On the permanence and periodic solutions of a plankton system with impulses and diffusion

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Abstract. In this paper, given the sudden changes of external environment and seasonal variations of climate, we investigate the non-toxic phytoplankton, toxin-producing phytoplankton and zooplankton model with periodic impulses and spatial diffusion. The sufficient conditions for ultimate boundedness of solutions and permanence of system are established by using theory of impulsive differential equations, comparison principle, upper-lower solution method and inequality techniques. Moreover, the existence and uniqueness of asymptotically stable periodic solution are studied with the help of auxiliary function. It is shown that the plankton populations will evolve periodically with time, provided that the system is permanent.

Keywords: marine ecosystem, toxin-producing phytoplankton, permanence, periodic solutions.

1. Introduction

In marine ecosystem, plankton lies at the basic trophic level of the entire food chain and therefore of all the food webs on Earth. Plankton can be primarily divided into autotrophic phytoplankton and grazing zooplankton. However, under eutrophication and suitable temperature, the red tide may outbreak due to the rapid growth of plankton and can make a huge negative impact on marine ecosystem, aquaculture and human health. Therefore it would be of great significance to using mathematical modeling to investigate the dynamic relationships between the phytoplankton and zooplankton populations, which can help to understand the occurrence mechanisms of red tide and further forecast the algae blooms. For more details, we refer to [1, 2, 3, 4].

Recent research has shown that toxin-producing phytoplankton plays an important role in the control of harmful algal blooms and can be used as a biocontrol agent [5, 6, 7]. Specifically, Chattopadhyay et al. [5] formulated a very

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general model by means of ordinary differential equations:

$$
\begin{cases}
\dfrac{\mathrm{d}P}{\mathrm{d}t} = rP\left(1 - \dfrac{P}{K}\right) - \alpha f(P)Z, \\
\dfrac{\mathrm{d}Z}{\mathrm{d}t} = \beta f(P)Z - \mu Z - \theta g(P)Z,
\end{cases} \tag{1}
$$

where $P(t)$ and $Z(t)$ are the densities of toxin-producing phytoplankton population and zooplankton population at time t, respectively. The functional responses $f(P)$ and $g(P)$ adopt Holling type I, II and III. Model (1) is related to field observations gathered in the Bay of Bengal. It is concluded that toxin-producing phytoplankton may terminate the planktonic blooms by decreasing the grazing pressure of zooplankton. It should be noted that there have been some meaningful results about the generalized cases of (1). For instance, Chattopadhyay et al. [8] later investigated the specific model with distributed delay and discrete delay respectively, and discussed the stability of equilibrium solution and existence of Hopf bifurcation. Wang et al. [9] also studied the Hopf-transcritical bifurcation for delayed model. Furthermore, Sharma et al. [10] also explored the effect of spatial diffusion on the stability of the plankton model. More related results can be found in [11, 12, 13, 14].

While the toxin-producing phytoplankton population is important in marine ecosystem, the role of non-toxic phytoplankton cannot be ignored. So, Chattopadhyay et al. [6] further proposed and analyzed the following non-toxic phytoplankton, toxin-producing phytoplankton and zooplankton model:

$$
\begin{cases}
\dfrac{\mathrm{d}P_1}{\mathrm{d}t} = rP_1\left(1 - \dfrac{P_1}{K}\right) - \alpha P_1 Z, \\
\dfrac{\mathrm{d}P_2}{\mathrm{d}t} = sP_2\left(1 - \dfrac{P_2}{K}\right) - \dfrac{\theta P_2 Z}{\gamma + P_2}, \\
\dfrac{\mathrm{d}Z}{\mathrm{d}t} = \beta P_1 Z - \mu Z - \dfrac{\theta_1 P_2 Z}{\gamma + P_2},
\end{cases} \tag{2}
$$

where $P_1(t)$ is the concentration of the non-toxic phytoplankton at time t, $P_2(t)$ is the concentration of the toxin-producing phytoplankton population at time t, and $Z(t)$ is the concentration of the zooplankton population at time t.

In model (2), the authors have neglected the effect of the competition between different phytoplankton groups. But it is now well established that competition among different species may be a reason for stable coexistence [15]. Huisman and Weissing [16] also showed that multi-species competition may generate oscillations and chaos and that these fluctuations create opportunities for the coexistence of many species. Then Pal et al. [17] modified model (2) as follows:

$$(3)\qquad \begin{cases} \dfrac{\mathrm{d}P_1}{\mathrm{d}t} = rP_1\left(1 - \dfrac{P_1 + \alpha P_2}{K_1}\right) - \dfrac{\alpha_1 P_1 Z}{1 + \beta_1 P_1 + \beta_2 P_2}, \\ \dfrac{\mathrm{d}P_2}{\mathrm{d}t} = sP_2\left(1 - \dfrac{P_2 + \beta P_1}{K_2}\right) - \dfrac{\alpha_2 P_2 Z}{1 + \beta_1 P_1 + \beta_2 P_2}, \\ \dfrac{\mathrm{d}Z}{\mathrm{d}t} = \dfrac{(\alpha_1' P_1 - \alpha_2' P_2)Z}{1 + \beta_1 P_1 + \beta_2 P_2} - \mu Z, \end{cases}$$

where α and β are the competition coefficients. It is shown that the high competition will result in planktonic bloom.

In reality, the plankton individual can freely float thanks to the wind and turbulence. So, some reaction-diffusion plankton ecological models have been proposed to simulate this phenomenon [18, 19]. Besides, the marine ecosystem is often deeply disturbed by human exploitation activities which might lead to the duration of abrupt changes [20, 21]. It is necessary to take into account this impulsive response. Recently, some research on the dynamics about impulsive and diffusive predator-prey models has appeared [22, 23, 24]. Nevertheless, there has been no result on the joint effects of spatial diffusion and impulsive response about the spatiotemporal dynamics of plankton system. Thus, motivated by the above work, in this paper, we mainly consider the following plankton system with periodic impulses and spatial diffusion:

$$(4)\qquad \begin{cases} \dfrac{\partial P}{\partial t} = d_1 \Delta P + r_1(t,x)P\left(1 - \dfrac{P}{K_1(t,x)}\right) & \\ \qquad -\alpha_1(t,x)PT - \dfrac{\beta_1(t,x)PZ}{a(t,x)+P}, & t \neq t_k, x \in \Omega, \\ \dfrac{\partial T}{\partial t} = d_2 \Delta T + r_2(t,x)T\left(1 - \dfrac{T}{K_2(t,x)}\right) & \\ \qquad -\alpha_2(t,x)PT - \dfrac{\theta_1(t,x)TZ}{b(t,x)+T}, & t \neq t_k, x \in \Omega, \\ \dfrac{\partial Z}{\partial t} = d_3 \Delta Z + \dfrac{\beta_2(t,x)PZ}{a(t,x)+P} & \\ \qquad + \dfrac{\theta_2(t,x)TZ}{b(t,x)+T} - m(t,x)Z^2 - \rho(t,x)T^{\frac{2}{3}}Z, & t \neq t_k, x \in \Omega, \\ P(t_k^+,x) = P(t_k,x)f_k(x, P(t_k,x), T(t_k,x), Z(t_k,x)), & t = t_k, x \in \Omega, \\ T(t_k^+,x) = T(t_k,x)g_k(x, P(t_k,x), T(t_k,x), Z(t_k,x)), & t = t_k, x \in \Omega, \\ Z(t_k^+,x) = Z(t_k,x)h_k(x, P(t_k,x), T(t_k,x), Z(t_k,x)), & t = t_k, x \in \Omega, \\ \dfrac{\partial P}{\partial \nu} = \dfrac{\partial T}{\partial \nu} = \dfrac{\partial Z}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases}$$

where $P(t,x)$, $T(t,x)$ and $Z(t,x)$ denote the densities of non-toxic phytoplankton, toxin-producing phytoplankton and zooplankton at time t and location x, respectively. The bounded space domain $\Omega \subset \mathbb{R}^n (n = 1, 2, 3)$ has smooth boundary $\partial\Omega$, Δ is the Laplace operator, $\partial/\partial\nu$ is the outward normal derivative. In system (4), the toxic phytoplankton and non-toxic phytoplankton compete for common survival resources, $d_i(i = 1, 2, 3)$ is the diffusivity of three plankton population respectively, the quadratic closure term is adopted to describe the mortality of zooplankton, the term $\rho T^{2/3}Z$ means that the toxic phytoplankton gathered in patches reduces the zooplankton's grazing.

