A subordination result and integral mean for a class of analytic functions defined by q-differintegral operator

B.A. Frasin*
Fuculty of Science
Department of Mathematics
Al Al-Bayt University
Mafraq
Jordan
bafrasin@yahoo.com

N. Ravikumar
PG Department of Mathematics
JSS College of Arts
Commerce and Science Mysore-570 025
India
ravisn.kumar@gmail.com

S. Latha
Department of Mathematics
Yuvaraj's College
University of Mysore
Mysore-570 009
India
drlatha@gmail.com

Abstract. In this paper, we derive a subordination result and integral mean for certain class of analytic functions defined by means of a fractional q-differintegral operator $\Omega^{\delta}_{q,z}f(z)$.
Keywords: analytic functions, univalent functions, subordinating factor sequence, q-difference operator, Hadamard product (or convolution).

1. Introduction

Let $\mathcal{A}$ denote the family of functions of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m \tag{1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$. Also denote by $\mathcal{T}$, the subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{m=2}^{\infty} |a_m| z^m \tag{2}$$

*. Corresponding author

which are univalent and normalized in $\mathcal{U}$. For $f \in \mathcal{A}$ and of the form (1) and $g(z) \in \mathcal{A}$ given by $g(z) = z + \sum_{m=2}^{\infty} b_m z^m$, we define the convolution (or Hadamard product) $f * g$ of two power series f and g by

$$(f * g)(z) = z + \sum_{m=2}^{\infty} a_m b_m z^m.$$

The fractional q-calculus is the extension of the ordinary fractional calculus in the q-theory. The theory of q-calculus operators in recent past have been applied in the areas of ordinary fractional calculus, optimal control problems and in finding solutions of the q-difference and q-integral equations, and in q-transform analysis and also in the geometric function theory of complex analysis. For more details on the subject, one may refer to [1, 5, 9, 13, 21].

The q-shifted factorial is defined for α, $q \in \mathbb{C}$ as a product of n factors by

$$(\alpha, q)_n = \begin{cases} 1, & n = 0 \\ (1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{n-1}), & n \in \mathbb{N}, \end{cases} \tag{3}$$

and in terms of the basic analogue of the gamma function

$$(q^{\alpha}; q)_n = \frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)}, \ (n > 0), \tag{4}$$

where the q-gamma functions [9, 10] is defined by

$$\Gamma_q(x) = \frac{(q;q)_{\infty}(1-q)^{1-x}}{(q^x;q)_{\infty}} \ (0 < q < 1). \tag{5}$$

Note that, if $|q| < 1$, the q-shifted factorial (3), remains meaningful for $n = \infty$ as a convergent infinite product

$$(\alpha; q)_{\infty} = \prod_{m=0}^{\infty} (1 - \alpha q^m).$$

Now, recall the following q-analogue definitions given by Gasper and Rahman [9]. The recurrence relation for q-gamma function is given by

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x), \text{ where, } [x]_q = \frac{(1-q^x)}{(1-q)}, \tag{6}$$

and called q-analogue of x.

Jackson's q-derivative and q-integral of a function f defined on a subset of $\mathbb{C}$ are, respectively, given by (see Gasper and Rahman [9])

$$D_q f(z) = \frac{f(z) - f(zq)}{z(1-q)} \ (z \neq 0, \, q \neq 0) \tag{7}$$

and

$$\int_0^z f(t)d_q(t) = z(1-q)\sum_{m=0}^{\infty} q^m f(zq^m). \tag{8}$$

In view of the relation

$$\lim_{q\to 1^-} \frac{(q^\alpha;q)_n}{(1-q)^n} = (\alpha)_n, \tag{9}$$

we observe that the q-shifted fractional (3), reduces to the familiar Pochhammer symbol $(\alpha)_n$, where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n+1)$.

Now, recall the definition of the fractional q-calculus operators of a complex-valued function $f(z)$, which were recently studied by Purohit and Raina [17].

