c = gh^{-1}c <^* a$. Otherwise, if $a <^* gh^{-1}c$, then $hg^{-1}a <^* c$, hence $hg^{-1}a \in \mathscr{Y}$, contrary to $hg^{-1}a \notin \mathscr{Y}$. We conclude that $gh^{-1}c <^* a$. This implies that $c <^* hg^{-1}a$, which derives that $c = c + hg^{-1}a \leq b$. Thereby, $(c,k) = (c, hg^{-1}) \leq (b, hg^{-1})$.

If $b <^* hg^{-1}a \in \mathscr{Y}$, then $gh^{-1}b <^* a$ and so $gh^{-1}b \in \mathscr{Y}$, which implies that $(b, hg^{-1}) \in \mathfrak{L}$. By equation (4.1), $(b,h)/(a,g) = (b, hg^{-1})$ and by the definition of $\circ$, $(b, hg^{-1}) \circ (a,g) = (b + hg^{-1}a, h) = (b,h) \leq (b,h)$. Let $(c,k) \in \mathfrak{L}$ such that $(c,k)(a,g) \leq (b,h)$. Then $kg < h$ or $kg = h$ and $c + ka \leq b$. We shall prove that $(c,k) \leq (b, hg^{-1})$. If $kg < h$, then $k < hg^{-1}$, hence $(c,k) \leq (b, hg^{-1})$. If $kg = h$ and $c + ka \leq b$, then $k = hg^{-1}$. If $c \leq^* hg^{-1}a$, then $c = c + hg^{-1}a \leq b$ which implies that $(c,k) = (c, hg^{-1}) \leq (b, hg^{-1})$. If $hg^{-1}a <^* c$, then $hg^{-1}a =$

$c+hg^{-1}a \leq b$. We can claim that $b > e$; whereas, if $b \leq e$, then since $hg^{-1}a \leq b$, $hg^{-1}a \leq e$, hence by Lemma 2.2(3), $b = hg^{-1}a + b = hg^{-1}a \wedge b = hg^{-1}a$, contrary to $b <^* hg^{-1}a$. Consequently, $b > e$. Furthermore, we have $c < b$. Otherwise, if $c \geq b$, then $c > e$. Since $b <^* hg^{-1}a <^* c$, by Lemma 2.2(2), $b = b + c = b \vee c = c$, contrary to $b <^* c$. We conclude that $c < b$, which implies that $(c, k) = (c, hg^{-1}) \leq (b, hg^{-1})$.

If $hg^{-1}a = b \geq e$, then $gh^{-1}b = a \in \mathscr{Y}$ and so $(b, hg^{-1}) \in \mathfrak{L}$. By equation (4.1), $(b, h)/(a, g) = (b, hg^{-1})$ and by the definition of $\circ$, $(b, hg^{-1}) \circ (a, g) = (b + hg^{-1}a, h) = (b, h)$. Let $(c, k) \in \mathfrak{L}$ such that $(c, k)(a, g) \leq (b, h)$. Then $kg < h$ or $kg = h$ and $c + ka \leq b$. We shall prove that $(c, k) \leq (b, hg^{-1})$. If $kg < h$, then $k < hg^{-1}$, hence $(c, k) \leq (b, hg^{-1})$. If $kg = h$ and $c + ka \leq b$, then $k = hg^{-1}$. On one hand, if $c <^* hg^{-1}a$, then $c = c + hg^{-1}a \leq b$ and so $(c, k) = (c, hg^{-1}) \leq (b, hg^{-1})$. On the other hand, if $b = hg^{-1}a <^* c$, then we can claim that $b > c$; whereas, if $b \leq c$, then since $b \geq e$, $c \geq e$, hence by Lemma 2.2(2), $b = b + c = b \vee c = c$, contrary to $b <^* c$. We conclude that $b > c$. This means that $(c, k) = (c, hg^{-1}) \leq (b, hg^{-1})$.