For system (4), the nonnegative initial conditions are also satisfied:

$$P(0,x) = P_0(t) \geq 0, \quad T(0,x) = T_0(t) \geq 0, \quad Z(0,x) = Z_0(t) \geq 0, \quad x \in \partial\Omega.$$

The plankton populations are disturbed at time t_k, where $\{t_k\}$ is a real sequence, and $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots$, $\lim_{k\to\infty} t_k = +\infty$.

The main purpose of this paper is to investigate the permanence and periodic solutions of system (4). This paper is organized as follows. In Section 2, some basic assumptions and useful results are given. In Section 3, the ultimate boundedness of solutions and permanence of system (4) are explored. In Section 4, the existence of a unique periodic solution for system (4) is established. Finally, some conclusions are drawn in Section 5.

2. Basic assumptions and preliminaries

Considering the seasonal variation and cyclical factors of biological rhythm, we first state the basic assumptions as follows:

(H_1) The functions $r_i(t,x)$, $K_i(t,x)$, $\alpha_i(t,x)$, $\beta_i(t,x)$, $\theta_i(t,x)$, $a(t,x)$, $b(t,x)$, $m(t,x)$ and $(\rho(t,x))$ are bounded and positive-valued on $\mathbb{R} \times \bar{\Omega}$, continuously differentiable in t and x for $i = 1, 2$;

(H_2) The functions $f_k(x,P,T,Z)$, $g_k(x,P,T,Z)$ and $h_k(P,T,Z)$ are continuously differentiable in all arguments and positive for any $k \in \mathbb{N}^+$;

(H_3) The functions $r_i(t,x)$, $K_i(t,x)$, $\alpha_i(t,x)$, $\beta_i(t,x)$, $\theta_i(t,x)$, $a(t,x)$, $b(t,x)$, $m(t,x)$ and $(\rho(t,x))$ are $\omega-$periodic in t for $i = 1, 2$;

(H_4) There exists a positive integer p such that $t_{t+p} = t_k + \omega$ for any $k \in \mathbb{N}^+$;

(H_5) The function sequences $\{f_k\}$, $\{g_k\}$ and $\{h_k\}$ satisfy $f_{k+p}(x,P,T,Z) = f_k(x,P,T,Z)$, $g_{k+p}(x,P,T,Z) = g_k(x,P,T,Z)$ and $h_{k+p}(x,P,T,Z) = h_k(x,P,T,Z)$ for any $k \in \mathbb{N}^+$.

For convenience, we give the following notations:

$$G = \mathbb{R}^+ \times \Omega, \quad \bar{G} = \mathbb{R}^+ \times \bar{\Omega}, \quad \Gamma_k = \{(t,x) | \, t \in (t_{k-1}, t_k), x \in \Omega\},$$

$$\Gamma = \bigcup_{k\in\mathbb{N}^+} \Gamma_k, \quad \bar{\Gamma}_k = \left\{(t,x) | \, t \in (t_{k-1}, t_k), x \in \bar{\Omega}\right\}, \quad \bar{\Gamma} = \bigcup_{k\in\mathbb{N}^+} \bar{\Gamma}_k.$$

For a bounded function $\phi(t,x)$, we denote

$$\phi^L = \inf_{t>0, x\in\Omega} \phi(t,x), \quad \phi^M = \sup_{t>0, x\in\Omega} \phi(t,x).$$

Denote by Υ a class of functions $\phi : \bar{G} \to \mathbb{R}$ with the following properties:

(i) $\phi(t,x) \in C^{1,2}_{t,x}(\Gamma_k)$, $\phi(t,x) \in C^{1,1}_{t,x}(\bar{\Gamma}_k)$ for all $k \in \mathbb{N}^+$;

(ii) $\lim\limits_{s\to t_k^-} \phi(s,x) = \phi(t_k,x)$, $\lim\limits_{s\to t_k^+} \phi(s,x) = \phi(t_k^+,x)$ for all $k \in \mathbb{N}^+$ and $x \in \Omega$.

The vector-valued function $(P(t,x),T(t,x),Z(t,x)) \in \Upsilon\times\Upsilon\times\Upsilon$ is defined as the solution of system (4) if it satisfies all the equalities.

In the following, we restate some significant lemmas from [25, 26, 27] which will be useful in the proof of our main results.

Consider the impulsive logistic differential equation:

$$\begin{cases} \frac{\mathrm{d}z}{\mathrm{d}t} = az(b-z), & t \neq t_k, \\ z(t_k+0) = z(t_k)\lambda_k\left(z(t_k)\right), & k \in \mathbb{N}^+, \end{cases} \tag{5}$$

where $z \in \mathbb{R}^+$, a and b are positive constants, the strictly increasing sequence $\{t_k\}$ satisfies assumption (H_4). For any positive integer k, λ_k are continuous and positive-valued functions such that $\lambda_{k+p}(z) = \lambda_k(z)$ for all $z \in \mathbb{R}^+$. It can be obtained that $t_{k+1} - t_k \geq \theta = \min_{i=0,1,2,\cdots p}(t_{i+1}-t_i)$ by assumption (H_4).

Denote $A = \frac{b}{1-e^{-ab\theta}}$, $B = \max_{i=0,1,2,\cdots,p}\max_{z\in[0,A]}\lambda_k(z)$, $C = \max\{A,B\}$.

Lemma 2.1 ([25]). *Every solution $z(t) = z(t,0,z_0)$ of (5) satisfies $0 < z(t) < C$ for $t \geq 0$ and $z_0 > 0$.*

Lemma 2.2 ([26]). *Suppose that vector functions $v(t,x) = (v_1(t,x),\cdots,v_m(t,x))$ and $w(t,x) = (w_1(t,x),\cdots,w_m(t,x))$, $m \geq 1$ satisfy the following conditions:*

(i) they are of class C^2 in x, $x \in \Omega$ and of class C^1 in $(t,x) \in [a,b]\times\bar{\Omega}$, where $\Omega \in \mathbb{R}^n$ is a bounded domain with smooth boundary;

(ii) $v_t - \mu\Delta v_t - g(t,x,v) \leq w_t - \mu\Delta w_t - g(t,x,w)$, where $(t,x) \in [a,b]\times\bar{\Omega}$ and $\mu = (\mu_1,\cdots,\mu_m) > 0$ (inequalities between vectors are satisfied coordinate-wise), vector function $g(t,x,u) = (g_1(t,x,u),\cdots,g_m(t,x,u))$ is continuously differentiable and quasi-monotonically increasing with respect to $\frac{\partial g_i(t,x,u_1,\cdots,u_m)}{\partial u_j} \geq 0$, $i,j = 1,2,\cdots m$, $i \neq j$;

(iii) $\frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0$, $(t,x) \in [a,b]\times\partial\Omega$.
Then $v(t,x) \leq w(t,x)$ for any $(t,x) \in [a,b]\times\bar{\Omega}$.