Definition 1.1. *The fractional q-integral operator $I^\delta_{q,z}$ of a function $f(z)$ of order $\delta\,(\delta > 0)$ is defined by*

$$I^\delta_{q,z} = D^{-\delta}_{q,z} f(z) = \frac{1}{\Gamma_q(\delta)}\int_0^z (z-tq)_{1-\delta} f(t) d_q t, \tag{10}$$

where $f(z)$ is a analytic in a simply connected region in the z-plane containing the origin. Here, the term $(z-tq)_{\delta-1}$ is a q-binomial function defined by

$$\begin{aligned}(z-tq)_{\delta-1} &= z^{\delta-1}\prod_{m=0}^{\infty}\left[\frac{1-(\frac{tq}{z})q^m}{1-(\frac{tq}{z})q^\delta+m-1}\right] \\ &= z^\delta\,{}_1\phi_0\left[q^{-\delta+1};-;q,\frac{tq^\delta}{z}\right].\end{aligned} \tag{11}$$

According to Gasper and Rahman [9], the series ${}_1\phi_0[\delta;-;q,z]$ is single-valued when $|arg(z)| < \pi$. Therefore, the function $(z-tq)_{\delta-1}$ in (11), is single-valued when $|arg(\frac{-tq^\delta}{z}| < \pi$, $|tq^{\frac{\delta}{z}}| < 1$, and $|arg(z)| < \pi$.

Definition 1.2. *The fractional q-derivative operator $D^\delta_{q,z}$ of a $f(z)$ of order $\delta(0 \le \delta < 1)$ is defined by*

$$D^\delta_{q,z} f(z) = D_{q,z} I^{1-\delta}_{q,z} f(z) = \frac{1}{\Gamma_q(1-\delta)} D_q \int_0^z (z-tq)_{-\delta} f(t) d_q t, \tag{12}$$

where $f(z)$ is suitably constrained and the multiplicity of $(z-tq)_{-\delta}$ is removed as in Definition 1.1 above.

Definition 1.3. *Under the hypotheses of Definition 1.2, the fractional q-derivative for the function $f(z)$ of order δ is defined by*

$$D^\delta_{q,z} f(z) = D^n_{q,z} I^{n-\delta}_{q,z} f(z), \tag{13}$$

where, $n-1 \le \delta < n$, $n \in \mathbb{N}_0$.

Now, we define a fractional q-differintegral operator $\Omega_{q,z}^{\delta} f(z)$ for the function $f(z)$ of the form (1), by

$$\begin{aligned} \Omega_q^{\delta} f(z) &= \Gamma_q(2-\delta) z^{\delta} D_{q,z}^{\delta} f(z) \\ &= z + \sum_{m=2}^{\infty} \frac{\Gamma_q(m+1)\Gamma_q(2-\delta)}{\Gamma_q(m+1-\delta)} a_m z^m, \end{aligned} \tag{14}$$

where in $D_{q,z}^{\delta}$ (14), represents, respectively, a fractional q-integral of $f(z)$ of order δ when $-\infty < \delta < 0$ and a fractional q-derivative of $f(z)$ of order δ when $0 < \delta < 2$. We note that $q \to 1^-$, the operator Ω_q^{δ} reduces the operator Ω^{δ} defined by Owa and Srivastava [14].

Recently, several authors investigated applications of fractional q-calculus operators by introducing certain new classes of functions which are analytic in the open disc, (see for example, [16, 18, 19, 27, 28]).

By making use of the concepts of fractional q- calculus, Ravikumar et al. [20] defined the following subclasses $\mathcal{S}_p^q(\alpha, \beta, \delta, b)$ of analytic function.