Case 3. $hg^{-1}a \leq^* b, hg^{-1}a < e$ and $gh^{-1}\psi(hg^{-1}a) \in \mathscr{Y}$. Then $(\psi(hg^{-1}a), hg^{-1}) \in \mathfrak{L}$ and by $(E3), hg^{-1}a <^* \psi(hg^{-1}a)$. Furthermore, we have $hg^{-1}a \leq b$. Otherwise, if $hg^{-1}a > b$, then $b < e$, hence by Lemma 2.2(3), $hg^{-1}a = hg^{-1}a + b = hg^{-1}a \wedge b = b$, contrary to $hg^{-1}a > b$. Consequently, $hg^{-1}a \leq b$. By equation (4.1), $(b, h)/(a, g) = (\psi(hg^{-1}a), hg^{-1})$ and by the definition of $\circ$, $(\psi(hg^{-1}a), hg^{-1}) \circ (a, g) = (\psi(hg^{-1}a) + hg^{-1}a, h) = (hg^{-1}a, h) \leq (b, h)$. Let $(c, k) \in \mathfrak{L}$ such that $(c, k)(a, g) \leq (b, h)$. Then $kg < h$ or $kg = h$ and $c + ka \leq b$. We shall prove that $(c, k) \leq (\psi(hg^{-1}a), hg^{-1})$. We need only to check the following subcases:

If $kg < h$, then $k < hg^{-1}$, hence $(c, k) \leq (\psi(hg^{-1}a), hg^{-1})$.

If $kg = h$ and $c+ka \leq b$, then $k = hg^{-1}$. If $c <^* hg^{-1}a$, then $c = c+hg^{-1}a = c + ka \leq b$. Furthermore, we have $c < e$. Otherwise, if $c \geq e$, then $b \geq c \geq e$. Since $c <^* hg^{-1}a \leq^* b$, by Lemma 2.2(2), $c = c+b = c \vee b = b$, contrary to $c <^* b$. Thus $c < e$. Since by the definition of ψ, $\psi(hg^{-1}a) \geq e$, $c < \psi(hg^{-1}a)$. Hence $(c, k) = (c, hg^{-1}) \leq (\psi(hg^{-1}a), hg^{-1})$. If $hg^{-1}a \leq^* c <^* \psi(hg^{-1}a)$, then by $(E5)$, $c < e$ and so $c < \psi(hg^{-1}a)$. Hence $(c, k) = (c, hg^{-1}) \leq (\psi(hg^{-1}a), hg^{-1})$. If $\psi(hg^{-1}a) <^* c$, then we can claim that $c < \psi(hg^{-1}a)$; whereas, if $c \geq \psi(hg^{-1}a)$, then since $\psi(hg^{-1}a) \geq e$, $c \geq e$, hence by Lemma 2.2(2), $\psi(hg^{-1}a) = \psi(hg^{-1}a) + c = \psi(hg^{-1}a) \vee c = c$, contrary to $\psi(hg^{-1}a) <^* c$. Consequently, $c < \psi(hg^{-1}a)$. This implies that $(c, k) = (c, hg^{-1}) \leq (\psi(hg^{-1}a), hg^{-1})$.