Lemma 2.3 ([27]). *Assume that ω and d are positive real numbers, a function $u(t,x)$ is continuous on $[0,\omega]\times\bar{\Omega}$, continuously differentiable in $x \in \bar{\Omega}$, with continuous derivatives $\frac{\partial^2 u}{\partial x_i \partial x_j}$ and $\frac{\partial u}{\partial t}$, and $u(t,x)$ satisfies the following inequalities:*

$$\begin{cases} \frac{\partial u}{\partial t} - d\Delta u + c(t,x)u \geq 0, & (t,x) \in (0,\omega]\times\Omega, \\ \frac{\partial u}{\partial \nu} = 0, & (t,x) \in (0,\omega]\times\partial\Omega, \\ u(0,x) \geq 0, & x \in \Omega, \end{cases}$$

where $c(t,x)$ is bounded on $(0,\omega]\times\Omega$. Then $u(t,x) \geq 0$ on $(0,\omega]\times\bar{\Omega}$. Moreover, $u(t,x)$ is strictly positive on $(0,\omega]\times\bar{\Omega}$ if $u(0,x) \not\equiv 0$.

Based on the above lemmas and upper-lower solution method, we can obtain the positive invariance of system (4), that is to say, the solutions of (4) always remain nonnegative.

Theorem 2.1. *Assume that* $(P_1(t), T_1(t), Z_1(t))$ *is the solution of impulsive ordinary differential system*

$$\begin{cases} \dfrac{dP(t)}{dt} = r_1(t)P(t)\left(1 - \dfrac{P(t)}{K_1(t)}\right), & t \neq t_k, \\ \dfrac{dT(t)}{dt} = r_2(t)T(t)\left(1 - \dfrac{T(t)}{K_2(t)}\right), & t \neq t_k, \\ \dfrac{dZ(t)}{dt} = \dfrac{\beta_2(t)P(t)Z(t)}{a(t)+P(t)} & \\ \qquad +\dfrac{\theta_2(t)T(t)Z(t)}{b(t)+T(t)} - m(t)Z^2(t) - \rho(t)T^{\frac{2}{3}}(t)Z(t), & t \neq t_k, \\ P(t_k^+) = P(t_k)f_k(P(t_k), T(t_k), Z(t_k)), & k = 1, 2, \cdots, \\ T(t_k^+) = T(t_k)g_k(P(t_k), T(t_k), Z(t_k)), & k = 1, 2, \cdots, \\ Z(t_k^+) = Z(t_k)h_k(P(t_k), T(t_k), Z(t_k)), & k = 1, 2, \cdots \end{cases}$$

and $(P_2(t), T_2(t), Z_2(t))$ *is the solution of impulsive ordinary differential system*

$$\begin{cases} \dfrac{dP(t)}{dt} = r_1(t)P(t)\left(1 - \dfrac{P(t)}{K_1(t)}\right) & \\ \qquad -\alpha_1(t)P(t)T(t) - \dfrac{\beta_1(t)P(t)Z(t)}{a(t)+P(t)}, & t \neq t_k, \\ \dfrac{dT(t)}{dt} = r_2(t)T(t)\left(1 - \dfrac{T(t)}{K_2(t)}\right) & \\ \qquad -\alpha_2(t)P(t)T(t) - \dfrac{\theta_1(t)T(t)Z(t)}{b(t)+T(t)}, & t \neq t_k, \\ \dfrac{dZ(t)}{dt} = \dfrac{\beta_2(t)P(t)Z(t)}{a(t)+P(t)} & \\ \qquad +\dfrac{\theta_2(t)T(t)Z(t)}{b(t)+T(t)} - m(t)Z^2(t) - \rho(t)T^{\frac{2}{3}}(t)Z(t), & t \neq t_k, \\ P(t_k^+) = P(t_k)f_k(P(t_k), T(t_k), Z(t_k)), & k = 1, 2, \cdots \\ T(t_k^+) = T(t_k)g_k(P(t_k), T(t_k), Z(t_k)), & k = 1, 2, \cdots \\ Z(t_k^+) = Z(t_k)h_k(P(t_k), T(t_k), Z(t_k)), & k = 1, 2, \cdots. \end{cases}$$

If $0 < P_2(0) \leq P_0(x) \leq P_1(0)$, $0 < T_2(0) \leq T_0(x) \leq T_1(0)$ *and* $0 < Z_1(0) \leq Z_0(x) \leq Z_2(0)$, *then system (4) has the unique solution* $(P(t,x), T(t,x), Z(t,x))$ *and*

$$P_2(t) \leq P(t,x) \leq P_1(t), \quad T_2(t) \leq T(t,x) \leq T_1(t), \quad Z_1(t) \leq Z(t,x) \leq Z_2(t).$$

Proof. It is only need to verify that $(P_1(t), T_1(t), Z_2(t))$ and $(P_2(t), T_2(t), Z_1(t))$ are a pair of upper-lower solutions to system (4).

In fact, direct calculation can lead to $\frac{\partial P_1(t)}{\partial t} - d_1 \Delta P_1(t) - r_1(t)P_1(t)(1 - \frac{P_1(t)}{K_1(t)}) + \alpha_1(t)P_1(t)T_1(t) + \frac{\beta_1(t)P_1(t)Z_1(t)}{a(t)+P_1(t)} = r_1(t)P_1(t)(1 - \frac{P_1(t)}{K_1(t)}) - r_1(t)P_1(t)(1 - \frac{P_1(t)}{K_1(t)}) + \alpha_1(t)P_1(t)T_1(t) + \frac{\beta_1(t)P_1(t)Z_1(t)}{a(t)+P_1(t)} \geq 0.$

Similarly, we also have

$$\begin{aligned}
&\frac{\partial T_1(t)}{\partial t} - d_2\Delta T_1(t) - r_2(t)T_1(t)\left(1-\frac{T_1(t)}{K_2(t)}\right)\\
&+\alpha_2(t)P_1(t)T_1(t) + \frac{\theta_1(t)T_1(t)Z_1(t)}{b(t)+T_1(t)} \geq 0,\\
&\frac{\partial P_2(t)}{\partial t} - d_1\Delta P_2(t) - r_1(t)P_2(t)\left(1-\frac{P_2(t)}{K_1(t)}\right)\\
&+\alpha_1(t)P_2(t)T_2(t) + \frac{\beta_1(t)P_2(t)Z_2(t)}{a(t)+P_2(t)} \leq 0,\\
&\frac{\partial T_2(t)}{\partial t} - d_2\Delta T_2(t) - r_2(t)T_2(t)\left(1-\frac{T_2(t)}{K_2(t)}\right)\\
&+\alpha_2(t)P_2(t)T_2(t) + \frac{\theta_1(t)T_2(t)Z_2(t)}{b(t)+T_2(t)} \leq 0,\\
&\frac{\partial Z_1(t)}{\partial t} - d_3\Delta Z_1(t) - \frac{\beta_2(t)P_1(t)Z_1(t)}{a(t)+P_1(t)}\\
&-\frac{\theta_2(t)T_1(t)Z_1(t)}{b(t)+T_1(t)} + m(t)Z_1^2(t) + \rho(t)T_1^{\frac{2}{3}}(t)Z_1(t) \leq 0,\\
&\frac{\partial Z_2(t)}{\partial t} - d_3\Delta Z_2(t) - \frac{\beta_2(t)P_2(t)Z_2(t)}{a(t)+P_2(t)}\\
&-\frac{\theta_2(t)T_2(t)Z_2(t)}{b(t)+T_2(t)} + m(t)Z_2^2(t) + \rho(t)T_2^{\frac{2}{3}}(t)Z_2(t) \geq 0.
\end{aligned}$$

It is seen that $(P_1(t), T_1(t), Z_2(t))$ and $(P_2(t), T_2(t), Z_1(t))$ are a pair of upper-lower solutions to system (4). This completes the proof.