Definition 1.4. *For* $-1 \leq \alpha < 1$, $\beta \geq 0$, $0 < \delta < 2$, $b \in \mathbb{C} - \{0\}$ *and* $0 < q < 1$, *let* $\mathcal{S}_p^q(\alpha, \beta, \delta, b)$ *be the subclass of* $\mathcal{A}$ *consisting of functions of the form* (1) *and satisfying the analytic criterion*

$$\Re\left\{1 - \frac{2}{b} + \frac{2}{b}.\frac{zD_q(\Omega_q^{\delta} f(z))}{\Omega_q^{\delta} f(z)}\right\} > \beta\left|\frac{2}{b}.\frac{zD_q(\Omega_q^{\delta} f(z))}{\Omega_q^{\delta} f(z)} - \frac{2}{b}\right| + \alpha, \ z \in \mathcal{U}. \tag{15}$$

It can be seen that, the special cases of the class $\mathcal{TS}_p^q[\alpha, \beta, \delta, b]$ as $q \to 1^-$ and for different choices of the parameters we get the results obtained by Altintas and Owa [3], Bharthi, Parvtham and Swaminathan [6], Padamanabhan and Jayamala [15], Owa and Srivastava [14], Kim and Ronning [11].

Before we state and prove our main result we need the following definitions and lemmas.

Definition 1.5 (Subordination principle)**.** *Let* $g(z)$ *be analytic and univalent in* $\mathcal{U}$*. If* $f(z)$ *is analytic in* $\mathcal{U}$, $f(0) = g(0)$, *and* $f(\mathcal{U}) \subset g(\mathcal{U})$, *then we see that the function* $f(z)$ *is subordinate to* $g(z)$ *in* $\mathcal{U}$, *and we write* $f(z) \prec g(z)$.

Definition 1.6 (Subordinating factor sequence)**.** *A sequence* $\{b_m\}_{m=1}^{\infty}$ *of complex numbers is called a subordinating factor sequence if, whenever* $f(z)$ *is analytic , univalent and convex in* $\mathcal{U}$, *we have the subordination given by*

$$\sum_{m=2}^{\infty} b_m a_m z^n \prec f(z) \qquad (z \in \mathcal{U}, \ a_1 = 1). \tag{16}$$

Lemma 1.1 ([29])**.** *The sequence* $\{b_m\}_{m=1}^{\infty}$ *is a subordinating factor sequence if and only if*

$$Re\left\{1 + 2\sum_{m=1}^{\infty} b_m z^m\right\} > 0, \qquad (z \in \mathcal{U}). \tag{17}$$

Lemma 1.2 ([20]). *If*

$$\sum_{m=2}^{\infty}[2(1+\beta)([m]_q-1)+b(1-\alpha)]K_q(m,\delta)(\delta)\,|a_m|\leq b(1-\alpha), \tag{18}$$

then $f(z)\in\mathcal{S}_p^q(\alpha,\beta,\delta,b)$, *where,* $K_q(m,\delta)=\frac{\Gamma_q(m+1)\Gamma_q(2-\delta)}{\Gamma_q(m+1-\delta)}$, $-1\leq\alpha<1,\ \beta\geq 0,\ 0<\delta<2,\ b>0$ *and* $0<q<1$.

The result is sharp for the function

$$f_m(z)=z-\frac{b(1-\alpha)}{[2(1+\beta)([m]_q-1)+b(1-\alpha)]K_q(m,\delta)(\delta)}z^m.$$

Let $\mathcal{S}_p^{*,q}(\alpha,\beta,\delta,b)$ denote the class of functions $f(z)\in\mathcal{A}$ whose coefficients satisfy the condition (18). We note that $\mathcal{S}_p^{*,q}(\alpha,\beta,\delta,b)\subseteq\mathcal{S}_p^q(\alpha,\beta,\delta,b)$.

Unless otherwise mentioned, we assume throughout this paper that $-1\leq\alpha<1,\ \beta\geq 0,\ 0<\delta<2,\ b>0$ and $0<q<1$.

2. Subordination result

Employing the techniques used by Aouf et al. [2], Attiya [4], Frasin [7], Frasin et al. [8], Singh [25], Srivastava and Attiya [26], and others, we state and prove the following theorem.