Case 4. $hg^{-1}a \in \widetilde{\mathscr{Y}}, hg^{-1}a <^* b$ and $hg^{-1}a > e$. Then by $(E2)$, $\varphi(hg^{-1}a) <^* hg^{-1}a <^* b$, hence $gh^{-1}\varphi(hg^{-1}a) <^* a$, so $gh^{-1}\varphi(hg^{-1}a) \in \mathscr{Y}$, which implies that $(\varphi(hg^{-1}a), hg^{-1}) \in \mathfrak{L}$. Furthermore, we have $\varphi(hg^{-1}a) < b$. Otherwise, if $\varphi(hg^{-1}a) \geq b$, then since by the definition of φ, $\varphi(hg^{-1}a) < e$, $b \leq \varphi(hg^{-1}a) < e$, hence by Lemma 2.2(3), $\varphi(hg^{-1}a) = \varphi(hg^{-1}a) + b = \varphi(hg^{-1}a) \wedge b = b$, contrary to $\varphi(hg^{-1}a) <^* b$. Thus $\varphi(hg^{-1}a) < b$. By equation (1), $(b, h)/(a, g) = (\varphi(hg^{-1}a), hg^{-1})$ and by the definition of $\circ$, $(\varphi(hg^{-1}a), hg^{-1}) \circ (a, g) = (\varphi(hg^{-1}a) + hg^{-1}a, h) = (\varphi(hg^{-1}a), h) \leq (b, h)$. Let $(c, k) \in \mathfrak{L}$ such that $(c,k)(a,g) \leq (b,h)$. Then $kg < h$ or $kg = h$ and $c + ka \leq b$. We shall prove that $(c,k) \leq (\varphi(hg^{-1}a), hg^{-1})$. We need only to check the following subcases:

If $kg < h$, then $k < hg^{-1}$, hence $(c,k) \leq (\varphi(hg^{-1}a), hg^{-1})$.

If $kg = h$ and $c + ka \leq b$, then $k = hg^{-1}$. We can claim that $hg^{-1}a >^* c$; whereas, if $hg^{-1}a \leq^* c$, then $hg^{-1}a = c + hg^{-1}a = c + ka \leq b$, which, together with $hg^{-1}a <^* b$ and $b > e$ by noting that $hg^{-1}a > e$, derives $hg^{-1}a = b + hg^{-1}a = b \vee hg^{-1}a = b$ by Lemma 2.2(2), contrary to $hg^{-1}a <^* b$. We conclude that $hg^{-1}a >^* c$. Furthermore, we have $c \leq^* \varphi(hg^{-1}a)$. Otherwise, if $\varphi(hg^{-1}a) <^* c <^* hg^{-1}a$, then by $(E4)$, $c > e$ and $c = c + hg^{-1}a \leq b$, hence $b > e$, which, together with $c <^* hg^{-1}a <^* b$, derives $c = c + b = c \vee b = b$ by Lemma 2.2(2), contrary to $c <^* b$. Consequently, $c \leq^* \varphi(hg^{-1}a)$. It follows that $c \leq^* \varphi(hg^{-1}a) <^* hg^{-1}a <^* b$ and so $c = c + hg^{-1}a = c + ka \leq b$. We can prove that $c < e$. Assume to the contrary that $c \geq e$. Then $b \geq c \geq e$, hence by Lemma 2.2(2), $c = c + b = c \vee b = b$, contrary to $c <^* b$. Now, we conclude that $c < e$. Since $\varphi(hg^{-1}a) < e$, by Lemma 2.2(3), $c = c + \varphi(hg^{-1}a) = c \wedge \varphi(hg^{-1}a) \leq \varphi(hg^{-1}a)$, which implies that $(c,k) = (c, hg^{-1}) \leq (\varphi(hg^{-1}a), hg^{-1})$.

Case 5. $hg^{-1}a \in \mathscr{Y} \backslash \widetilde{\mathscr{Y}}$ and $hg^{-1}a <^* b$ or $hg^{-1}a \notin \mathscr{Y}, gh^{-1}b \notin \mathscr{Y}$ and $hg^{-1}\mathscr{Y} \cap \widetilde{\mathscr{Y}} = \emptyset$. We need only to check the following subcases:

If $hg^{-1}a \in \mathscr{Y} \backslash \widetilde{\mathscr{Y}}$ and $hg^{-1}a <^* b$, then by $(E1)$, $\top$ and $\overline{hg^{-1}}$ exist. By the proof of Case 5 of $(b,h)/(a,g) = \max\{(c,k) \in \mathfrak{L} | (c,k)(a,g) \leq (b,h)\}$ for all $(a,g),(b,h) \in \mathfrak{L}$, we have $g\top = \top$ for all $g \in G$ and $\top \leq^* d$ for all $d \in \mathscr{Y}$, hence $(\overline{hg^{-1}})^{-1}\top = \top \in \mathscr{Y}$, which implies that $(\top, \overline{hg^{-1}}) \in \mathfrak{L}$. Since $\overline{hg^{-1}} \prec hg^{-1}$, $\overline{hg^{-1}}g \prec h$. By equation (4.1), $(b,h)/(a,g) = (\top, \overline{hg^{-1}})$ and by the definition of $\circ$, $(\top, \overline{hg^{-1}}) \circ (a,g) = (\top + \overline{hg^{-1}}a, \overline{hg^{-1}}g) = (\top, \overline{hg^{-1}}g) \leq (b,h)$. Let $(c,k) \in \mathfrak{L}$ such that $(c,k)(a,g) \leq (b,h)$. Then $kg < h$ or $kg = h$ and $c + ka \leq b$. We shall prove that $(c,k) \leq (\top, \overline{hg^{-1}}g)$. Assume that $kg = h$ and $c + ka \leq b$. Then $k = hg^{-1}$ and $c + hg^{-1}a = c + ka \leq b$. If $c \leq^* hg^{-1}a$, then $c <^* b$ and $c = c + hg^{-1}a \leq b$. Since $hg^{-1}a \in \mathscr{Y} \backslash \widetilde{\mathscr{Y}}$, $c \in \mathscr{Y} \backslash \widetilde{\mathscr{Y}}$, so $c > e$, which implies that $b > e$. By Lemma 2.2(2), $c = c + b = c \vee b = b$, contrary to $c <^* b$. If $hg^{-1}a <^* c$, then $hg^{-1}a = c + hg^{-1}a \leq b$. Since $hg^{-1}a \in \mathscr{Y} \backslash \widetilde{\mathscr{Y}}$, $hg^{-1}a > e$ and so $b > e$. Note that $hg^{-1}a <^* b$, by Lemma 2.2(2), $hg^{-1}a = hg^{-1}a + b = hg^{-1}a \vee b = b$, contrary to $hg^{-1}a <^* b$. We conclude that $kg < h$. This implies that $k < hg^{-1}$ and so $(c,k) \leq (\top, \overline{hg^{-1}})$.