Proposition 2.1. *System (4) is positively invariant.*

Proof. Set $P_2(t) = 0$, $T_2(t) = 0$, $Z_1(t) = 0$, $P_1(t) = \bar{P}(t)$, $T_1(t) = \bar{T}(t)$, $Z_2(t) = \bar{Z}(t)$, then $\big(\bar{P}(t), \bar{T}(t), \bar{Z}(t)\big)$ and $(0, 0, 0)$ are a pair of upper-lower solutions to system (4).

Thus, the result follows.

3. Permanence

Here, we first give the definitions of the ultimate boundedness of solutions and permanence of system (4).

Definition 3.1. *Solutions of system (4) are said to be ultimately bounded if there exist positive constants N_1, N_2 and N_3, such that for every solution $(P(t,x,P_0,T_0,Z_0)$,$T(t,x,P_0,T_0,Z_0)$,$Z(t,x,P_0,T_0,Z_0))$, there exists a moment of time $\bar{t} = \bar{t}(P_0,T_0,Z_0) > 0$ such that*

$$P(t,x,P_0,T_0,Z_0) \leq N_1, \quad T(t,x,P_0,T_0,Z_0) \leq N_2, \quad Z(t,x,P_0,T_0,Z_0) \leq N_3$$

for all $x \in \bar{\Omega}$ and $t \geq \bar{t}$.

Definition 3.2. *System* (4) *is said to be permanent if there exist positive constants* m_i *and* N_i, $i = 1, 2, 3$ *such that for every solution with non-negative initial functions that are not identically zero, there exists a moment of time* $\tilde{t} = \tilde{t}(t, x, P_0, T_0, Z_0)$ *such that*

$$\begin{aligned} m_1 &\le P(t, x, P_0, T_0, Z_0) \le N_1, \\ m_2 &\le T(t, x, P_0, T_0, Z_0) \le N_2, \\ m_3 &\le Z(t, x, P_0, T_0, Z_0) \le N_3 \end{aligned} \tag{6}$$

for all $x \in \bar{\Omega}$ *and* $t \ge \tilde{t}$.

In the following, we first confirm the ultimate boundedness of solutions.

Theorem 3.1. *If the assumptions* (H_1)-(H_5) *are satisfied, and the following conditions hold:*

(i) *there exists a positive-valued function* $\eta_1(M)$ *such that* $f_k(x, P, T, Z) \le \eta_1(M_1)$ *for* $k \in \mathbb{N}^+$, $P \le M_1$, $T \ge 0$, $Z \ge 0$ *and* $x \in \Omega$;

(ii) *there exists a positive-valued function* $\eta_2(M)$ *such that* $g_k(x, P, T, Z) \le \eta_2(M_2)$ *for* $k \in \mathbb{N}^+$, $P \ge 0$, $T \le M_2$, $Z \ge 0$ *and* $x \in \Omega$;

(iii) *there exists a positive-valued function* $\eta_3(M)$ *such that* $h_k(x, P, T, Z) \le \eta_3(M_3)$ *for* $k \in \mathbb{N}^+$, $P \ge 0$, $T \ge 0$, $Z \le M_3$ *and* $x \in \Omega$.

Then all solutions of (4) are ultimately bounded.

Proof. Let $\bar{P}(t, x, P_0)$ be a solution of the equation

$$\frac{\partial \bar{P}}{\partial t} - d_1 \Delta \bar{P} - \bar{P}\left(r_1^M - \frac{r_1^L}{K_1^M}\bar{P}\right) = 0. \tag{7}$$

From the inequality

$$\begin{aligned} 0 &= \frac{\partial P}{\partial t} - d_1 \Delta P - r_1(t,x)P\left(1 - \frac{P}{K_1(t,x)}\right) + \alpha_1(t,x)PT + \frac{\beta_1(t,x)PZ}{a(t,x)+P} \\ &\ge \frac{\partial P}{\partial t} - d_1 \Delta P - P\left(r_1^M - \frac{r_1^L}{K_1^M}P\right), \end{aligned}$$

we have

$$\begin{aligned} 0 &= \frac{\partial \bar{P}}{\partial t} - d_1 \Delta \bar{P} - \bar{P}\left(r_1^M - \frac{r_1^L}{K_1^M}\bar{P}\right) \\ &\ge \frac{\partial P}{\partial t} - d_1 \Delta P - P\left(r_1^M - \frac{r_1^L}{K_1^M}P\right). \end{aligned}$$

Using Lemma 2.2, it can be concluded that $P(t, x, P_0, T_0, Z_0) \le \bar{P}(t, M_P)$, where $M_P \ge \max_{x \in \bar{\Omega}} = \|P_0(x)\|_C$. As $\bar{P}(t, M_P)$ is independent of the spatial variable x, function $\bar{P}(t, M_P)$ is the solution of ordinary differential equation

$$\frac{\mathrm{d}\bar{P}}{\mathrm{d}t} = \bar{P}\left(r_1^M - \frac{r_1^L}{K_1^M}\bar{P}\right).$$

Moreover, we also have

$$\|P(t_k^+,x,P_0,T_0,Z_0)\|_C = \|P(t_k,x,P_0,T_0,Z_0)f_k(x,P(t_k,x,P_0,T_0,Z_0),\\ T(t_k,x,P_0,T_0,Z_0),Z(t_k,x,P_0,T_0,Z_0))\|_C\\ \le \bar{P}(t_k,M_P)\cdot\eta_1\left(\bar{P}(t_k,M_P)\right).$$

According to Lemma 2.1, all solutions of impulsive differential equation

$$\begin{cases}\dfrac{\mathrm{d}\bar{P}}{\mathrm{d}x} = \bar{P}\left(r_1^M - \dfrac{r_1^L}{K_1^M}\bar{P}\right),\\ \bar{P}(t_k^+) = \bar{P}(t_k)\eta_1(\bar{P}(t_k))\end{cases}$$

are ultimately bounded. Then for the first equation of (4) and its corresponding impulsive condition, there exists a positive constant N_1, such that $P(t,x)\le N_1$ for $t>\bar{t}_1$.

With the same method, there exists a positive constant N_2, such that $T(t,x)\le N_2$ for $t>\bar{t}_2$; there exists a positive constant N_3, such that $Z(t,x)\le N_3$ for $t>\bar{t}_3$. This proves the theorem.