Theorem 2.1. *Let the function* $f(z)$ *be defined by (1) be in the class* $\mathcal{S}_p^{*,q}(\alpha,\beta,\delta,b)$. *Also let* $\mathcal{C}$ *denote the familiar class of functions* $f(z)\in\mathcal{A}$ *which are also univalent and convex in* $\mathcal{U}$. *Then*

$$\frac{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}{2[b(1-\alpha)+(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)]}(f*g)(z)\prec g(z), \tag{19}$$
$$(z\in\mathcal{U};,g\in\mathcal{C}),$$

and

$$Re(f(z))>-\frac{b(1-\alpha)+(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)},(z\in\mathcal{U}). \tag{20}$$

The constant $\frac{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}{2[b(1-\alpha)+(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)]}$ *is the best estimate.*

Proof. Let $f(z)\in\mathcal{S}_p^{*,q}(\alpha,\beta,\delta,b)$ and let $g(z)=z+\sum_{m=2}^{\infty}c_mz^m\in\mathcal{C}$. Then

$$\frac{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}{2[b(1-\alpha)+(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)]}(f*g)(z)$$
$$=\frac{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}{2[b(1-\alpha)+(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)]}\left(z+\sum_{m=2}^{\infty}a_mc_mz^m\right). \tag{21}$$

Thus, by Definition 1.6, the assertion of our theorem will hold if the sequence

$$\left\{\frac{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}{2[b(1-\alpha)+(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)]}a_m\right\}_{m=1}^{\infty} \tag{22}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1.1, this will be the case if and only if

$$\mathrm{Re}\left\{1+2\sum_{n=1}^{\infty}\frac{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}{2[b(1-\alpha)+(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)]}a_m z^m\right\}>0, \tag{23}$$

$(z \in \mathcal{U})$.

Now,

$$\begin{aligned}
&\mathrm{Re}\left\{1+\frac{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}{b(1-\alpha)+(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}\sum_{m=1}^{\infty}a_m z^m\right\}\\
=\;&\mathrm{Re}\left\{1+\frac{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}{b(1-\alpha)+(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}z+\right.\\
&+\frac{1}{b(1-\alpha)+(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}\\
&\left.\cdot\sum_{m=2}^{\infty}(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)a_m z^m\right\}\\
\geq\;&1-\left\{\frac{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}{b(1-\alpha)+(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}r\right.\\
&-\frac{1}{b(1-\alpha)+(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}\\
&\left.\cdot\sum_{m=2}^{\infty}[2(1+\beta)([m]_q-1)+b(1-\alpha)]K_q(m,\delta)(\delta)a_m r^m\right\}
\end{aligned}$$

(because $[2(1+\beta)([m]_q-1)+b(1-\alpha)]K_q(m,\delta)(\delta)$ is increasing function of m, $(m \geq 2)$)

$$\begin{aligned}
&>1-\frac{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}{b(1-\alpha)+(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}r\\
&-\frac{b(1-\alpha)}{b(1-\alpha)+(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}r>0, \qquad (|z|=r<1).
\end{aligned}$$

Thus (23) holds true in $\mathcal{U}$. This proves the inequality (19). The inequality (20) follows by taking the convex function $g(z) = \frac{z}{1-z} = z + \sum_{m=2}^{\infty} z^m$ in (19). To prove the sharpness of the constant $\frac{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}{2[b(1-\alpha)+(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)]}$, we consider the function $f_2(z) \in \mathcal{S}_p^{*,q}(\alpha,\beta,\delta,b)$ given by

$$f_2(z) = z - \frac{b(1-\alpha)}{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}z^2.$$

Thus from the (19), we have

$$\frac{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}{2[b(1-\alpha)+(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)]}f_2(z) \prec \frac{z}{1-z}. \tag{24}$$

It can easily verified that

$$\min\left\{\mathrm{Re}\left(\frac{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}{2[b(1-\alpha)+(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)]}f_2(z)\right)\right\} = -\frac{1}{2}, \tag{25}$$

$(z \in \mathcal{U})$. This shows that the constant $\frac{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}{2[b(1-\alpha)+(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)]}$ is best possible. □