If $hg^{-1}a \notin \mathscr{Y}, gh^{-1}b \notin \mathscr{Y}$ and $hg^{-1}\mathscr{Y} \cap \widetilde{\mathscr{Y}} = \emptyset$, then $hg^{-1}\mathscr{Y} \cap \mathscr{Y} \subseteq \mathscr{Y} \backslash \widetilde{\mathscr{Y}}$. Note that $hg^{-1}\mathscr{Y} \cap \mathscr{Y} \neq \emptyset$, $\mathscr{Y} \backslash \widetilde{\mathscr{Y}} \neq \emptyset$, which implies that $\top$ and $\overline{hg^{-1}}$ exist by $(E1)$. By the proof of Case 5 of $(b,h)/(a,g) = \max\{(c,k) \in \mathfrak{L} | (c,k)(a,g) \leq (b,h)\}$ for all $(a,g),(b,h) \in \mathfrak{L}$, we have $g\top = \top$ for all $g \in G$ and $\top \leq^* d$ for all $d \in \mathscr{Y}$, hence $(\overline{hg^{-1}})^{-1}\top = \top \in \mathscr{Y}$, which implies that $(\top, \overline{hg^{-1}}) \in \mathfrak{L}$. Since $\overline{hg^{-1}} \prec hg^{-1}$, $\overline{hg^{-1}}g \prec h$. By equation (4.1), $(b,h)/(a,g) = (\top, \overline{hg^{-1}})$ and by the definition of $\circ$, $(\top, \overline{hg^{-1}}) \circ (a,g) = (\top + \overline{hg^{-1}}a, \overline{hg^{-1}}g) = (\top, \overline{hg^{-1}}g) \leq (b,h)$. Let $(c,k) \in \mathfrak{L}$ such that $(c,k)(a,g) \leq (b,h)$. Then $kg < h$ or $kg = h$ and $c + ka \leq b$. We shall prove that $(c,k) \leq (\top, \overline{hg^{-1}}g)$. Assume that $kg = h$ and $c + ka \leq b$. Then $k = hg^{-1}$ and $c + hg^{-1}a = c + ka \leq b$. We can claim that $c <^* b$; whereas, if $b \leq^* c$, then $gh^{-1}b \leq^* gh^{-1}c$, which together with $gh^{-1}c = k^{-1}c \in \mathscr{Y}$, derives $gh^{-1}b \in \mathscr{Y}$, contrary to $gh^{-1}b \notin \mathscr{Y}$. Consequently, $c <^* b$. Similarly, $k^{-1}c = gh^{-1}c <^* a$, which implies that $c = hg^{-1}(gh^{-1}c + a) = c + hg^{-1}a \leq b$. Furthermore, we have $c < e$. Otherwise, if $c \geq e$, then $b \geq c \geq e$, hence by Lemma 2.2(2), $c = c + b = c \vee b = b$, contrary to $c <^* b$. Thus $c < e$. This means that $hg^{-1}\mathscr{Y} \cap \widetilde{\mathscr{Y}} \neq \emptyset$, contrary to $hg^{-1}\mathscr{Y} \cap \widetilde{\mathscr{Y}} = \emptyset$. We conclude that $kg < h$. This implies that $k < hg^{-1}$ and so $(c, k) \leq (\top, \overline{hg^{-1}})$.

Similarly, $(a, g)/(b, h) = \max\{(c, k) \in \mathfrak{L} | (a, g)(c, k) \leq (b, h)\}$ for all (a, g), $(b, h) \in \mathfrak{L}$. □

We will prove that any $E-$unitary inverse residuated chain is isomorphic to some $EIRC(\mathfrak{L}; G, \mathscr{X}, \mathscr{Y}; \varphi, \psi)$. In what follows, let $\mathfrak{L} = P(G, \mathscr{X}, \mathscr{Y})$ be an E-unitary inverse residuated chain. By Proposition 3.1(3) and (4), we can define two mappings φ and ϕ as follows:

$$\varphi : \widetilde{\mathscr{Y}}^{+} \backslash \{e\} \to \mathscr{Y}^{-}; \quad a \mapsto \max\{b \in \mathscr{Y}^{-} | b <^* a\} \text{ for } \leq^*$$

and

$$\phi : \mathscr{Y}^{-} \backslash \{e\} \to \widetilde{\mathscr{Y}}^{+}; \quad b \mapsto \min\{a \in \widetilde{\mathscr{Y}}^{+} | b <^* a\} \text{ for } \leq^* .$$

Lemma 4.4. *$(\mathfrak{L}; G, \mathscr{X}, \mathscr{Y}; \varphi, \psi)$ is an EIRC-system.*

Proof. By Proposition 3.1(2-4), conditions$(E1 - 5)$ hold. By Theorem 2.1, condition $(E6)$ holds. By Proposition 3.3(2), condition $(E7)$ holds. □

Theorem 4.5. *$\mathfrak{L}$ is equal to $EIRC(\mathfrak{L}; G, \mathscr{X}, \mathscr{Y}; \varphi, \psi)$.*

Proof. It is clear. □

By Theorems 4.3 and 4.5, we have the following theorem:

Theorem 4.6. *Let $(G, \mathscr{X}, \mathscr{Y}; \varphi, \psi)$ be an EIRC-system. Then $EIRC(\mathfrak{L}; G, \mathscr{X}, \mathscr{Y}; \varphi, \psi)$ is an E-unitary inverse residuated chain. Conversely, any E-unitary inverse residuated chain can be constructed in this manner.*

Acknowledgements

This work is supported by grants of the NSF of China # 11171294 , China # 11571158, Fujian Province # 2020J01799.