Theorem 3.2. *If the assumptions $(H_1)-(H_5)$ are satisfied, and the following conditions hold:*

(i) *all solutions of (4) are ultimately bounded,that is, there exist positive constants N_i, $i=1,2,3$ and a moment of time $\bar{t}_0$, such that $P(t,x)\le N_1$, $T(t,x)\le N_2$ and $Z(t,x)\le N_3$ for all $x\in\bar{\Omega}$ and $t>\bar{t}_0$;*

(ii) *the following inequalities*

$$\sum_{k=1}^{p}\ln\inf_{x\in\Omega,(P,T,Z)\in S} f_k(x,P,T,Z)+\omega\left(r_1^L-\alpha_1^M N_2-\frac{\beta_1^M N_3}{a^L}\right)>0,$$

$$\sum_{k=1}^{p}\ln\inf_{x\in\Omega,(P,T,Z)\in S} g_k(x,P,T,Z)+\omega\left(r_2^L-\alpha_2^M N_1-\frac{\beta_2^M N_3}{b^L}\right)>0,$$

and

$$\sum_{k=1}^{p}\ln\inf_{x\in\Omega,(P,T,Z)\in S} h_k(x,P,T,Z)+\omega\left(\frac{\beta_2^L\sigma_1}{a^M+\sigma_1}+\frac{\theta_2^L\sigma_2}{b^M+\sigma_2}-\rho^M N_2^{\frac{2}{3}}\right)>0$$

hold, where $S=\{(P,T,Z): 0<P\le N_1,\ 0<T\le N_2\ 0<Z\le N_3\}$, σ_1 and σ_2 are constants. Then system (4) is permanent.

Proof. Lemma 2.3 indicates that if initial functions $P_0(x)$, $T_0(x)$ and $Z_0(x)$ are nonnegative and not identically zero, then we have

$$P(t,x,P_0,T_0,Z_0)>0,\quad T(t,x,P_0,T_0,Z_0)>0,\quad Z(t,x,P_0,T_0,Z_0)>0.$$

Without loss of generality, set

$$\min_{x\in\bar{\Omega}} P_0(x) = m_P > 0, \quad \min_{x\in\bar{\Omega}} T_0(x) = m_T > 0, \quad \min_{x\in\bar{\Omega}} Z_0(x) = m_Z > 0.$$

From the inequality

$$\begin{aligned} 0 &= \frac{\partial P}{\partial t} - d_1\Delta P - r_1(t,x)P\left(1 - \frac{P}{K_1(t,x)}\right) + \alpha_1(t,x)PT + \frac{\beta_1(t,x)PZ}{a(t,x)+P} \\ &\le \frac{\partial P}{\partial t} - d_1\Delta P - P\left(r_1^L - \alpha_1^M N_2 - \frac{\beta_1^M N_3}{a^L} - \frac{r_1^M P}{K_1^L}\right), \end{aligned}$$

we have

$$\begin{aligned} 0 &= \frac{\partial \hat{P}}{\partial t} - d_1\Delta \hat{P} - \hat{P}\left(r_1^L - \alpha_1^M N_2 - \frac{\beta_1^M N_3}{a^L} - \frac{r_1^M \hat{P}}{K_1^L}\right) \\ &\le \frac{\partial P}{\partial t} - d_1\Delta P - P\left(r_1^L - \alpha_1^M N_2 - \frac{\beta_1^M N_3}{a^L} - \frac{r_1^M P}{K_1^L}\right). \end{aligned}$$

Afterwards, according to Lemma 2.2, we have $P(t,x,P_0,T_0,Z_0) \ge \hat{P}(t,m_P)$ for $0 \le t \le t_1$. At the first impulsive moment of time t_1, it can be obtained

$$P(t_1^+,x,P_0,T_0,Z_0) \ge \hat{P}(t,m_P)\inf_{x\in\Omega,(P,T,Z)\in S} f_1(x,P,T,Z).$$

Thus, the solution of logistic differential equation with periodic impulses

$$\begin{cases} \dfrac{\mathrm{d}\hat{P}}{\mathrm{d}t} = \hat{P}\left(r_1^L - \alpha_1^M N_2 - \dfrac{\beta_1^M N_3}{a^L} - \dfrac{r_1^M \hat{P}}{K_1^L}\right), \\ \hat{P}(t_k^+) = \hat{P}(t_k) \inf\limits_{x\in\Omega,(P,T,Z)\in S} f_1(x,P,T,Z) \end{cases} \tag{8}$$

is a lower bound of aforementioned solution $P(t,x,P_0,T_0,Z_0)$.

By using the Theorem 2.1 in [28], equation (8) has the unique piece-wise continuous and strictly positive periodic solution $\hat{P}^*(t)$ such that each solution of (8) has the property $\lim_{t\to\infty}\hat{P}(t,m_P) = \hat{P}^*(t)$. Therefore, there exists a positive constant σ_1 such that for every solution $\hat{P}(t,m_P)$ of (8) we have $\hat{P}(t,m_P) \ge \sigma_1$ when $t > \hat{t}_1 = \hat{t}_1(m_P) > 0$. And because that $\hat{P}(t,m_P)$ is a lower bound of $P(t,x,P_0,T_0,Z_0)$, we can get $P(t,x,P_0,T_0,Z_0) \ge \sigma_1$ when $t > \hat{t}_1$.

Using the same procedures, there exists a positive constant σ_2 such that $T(t,x,P_0,T_0,Z_0) \ge \sigma_2$ when $t > \hat{t}_2$.

At last we consider the third equation of (4) and have

$$\begin{aligned} 0 &= \frac{\partial Z}{\partial t} - d_3\Delta Z - Z\left[\frac{\beta_2(t,x)P}{a(t,x)+P} + \frac{\theta_2(t,x)T}{b(t,x)+T} - m(t,x)Z - \rho(t,x)T^{\frac{2}{3}}\right] \\ &\le \frac{\partial Z}{\partial t} - d_3\Delta Z - Z\left[\frac{\beta_2^L\sigma_1}{a^M+\sigma_1} + \frac{\theta_2^L\sigma_2}{b^M+\sigma_2} - \rho^M N_2^{\frac{2}{3}} - m^M Z\right]. \end{aligned}$$

Repeating the previous steps will result that there exists a positive constant σ_3 such that $Z(t,x,P_0,T_0,Z_0) \ge \sigma_3$ when $t > \hat{t}_3$. Thus Theorem 3.2 holds.

4. The existence and uniqueness of periodic solutions

In this section, we mainly establish the existence and uniqueness of asymptotically stable solution for system (4) by constructing suitable auxiliary function and using inequality techniques.