3. Integral means inequalities

Lemma 3.1 ([12]). *If the functions f and g are analytic in $\mathcal{U}$ with $g \prec f$, then for $\eta > 0$, and $0 < r < 1$,*

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^\eta d\theta \le \int_0^{2\pi} \left|f(re^{i\theta})\right|^\eta d\theta. \tag{26}$$

In [22], Silverman found that the function $F(z) = z - \frac{z^2}{2}$ is often extremal over the family $\mathcal{T}$ and applied this function to resolve his integral means inequality, conjectured in [23] and settled in [24], that

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^\eta d\theta \le \int_0^{2\pi} \left|F(re^{i\theta})\right|^\eta d\theta,$$

for all $f \in \mathcal{T}$, $\eta > 0$ and $0 < r < 1$. In [24], Silverman also proved his conjecture for the subclasses $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$ of $\mathcal{T}$, where $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$ are starlike functions of order α and convex functions of order α, respectively.

Applying Lemma 3.1 and Lemma 1.2 , we prove the following result.

Theorem 3.1. *Suppose that $f \in \mathcal{S}_p^{*,q}(\alpha,\beta,\delta,b)$ and $f_2(z)$ is defined by*

$$f_2(z) = z - \frac{b(1-\alpha)}{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}z^2.$$

Then for $z = re^{i\theta}$, $0 < r < 1$, we have

$$\int_0^{2\pi} |f(z)|^\eta d\theta \le \int_0^{2\pi} |f_2(z)|^\eta d\theta, \qquad (\eta > 0). \tag{27}$$

Proof. For $f(z) = z - \sum_{m=2}^{\infty} |a_m| z^m$, (27) is equivalent to proving that

$$\int_0^{2\pi} \left|1 - \sum_{m=2}^{\infty} |a_m| z^{m-1}\right|^\eta d\theta \le \int_0^{2\pi} \left|1 - \frac{b(1-\alpha)}{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}z\right|^\eta d\theta.$$

By Lemma 3.1, it suffices to show that

$$1-\sum_{m=2}^{\infty}|a_m|z^{m-1} \prec 1-\frac{b(1-\alpha)}{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}z.$$

Letting

$$1-\sum_{m=2}^{\infty}|a_m|z^{m-1} = 1-\frac{b(1-\alpha)}{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}w(z), \tag{28}$$

and using (1), we obtain $w(z)$ is analytic in $\mathcal{U}, w(0)=0$, and

$$\begin{aligned}|w(z)| &= \left|\sum_{m=2}^{\infty}\frac{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}{b(1-\alpha)}|a_m|z^{m-1}\right| \\ &\leq |z|\sum_{m=2}^{\infty}\frac{[2(1+\beta)([m]_q-1)+b(1-\alpha)]K_q(m,\delta)(\delta)}{b(1-\alpha)}|a_m| \\ &\leq |z|.\end{aligned}$$

This completes the proof by Theorem 3.1. □

References

[1] M. H. Abu-Risha, M. H. Annaby, M. E. H. Isail, Z. S. Mansour, *Linear q-difference equations*, Z. Anal. Anwend., 26 (2007), 481-494.

[2] M. K. Aouf, A. Shamandy, A. O. Mostafa, F. El-Emam, *Subordination results associated with β-uniformly convex and starlike functions*, Proc. Pakistan Acad. Sci., 46 (2009), 97-101.

[3] O. Altintas, S. Owa, *On subclasses of univalent functions with negative coefficients*, Pusan Kyongnam Math. J, 4 (1988), 41-56.

[4] A. A. Attiya, *On some application of a subordination theorems*, J. Math. Anal. Appl., 311 (2005), 489-494.

[5] G. Bangerzako, *Variational calculus on q-nonuniform lattices*, J. Math. Anal. Appl., 306 (2005), 161-179.

[6] R. Bharthi, R. Paravatham, A. Swaminathan, *On subclasses of uniformaly convex functions and corresponding class of starlike functions*, Tamkang J. Math., 26 (1997), 17-32.