References

[1] G.M.S. Gomes, E. Giraldes, D.B. McAlister, *On a class of lattice ordered inverse semigroups*, Journal of Algebra, 230 (2000), 496-517.

[2] T. Saitô, *Proper ordered inverse semigroups*, Pacific J. Math., 15 (1965), 649-666.

[3] D.B. McAlister, *Groups, semilattices, and inverse semigroups, I,II,* Trans. Ameri. Math. Soc., 192 (1973), 1-18, 196 (1974), 351-370.

[4] D. Stanovský, *Commutative idempotent residuated lattices,* Czechoslovak Mathematical Journal, 57 (2007), 191-200.

[5] W. Chen, X. Zhao, *The structure of idempotent residuated chains,* Czechoslovak Mathematical Journal, 59 (2009), 453-479.

[6] W. Chen, X. Zhao, X.J. Guo, *Conical residuated lattice-ordered idempotent monoids,* Semigroup Forum, 79 (2009), 244-278.

[7] W. Chen, Y. Chen, *Variety generated by conical residuated lattice-ordered idempotent monoids,* Semigroup Forum, 2019, https://doi.org/10.1007/s00233-019-10014-3

[8] W. Chen, *The property of totally ordered E-unitary inverse residuated lattices,* Journal of Minnan Normal University (Natural Science Edition), 2 (2018), 1-8.

[9] J.G. Raftery, *Representable idempotent commutative residuated lattices,* Trans. Ameri. Math. Soc., 359 (2007), 4405-4427.

[10] J. Kühr, *Representable pseudo-BCK-algebras and integral residuated lattices,* Journal of Algebra, 317 (2007), 354-364.

[11] P. Jipsen, C. Tsinakis, *A survey of residuated lattices*, Ordered Algebraic Structures (J. Martinez, ed.), Kluwer Academic Publishers, Dordrecht, 2002, 19-56.

[12] P. Bahls, J. Cole, N. Galatos, P. Jipsen, C. Tsinakis, *Cancellative residuated lattices*, Algebra Universalies, 50 (2003), 83-106.

[13] K. Blount, C. Tsinakis, *The structure of residuated lattices,* Internat. J. Algebra Comput., 13 (2003), 437-461.

[14] J. Hart, L. Rafter, C. Tsinakis, *The structure of commutative residuated lattices,* Internat. J. Algebra Comput., 12 (2002), 509-524.

[15] N. Galatos, P. Jipsen, T. Kowalski, H. Ono, *Residuated Lattices: an algebraic glimpse at substructural logics*, Studies in Logics and the Foundations of Mathematics, Elsevier, 2007.

[16] T.S. Blyth, M.F. Janowitz, *Residuation theory*, Pergamon Press, Oxford, NewYork, 1972.

[17] S. Burris, H.P. Sankappanavar, *A course in universal algebra*, GTM78, Springer, 1981.

[18] J.M. Howie, *Fundamentals of semigroup theory*, London Mathematical Society Monographs, New series, vol. 12, Oxford Univ. Press, New York, 1995.

[19] G. Birkhoff, *Lattice theory*, American Mathematical Society Colloquium Publications, Vol.25, Am. Math. Soc., Providence, 1967.

[20] T. Saitô, *Ordered inverse semigroups,* Trans. Amer. Math. Soc., 153 (1971), 99-138.

[21] M. Ward, R.P. Dilworth, *Residuated lattices,* Proceedings of the National Academy of Sciences, 24 (1938), 162-164.

Accepted: 31.03.2019