Theorem 4.1. *If the assumptions* $(H_1)-(H_5)$ *are satisfied and system (4) is permanent, the following inequality*

$$\sum_{k=1}^{p} \ln L_k + \omega\lambda_M < 0$$

holds, where

$$L_k = \max_{P,T,Z\in\Xi,x\in\Omega} 2\left\{ (f_k)^2 + (g_k)^2 + (h_k)^2 + \left(N\frac{\partial f_k}{\partial P}\right)^2 + \left(N\frac{\partial f_k}{\partial T}\right)^2 + \left(N\frac{\partial f_k}{\partial Z}\right)^2 \right.$$
$$\left. + \left(N\frac{\partial g_k}{\partial P}\right)^2 + \left(N\frac{\partial g_k}{\partial T}\right)^2 + \left(N\frac{\partial g_k}{\partial Z}\right)^2 + \left(N\frac{\partial h_k}{\partial P}\right)^2 + \left(N\frac{\partial h_k}{\partial T}\right)^2 + \left(N\frac{\partial h_k}{\partial Z}\right)^2 \right\},$$

λ_M *is the maximal eigenvalue of the matrix*

$$\begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{pmatrix},$$

where

$$\begin{aligned}
E_{11} &= 2\left[r_1^M - 2\sigma\frac{r_1^L}{K_1^M} - \alpha_1^L\sigma - \frac{a^L\beta_1^L\sigma}{(a^M+N)^2} \right],\\
E_{22} &= 2\left[r_2^M - 2\sigma\frac{r_2^L}{K_2^M} - \alpha_2^L\sigma - \frac{b^L\theta_1^L\sigma}{(b^M+N)^2} \right],\\
E_{33} &= 2\left[\frac{\beta_2^M N(a^M+N)}{(a^L+\sigma)^2} + \frac{\theta_2^M N(b^M+N)}{(b^L+\sigma)^2} - \rho^L\sigma^{\frac{2}{3}} \right],\\
E_{12} &= E_{21} = N\left(\alpha_1^M + \alpha_2^M\right),\\
E_{13} &= E_{31} = \frac{N\left(a^M\beta_1^M + \beta_1^M N + a^M\beta_2^M\right)}{(a^L+\sigma)^2},\\
E_{23} &= E_{32} = \frac{N\left(b^M\theta_1^M + \theta_1^M N + b^M\theta_2^M\right)}{(b^L+\sigma)^2}.
\end{aligned}$$

Then system (4) has the unique globally asymptotically stable and strictly positive piecewise continuous $\omega-$*periodic solution.*

Proof. Due to the permanence of system (4), there exist positive constants N, σ, and a moment of time $\bar{t}$ such that every solution of (4) satisfies

$$(P(t,x),T(t,x),Z(t,x)) \in \Xi = \{(P,T,Z):\ \sigma \le P \le N, \sigma \le T \le N, \sigma \le Z \le N\}$$

for any $t > \bar{t}$ and with nonnegative initial conditions not identically equal to zero.

Assuming that $(P(t,x),T(t,x),Z(t,x))$ and $\left(\bar{P}(t,x),\bar{T}(t,x),\bar{Z}(t,x)\right)$ are two solutions of system (4), whose upper and lower bounds are σ and N respectively.

Construct the function

$$V(t)=\int_\Omega \left[P(t,x)-\bar{P}(t,x)\right]^2 \mathrm{d}x+\int_\Omega \left[T(t,x)-\bar{T}(t,x)\right]^2 \mathrm{d}x+\int_\Omega \left[Z(t,x)-\bar{Z}(t,x)\right]^2 \mathrm{d}x.$$

Calculating the derivative of $V(t)$ with respect to t leads to

$$\begin{aligned}\frac{\mathrm{d}V(t)}{\mathrm{d}t} &= 2\int_\Omega (P-\bar{P})\left(\frac{\partial P}{\partial t}-\frac{\partial \bar{P}}{\partial t}\right)\mathrm{d}x + 2\int_\Omega (T-\bar{T})\left(\frac{\partial T}{\partial t}-\frac{\partial \bar{T}}{\partial t}\right)\mathrm{d}x \\ &\quad + 2\int_\Omega (Z-\bar{Z})\left(\frac{\partial Z}{\partial t}-\frac{\partial \bar{Z}}{\partial t}\right)\mathrm{d}x \\ &\triangleq I_1+I_2+I_3+I_4,\end{aligned}$$

where

$$\begin{aligned}I_1 &= 2d_1\int_\Omega \left(P-\bar{P}\right)\Delta\left(P-\bar{P}\right)\mathrm{d}x + 2d_2\int_\Omega \left(T-\bar{T}\right)\Delta\left(T-\bar{T}\right)\mathrm{d}x \\ &\quad + 2d_3\int_\Omega \left(Z-\bar{Z}\right)\Delta\left(Z-\bar{Z}\right)\mathrm{d}x \\ &= -2d_1\int_\Omega \nabla^2\left(P-\bar{P}\right)\mathrm{d}x - 2d_2\int_\Omega \nabla^2\left(T-\bar{T}\right)\mathrm{d}x - 2d_3\int_\Omega \nabla^2\left(Z-\bar{Z}\right)\mathrm{d}x \\ &\le -2d_1\int_\Omega \left|\nabla\left(P-\bar{P}\right)\right|^2\mathrm{d}x - 2d_2\int_\Omega \left|\nabla\left(T-\bar{T}\right)\right|^2\mathrm{d}x - 2d_3\int_\Omega \left|\nabla\left(Z-\bar{Z}\right)\right|^2\mathrm{d}x \\ &\le 0,\end{aligned}$$

$$\begin{aligned}I_2 &= 2\int_\Omega (P-\bar{P})\left[r_1P\left(1-\frac{P}{K_1}\right)-\alpha_1PT-\frac{\beta_1PZ}{a+P}\right. \\ &\quad \left. -r_1\bar{P}\left(1-\frac{\bar{P}}{K_2}\right)+\alpha_1\bar{P}\bar{T}+\frac{\beta_1\bar{P}\bar{Z}}{a+\bar{P}}\right]\mathrm{d}x \\ &= 2\int_\Omega (P-\bar{P})\left[r_1(P-\bar{P})-\frac{r_1}{K_1}(P^2-\bar{P}^2)-\alpha_1PT+\alpha_1\bar{P}T+\alpha_1\bar{P}T+\alpha_1\bar{P}\bar{T}\right. \\ &\quad \left. +\frac{a\beta_1\bar{P}\bar{Z}+\beta_1P\bar{P}\bar{Z}-a\beta_1PZ-\beta_1P\bar{P}Z}{(a+P)(a+\bar{P})}\right]\mathrm{d}x \\ &= 2\int_\Omega \left[r_1(P-\bar{P})^2-\frac{r_1}{K_1}(P+\bar{P})(P-\bar{P})-\alpha_1T(P-\bar{P})^2-\alpha_1\bar{P}(P-\bar{P})(T-\bar{T})\right. \\ &\quad \left. -\frac{a\beta_1(P-\bar{P})\left[\bar{Z}(P-\bar{P})+P(Z-\bar{Z})\right]+\beta_1P\bar{P}(P-\bar{P})(Z-\bar{Z})}{(a+P)(a+\bar{P})}\right]\mathrm{d}x,\end{aligned}$$

$$
\begin{aligned}
I_3 &= 2\int_\Omega (T-\bar{T})\left[r_2T\left(1-\frac{T}{K_2}\right)-\alpha_2PT-\frac{\theta_1TZ}{b+T}\right.\\
&\quad\left.-r_2\bar{T}\left(1-\frac{\bar{T}}{K_2}\right)+\alpha_2\bar{P}\bar{T}+\frac{\theta_1\bar{T}\bar{Z}}{b+\bar{T}}\right]\mathrm{d}x\\
&= 2\int_\Omega\left[r_2(T-\bar{T})^2-\frac{r_2}{K_2}(T+\bar{T})(T-\bar{T})^2-\alpha_2P(T-\bar{T})^2-\alpha_2\bar{T}(P-\bar{P})(T-\bar{T})\right.\\
&\quad\left.-\ \frac{b\theta_1(T-\bar{T})\left[\bar{Z}(T-\bar{T})+T(Z-\bar{Z})\right]+\theta_1T\bar{T}(T-\bar{T})(Z-\bar{Z})}{(b+T)(b+\bar{T})}\right]\mathrm{d}x
\end{aligned}
$$