[7] B. A, Frasin, *Subordination results for a class of analytic functions defined by a linear operator,* J. Inequal. Pure Appl. Math., 7 (2006), 1-7.

[8] B. A. Frasin, Y. Talafha, Tariq Al-Hawary, *Subordination results for classes of functions of reciprocal order*, Tamsui Oxford Journal of Mathematical Sciences, 30 (2014), 81-89.

[9] G. Gasper, M. Rahman, *Basic hypergeometric series*, 35 of Encyclopedia of Mathematics and Its Applications, Cambridge University Press, Cambridge, Mass, USA, 1990.

[10] F. H. Jackson, *On q-functions and a certain difference operator,* Transactions of the Royal Society of Edinburgh, 46 (1909), 253-281.

[11] Y. C. Kim, F. Ronning, *Integral transforms of certain subclasses of analytic function,* J. Math. Anal. Appl., 258 (2001), 466-657.

[12] J. E. Littlewood, *On inequalities in theory of functions,* Proc. London Math. Soc., 23 (1925), 481-519.

[13] Z. S. I. Mansour, *Linear sequential q-difference equations of fractional order*, Fract. Calc. Appl. Anal., 12 (2009), 159-178.

[14] S. Owa, H. M. Srivastava, *Univalent and starlike generalized hypergeometric functions*, Canadian Journal of Mathematics, 39 (1987), 1057-1077.

[15] K. S. Padmanabhan, M. Jayamala, *A class of univalent functions with varying arguments*, Internat. J. Math. and Math. Sci., 15 (1992), 517-522.

[16] S. D. Purohit, *A new class of multivalently analytic functions associated with fractional q-calculus operators*, Fractional Differential Calculus, 2 (2012), 129-138.

[17] S. D. Purohit, R. K. Raina, *Certain subclasses of analytic functions associated with fractional q-calculus operators*, Mathematica Scandinavica., 109 (2011), 55-70.

[18] C. Ramachandran, T. Soupramanien, B. A. Frasin, *New subclasses of analytic function associated with q-difference operator*, European Journal of Pure and Applied Mathematics, 10 (2017), 348-362.

[19] N. Ravikumar, *Certain classes of analytic functions defined by fractional q-calculus operator*, Acta Univ. Sapientiae, Mathematica, 10 (2018), 178-188.

[20] N. Ravikumar, S. Latht, B. A. Frasin, *On certain subclasses of analytic functions with Riemann Liouville q-derivative distribution series*, Journal of Fractional Calculus and Applications, 10 (2019), 216-224.

[21] P. M. Rajkovic, S. D. Marinkovic, M. S. Stankovic, *Fractional integrals and derivatives in q-calculus*, Appl. Anal. Discrete Math., 1 (2007), 311-323.

[22] H. Silverman, *Univalent functions with negative coefficients,* Proc. Amer. Math. Soc., 51 (1975), 109-116.

[23] H. Silverman, *A survey with open problems on univalent functions whose coefficients are negative*, Rocky Mt. J. Math., 21 (1991), 1099-1125.

[24] H. Silverman, *Integral means for univalent functions with negative coefficients*, Houston J. Math., 23 (1997), 169-174.

[25] S. Singh, *A subordination theorems for spirallike functions*, IJMMS, 24 (2000), 433-435.

[26] H. M. Srivastava, A. A. Attiya, *Some subordination results associated with certain subclasses of analytic functions*, J. Inequal. Pure Appl. Math., 5 (2004), 1-6.

[27] K. Vijaya, *Certain Class of Analytic Functions Based on q-difference operator*, arXiv:1709.04138v1 [math.CV] 13 Sep 2017.

[28] K. Vijaya, G. Murugusundaramoorthy, S. Yalçın, *Certain class of analytic functions involving Salagean type q-difference operator*, Konuralp Journal of Mathematics, 6 (2018), 264-271.

[29] H. S. Wilf, *Subordinating factor sequence for convex maps of the unit circle*, Proc. Amer. Math. Soc., 12 (1961), 689-693.

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