An error estimate of a nonmatching grids method for a biharmonic equation

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Abstract. Motivated by the work of Boulaaras and Haiour in [7], we provide a maximum norm analysis of Schwarz alternating method for biharmonic equation with repect to the mixed boundary condition, where an optimal error analysis each subdomain between the discrete Schwarz sequence and the continuous solution of bilaplace equations is established.

Keywords: maximum norm analysis, nonmatching grids method, Schwarz sequence, bilaplace differential equations.

1. Introduction

Schwarz method has been invented by Herman Amandus Schwarz in 1890. This method has been used to solve the stationary or evolutionary boundary value problems on domains which consists of two or more overlapping subdomains(see[1,2,5,6,9]). The solution is approximated by a infinite sequence of function which results from solving a sequence of stationary or evolutionary boundary value problems in each of the sub-domain. In this work, we provide a maximum norm analysis of an overlapping Schwarz method on non-matching grids for bilaplace equation related to impulse control problem.

We consider the following elliptic equation : find $u \in H^2(\Omega)$ solution of

$$\begin{cases} \Delta^2 u + \alpha u = f, \quad \text{in } \Omega, \\ \Delta u + \rho \dfrac{\partial u}{\partial \eta} = 0 \text{ in } \Gamma, \\ \dfrac{\partial(\Delta u)}{\partial \eta} = \varphi \text{ in } \Gamma_0, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^2$ with boundary Γ and α is a positive constant such that satisfies

$$\beta \leq \alpha, \quad \beta > 0. \tag{1.2}$$

f is a regular function.

Let $(.,.)_{\Omega}$ be the scalar product in $L^2(\Omega)$ and $(.,.)_{\Gamma_0}$ be the scalar product in $L^2(\Gamma_0)$, where Γ_0 is the part of the boundary defined as

$$\Gamma_0 = \left\{ x \in \partial\Omega = \Gamma \text{ such that } \forall \xi \ > 0,\ x + \xi \notin \bar{\Omega} \right\}.$$

In [7] Boulaaras and Haiour provided a maximum norm analysis of a finite element Schwarz alternating method for a nonlinear parabolic partial differential equations on two overlapping subdomains with nonmatching grids. They considered a domain which is the union of two overlapping subdomains where each subdomain has its own independently generated grid. The two meshes being mutually independent on the overlap region, a triangle belonging to one triangulation does not necessarily belong to the other one. Then according to Lipschitz assumption, they proved on each subdomain an optimal error estimate on uniform norm between the discrete Schwarz sequence and the exact solution of a nonlinear parabolic partial differential equations. In this paper, the same approach can be extended to other types as a linear parabolic partial differential equations see [2] and singularly perturbed advection-diffusion equations (see [11]) using the overlapping domain decomposition method, where we applied it in a full discrete (see [3,4,7]).

In [7], the authors studied the overlapping domain decomposition method combined with a finite element approximation for Laplace equuation, where an overlapping Schwarz method on nonmatching grids has been used on uniform norm of and they also proved the geometric convergence on every subdomain.

In this paper, similar to that in [7], we can extend the study to bilaplace equation with mixed boundary conditions, where we provide a maximum norm analysis of the finite element Schwarz alternating method of the presented problem on two overlapping subdomains with nonmatching grids. We consider a domain which is the union of two overlapping subdomains where each subdomain has its own independently generated grid. The two meshes being mutually independent on the overlap region, a triangle belonging to one triangulation does not necessarily belong to the other one.

With respect to the stability analysis which has been given by our previous work in [7], we establish on each subdomain, an optimal error analysis between the discrete Schwarz sequence and the continuous solution of bilaplace equations. Also the geometric convergence is proved.

2. Nonlinear elliptic equation

We can reformulate (1.1) into the following variational equation by using Green's formula: find $u \in \mathcal{H}$ solution of

$$\begin{cases} \mathcal{A}(u,v) = (f,v)_{\Omega} + (\varphi, v)_{\Gamma_0}, \\ \Delta u + \rho \dfrac{\partial u}{\partial \eta} = 0 \text{ on } \Gamma, \\ \dfrac{\partial(\Delta u)}{\partial \eta} = \varphi \text{ in } \Gamma_0, \end{cases} \tag{2.1}$$

where $\mathcal{A}(.,.)$ is the bilinear form defined as:

$$u, v \in H^2(\Omega) : \mathcal{A}(u,v) = (\Delta u, \Delta v)_\Omega + \rho (\partial_\eta u, \partial_\eta v)_\Gamma + \alpha (u,v). \tag{2.2}$$