and

$$
\begin{aligned}
&I_4 = 2\int_\Omega (Z-\bar{Z})\left[\frac{\beta_2PZ}{a+P}+\frac{\theta_2TZ}{b+T}-mZ^2-\rho T^{\frac{2}{3}}Z\right.\\
&\left.-\frac{\beta_2\bar{P}\bar{Z}}{a+\bar{P}}-\frac{\theta_2\bar{T}\bar{Z}}{b+\bar{T}}+m\bar{Z}^2+\rho\bar{T}^{\frac{2}{3}}\bar{Z}\right]\mathrm{d}x\\
&= 2\int_\Omega\left[\frac{a\beta_2(Z-\bar{Z})\left[\bar{Z}(P-\bar{P})+P(Z-\bar{Z})\right]+\beta_2P\bar{P}(Z-\bar{Z})^2}{(a+P)(a+\bar{P})}-\rho T^{\frac{2}{3}}(Z-\bar{Z})^2\right.\\
&\left.-\rho\bar{Z}(Z-\bar{Z})\left(T^{\frac{2}{3}}-\bar{T}^{\frac{2}{3}}\right)+\frac{b\theta_2(Z-\bar{Z})\left[\bar{Z}(T-\bar{T})+T(Z-\bar{Z})\right]+\theta_2T\bar{T}(Z-\bar{Z})^2}{(b+T)(b+\bar{T})}\right]\mathrm{d}x.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
I_2+I_3+I_4 \le\ & 2\int_\Omega (P-\bar{P})^2\left[r_1^M-2\sigma\frac{r_1^L}{K_1^M}-\alpha_1^L\sigma-\frac{a^L\beta_1^L\sigma}{(a^M+N)^2}\right]\mathrm{d}x\\
&+2\int_\Omega (T-\bar{T})^2\left[r_2^M-2\sigma\frac{r_2^L}{K_2^M}-\alpha_2^L\sigma-\frac{b^L\theta_1^L\sigma}{(b^M+N)^2}\right]\mathrm{d}x\\
&+2\int_\Omega (Z-\bar{Z})^2\left[\frac{\beta_2^MN(a^M+N)}{(a^L+\sigma)^2}+\frac{\theta_2^MN(b^M+N)}{(b^L+\sigma)^2}-\rho^L\sigma^{\frac{2}{3}}\right]\mathrm{d}x\\
&+2\int_\Omega \left|(P-\bar{P})(T-\bar{T})\right|\left[N\left(\alpha_1^M+\alpha_2^M\right)\right]\mathrm{d}x\\
&+2\int_\Omega \left|(P-\bar{P})(Z-\bar{Z})\right|\frac{N\left(a^M\beta_1^M+\beta_1^MN+a^M\beta_2^M\right)}{(a^L+\sigma)^2}\mathrm{d}x\\
&+2\int_\Omega \left|(T-\bar{T})(Z-\bar{Z})\right|\frac{N\left(b^M\theta_1^M+\theta_1^MN+b^M\theta_2^M\right)}{(b^L+\sigma)^2}\mathrm{d}x.
\end{aligned}
$$

From the conditions of this theorem, we have

$$
\frac{\mathrm{d}V(t)}{\mathrm{d}t}\le\lambda_M\int_\Omega\left[(P-\bar{P})^2+(T-\bar{T})^2+(Z-\bar{Z})^2\right]\mathrm{d}x.
$$

That is $\frac{\mathrm{d}V(t)}{\mathrm{d}t} \leq \lambda_M V(t)$, and $V(t_{k+1}) \leq V(t_k)\exp\{\lambda_M(t_{k+1}-t_k)\}$. Thus,

$$\begin{aligned} V(t_{k+1}^+) \leq & \int_\Omega \left[Pf_k(P,T,Z) - \bar{P}f_k(\bar{P},\bar{T},\bar{Z})\right]^2 \mathrm{d}x \\ & + \int_\Omega \left[Tg_k(P,T,Z) - \bar{T}g_k(\bar{P},\bar{T},\bar{Z})\right]^2 \mathrm{d}x \\ & + \int_\Omega \left[Zh_k(P,T,Z) - \bar{Z}h_k(\bar{P},\bar{T},\bar{Z})\right]^2 \mathrm{d}x \\ & \leq L_{k+1}V(t_{k+1}) \leq L_{k+1}\exp\{\lambda_M(t_{k+1}-t_k)\}V(t_k^+). \end{aligned}$$

Next, we estimate the variation of function $V(t)$ over one period. From

$$V(t+\omega) \leq L^* V(t) = \prod_{k=1}^{p} L_k \exp\{\lambda_M \omega\} V(t),$$

then we have

$$L^* = \prod_{k=1}^{p} L_k \exp\{\lambda_M \omega\} < 1.$$

Therefore, $V(n\omega + s) \leq L^* V(s) \to 0$ when $n \to \infty$, and

$$\lim_{t\to\infty} \left\|P(t,x) - \bar{P}(t,x)\right\| = \lim_{t\to\infty} \left\|T(t,x) - \bar{T}(t,x)\right\| = \lim_{t\to\infty} \left\|Z(t,x) - \bar{Z}(t,x)\right\| = 0,$$

where $\|\cdot\|$ is the norm of space $L_2(\Omega)$.

With the help of Theorem 9 in [25], the solutions of (4) are bounded in the space $C^{1+\nu}$. We have

$$\begin{aligned} & \lim_{t\to\infty} \sup_{x\in\Omega} \left\|P(t,x) - \bar{P}(t,x)\right\| \\ & = \lim_{t\to\infty} \sup_{x\in\Omega} \left\|T(t,x) - \bar{T}(t,x)\right\| = \lim_{t\to\infty} \sup_{x\in\Omega} \left\|Z(t,x) - \bar{Z}(t,x)\right\| = 0. \end{aligned}$$

Consider the sequence $(P(k\omega, x, P_0, T_0, Z_0), T(k\omega, x, P_0, T_0, Z_0), Z(k\omega, x, P_0, T_0, Z_0)) = w(k\omega, w_0), k \in \mathbb{N}^+$, again by the Theorem 9 in [25], this sequence is compact in the space $C(\bar{\Omega}) \times C(\bar{\Omega}) \times C(\bar{\Omega})$. Let $\bar{w}$ be a limit point of this sequence, $\lim_{n\to\infty} w(k_n\omega, w_0) = \bar{w}$, where $\lim_{n\to\infty} k_n = \infty$. Then $w(\omega, \bar{w}) = \bar{w}$. In fact, from $w(\omega, w(k_n\omega, w_0)) = w(k_n\omega, w(\omega, w_0))$ and $\lim_{n\to\infty}[w(k_n\omega, w(\omega, w_0)) - w(k_n\omega, w_0)] = 0$, we have

$$\begin{aligned} \|w(\omega,\bar{w}) - \bar{w}\|_C \leq & \|w(\omega,\bar{w}) - w(\omega, w(k_n\omega, w_0))\|_C \\ & + \|w(\omega, w(k_n\omega, w_0)) - w(k_n\omega, w_0)\|_C \\ & + \|w(k_n\omega, w_0) - \bar{w}\|_C \to 0, \quad (n\to\infty). \end{aligned}$$

We claim that sequence $w(k\omega, w_0)$ has the unique limit point for all $k \in \mathbb{N}^+$. And if not, it is assumed that this sequence has two different limit points:

$$\lim_{n\to\infty} w(k_n\omega, w_0) = \bar{w}, \quad \lim_{n\to\infty} w(k_n\omega, w_0) = \tilde{w}.$$

The inequality $\|\bar{w} - \tilde{w}\|_C \leq \|\bar{w} - w(k_n\omega, w_0)\|_C + \|w(k_n\omega, w_0 - \tilde{w}\|_C \to 0$ can deduce $\|\bar{w} - \tilde{w}\|_C$=0. Hence, $\bar{w} = \tilde{w}$. In conclusion, system (4) has a unique periodic solution and it is globally asymptotically stable. The proof is completed.

5. Conclusions

In this paper, we have proposed a reaction-diffusion plankton model with periodic impulses. By using the basic theory of impulsive differential equation, comparison principle, upper-lower solution method and inequality techniques, the sufficient conditions for ultimate boundedness of solutions and permanence of the system have been established. Theorem 4.1 manifests that the periodic solution is unique and globally asymptotically stable if the system is permanent. It is not difficult to find that, in real plankton ecosystem, various population numbers may vary periodically due to the comprehensive influence of seasonal variation and biological rhythm.

Acknowledgement

This work was supported by the National Natural Science Foundation of China (12061033) and the Natural Science Research Projects of Universities in Anhui Province (KJ2020A0016 and KJ2020A0002).

References

[1] C. Jost, C.A. Lawrence, F. Campolongo, W. Bund, S. Hillm, D.L. DeAngelis, *The effects of mixotrophy on the stability and dynamics of a simple planktonic food web model*, Theoretical Population Biology, 66 (2004), 37-51.

[2] Y. Sekerci, S. Petrovskiii, *Mathematical modelling of spatiotemporal dynamics of oxygen in a plankton system*, Mathematical Modelling of Natural Phenomena, 10 (2015), 96-114.

[3] J.T. Zhao, J.P. Tian, J.J. Wei, *Minimal model of plankton systems revisited with spatial diffusion and maturation delay*, Bulletin of Mathematical Biology, 78 (2016), 381-412.

[4] J. Dhar, R.S. Baghel, *Role of dissolved oxygen on the plankton dynamics in spatio-temporal domain*, Modeling Earth Systems and Environment, 2 (2016), No. 6.

[5] J. Chattopadhyay, R.R. Sarkar, S. Mandal, *Toxin-producing plankton may act as a biological control for planktonic blooms—field study and mathematical modelling*, Journal of Theoretical Biology, 215 (2002), 333-344.

[6] J. Chattopadhyay, R.R. Sarkar, S. Pal, *Mathematical modelling of harmful algal blooms supported by experimental findings*, Ecological Complexity, 1 (2004), 225-235.

[7] S. Chakraborty, S. Chatterjee, E. Venturino, J. Chattopadhyay, *Recurring plankton bloom dynamics modeled via toxin-producing phytoplankton*, Journal of Biological Physics, 33 (2007), 271-290.

[8] J. Chattopadhyay, R.R. Sarkar, A. Abdllaoui, *A delay differential equation model on harmful algal blooms in the presence of toxic substances*, IMA Journal of Mathematics Applied in Medicine and Biology, 19(2002), 137-161.

[9] Y. Wang, H.B. Wang, W.H. Jiang, *Hopf-transcritical bifurcation in toxic phytoplankton-zooplankton model with delay*, Journal of Mathematical Analysis and Applications, 415 (2014), 574-594.

[10] A.K. Sharma, A. Sharma, K. Agnihotri, *Spatiotemporal dynamic of toxin producing phytoplankton(TPP)-zooplankton interaction*, International Journal of Mathematical Modelling and Computations, 6 (2016), 189-197.

[11] Z.C. Jiang, W.B. Ma, D. Li, *Dynamical behavior of a delay differential equation system on toxin producing phytoplankton and zooplankton interaction*, Japan Journal of Industrial and Applied Mathematics, 31 (2014), 583-609.

[12] S. Roy, J. Chattopadhyay, *Enrichment and ecosystem stability: Effect of toxic food*, BioSystems, 90 (2007), 151-160.

[13] H.J. Guo, L.S. Chen, X.Y. Song, *Qualitative analysis of impulsive state feedback control to an algae-fish system with bistable property*, Applied Mathematics and Computation, 271 (2015), 905-922.

[14] X.W. Yu, S.L. Yuan, T.H. Zhang, *Survival and ergodicity of a stochastic phytoplankton-zooplankton model with toxin-producing phytoplankton in an impulsive polluted environment*, Applied Mathematics and Computation, 347 (2019), 249-264.

[15] J. Passarge, S. Hol, M. Escher, J. Huisman, *Competition for nutrient and light: stable coexistence, alternative stable states, or competitive exclusions*? Ecological Monographs, 76 (2006), 57-72.

[16] J. Huisman, F.J. Weissing, *Biodiversity of plankton by species oscillations and chaos*, Nature, 402 (1999), 407-410.

[17] S. Pal, S. Chatterjee, K Das, J.Chattopadhyay, *Role of competition in phytoplankton population for the occurrence and control of plankton bloom in the presence of environmental fluctuations*, Ecological Modelling, 220 (2009), 96-110.

[18] S. Chakraborty, P.K. Tiwari, A.K. Misra, J. Chattopadhyay, *Spatial dynamics of a nutrient-phytoplankton system with toxic effect on phytoplankton*, Mathematical Biosciences, 264 (2015), 94-100.

[19] R.J. Han, B.X. Dai, *Spatiotemporal pattern formation and selection induced by nonlinear cross-diffusion in a toxic-phytoplankton-zooplankton model with Allee effect*, Nonlinear Analysis: Real World Applications, 45 (2019), 822-853.

[20] C.J. Xu, Q.M. Zhang, P.L. Li, *Almost periodic solution analysis in a two-species competitive model of plankton alleopathy with impulses*, Journal of Applied Mathematics and Computing, 50 (2016), 437-452.

[21] L.J. Chen, J.T. Sun, F.D. Chen, L. Zhao, *Extinction in a Lotka-Volterra competitive system with impulse and the effect of toxic substances*, Applied Mathematical Modelling, 40 (2016), 2015-2024.

[22] Q. Wang, Z.J. Wang, Y. Wang, *An impulsive ratio-dependent $n+1$-species predator-prey model with diffusion*, Nonlinear Analysis: Real World Applications, 11 (2010), 2164-2174.

[23] C.L. Li, X.Q. Guo, D.M. He, *An impulsive diffusion predator-prey system in three-species with Beddington-DeAngelis response*, Journal of Applied Mathematics and Computing, 43 (2013), 235-248.

[24] Z.J. Liu, L. Zhang, P. Bi, J.H. Pang, B. Li, C.L. Fang, *On the dynamics of one-prey-n-predator impulsive reaction-diffusion predator-prey system with ratio-dependent functional response*, Journal of Biological Dynamics, 12 (2018), 551-576.

[25] M.U. Akhmet, M. Beklioglu, T. Ergence, V.I. Tkachenko, *An impulsive ratio-dependent predator-prey system with diffusion*, Nonlinear Analysis: Real World Applications, 7 (2006), 1255-1267.

[26] W. Walter, *Differential inequalities and maximum principles: theory, new methods and applications*, Nonlinear Analysis: Theory, Methods and Applications, 30 (1997), 4695-4711.

[27] H.L. Smith, *Dynamics of competition*, Springer, Berlin, 1999.

[28] X.N. Liu, L.S. Chen, *Global dynamics of the periodic logistic system with periodic impulsive perturbations*, Journal of Mathematical Analysis and Applications, 289 (2004), 279-291.

Accepted: 05.02.2019