Let $V_h^{(\varphi)}$ the discrete space defined as

$$V_h^{(\varphi)} = \{v \in \mathcal{H} \text{ such that } v_h\,|_K = P_1,\ k \in \tau_h, \frac{\partial v_h}{\partial \eta} = \pi_h \varphi \text{ in } \Gamma_0, \tag{2.3}$$
$$v_h = 0 \text{ in } \Gamma \backslash \Gamma_0\},$$

where Ω be decomposed into triangles and τ_h denotes the set of those elements, where $h > 0$ is the mesh size. With the family τ_h is regular and quasi-uniform, and the usual basis of affine functions φ_i $i = \{1, ..., m(h)\}$ defined by $\varphi_i(M_j) = \delta_{ij}$ where M_j is a vertex of the considered triangulation, with P_1 Lagrangian polynomial of degree less than or equal to 1 and π_h is an interpolation operator on Γ_0. We consider r_h be the usual interpolation operator defined by

$$r_h v = \sum_{i=1}^{m(h)} v(M_i)\, \varphi_i(x).$$

2.1 The Discrete Maximum Principle assumption (DMP)

We assume the matrices whose coefficients $\mathcal{A}(\varphi_i, \varphi_j)$ are M-matrix ([12] and [13]). For convenience in all the sequels, C will be a generic constant independent on h.It can be approximated the problem (1.1) by a following elliptic equation $v \in H^2(\Omega)$

$$\mathcal{A}(u,v) = (f,v)_\Omega + (\varphi, v)_{\Gamma_0}. \tag{2.4}$$

We discretize in space, i.e., we approach the space $\mathcal{H}$ by a space discretization of finite dimensional $V_h \subset \mathcal{H}$, we get the following of elliptic equation

$$\mathcal{A}(u_h, v_h) = (f, v_h)_\Omega + (\varphi, v_h)_{\Gamma_0}. \tag{2.5}$$

Theorem 1 ([7]). *Under suitable regularity of the solution of problem* (1.1), *there exists a constant C independent of h such that*

$$\|\zeta_h^\infty - \zeta\| \le Ch^2 |\log h|. \tag{2.6}$$

Lemma 1 ([18]). *Let $w \in \mathcal{H} \cap C(\bar{\Omega})$ satisfies $\mathcal{A}(w,\phi) + a(w,\phi) \ge 0$ for all nonnegative $\phi \in H^1(\Omega)$ and $w \ge 0$ on Γ, then $w \ge 0$ on $\bar{\Omega}$.*

Notation 1. $(f, \varphi); (\widetilde{f}, \widetilde{\varphi})$ be a pair of data and $\zeta = \partial(f,\varphi); \widetilde{\zeta} = \partial(\widetilde{f}, \widetilde{\varphi})$ the corresponding solutions to (2.4) .

Proposition 1. *Under the previous notation, we have*

$$\|\zeta_h - \zeta\|_{L^2(\Omega)} \le \max\{\left(\frac{1}{\beta}\right) \left\|f - \widetilde{f}\right\|_{L\infty(\Omega)}, \|\varphi - \widetilde{\varphi}\|_{L\infty(\Omega)}\}. \tag{2.7}$$

Proof. First, putting

$$\mu = \max\{\left(\frac{1}{\beta}\right)\left\|f-\widetilde{f}\right\|_{L\infty(\Omega)}, \|\varphi-\widetilde{\varphi}\|_{L\infty(\Gamma)}\}, \tag{2.8}$$

then

$$\begin{cases} \widetilde{f} \leq f + \left\|f-\widetilde{f}\right\|_{L\infty(\Omega)} \leq f + \left(\frac{a}{\beta}\right)\left\|f-\widetilde{f}\right\|_{L\infty(\Omega)} \\ \leq f + a\max\{\left(\frac{1}{\beta}\right)\left\|f-\widetilde{f}\right\|_{L\infty(\Omega)}, \|\varphi-\widetilde{\varphi}\|_{L\infty(\Gamma)}\} \leq f + a\mu. \end{cases}$$

So,

$$\mathcal{A}\left(\widetilde{\zeta},\phi\right) \leq \mathcal{A}\left(\zeta,\phi\right) + a\left(\mu,\phi\right), \text{ for all } \phi \geq 0,\, \phi \in \mathcal{H} \tag{2.9}$$

and thus

$$\mathcal{A}\left(\widetilde{\zeta},\phi\right) \leq \mathcal{A}\left(\zeta+\mu,\phi\right) = \left(f + a\mu,\phi\right).$$

On the other hand, we have

$$\zeta + \phi - \widetilde{\zeta} \geq 0 \text{ on } \Gamma_0. \tag{2.10}$$

So,

$$\mathcal{A}(\zeta + \phi - \widetilde{\zeta}) \geq 0. \tag{2.11}$$

By using the result of Lemma 1, we get

$$\widetilde{\zeta} + \phi - \zeta \geq 0 \text{ on } \overline{\Omega}. \tag{2.12}$$

Similarly, interchanging the roles of the couples (f,φ) and $(\widetilde{f},\widetilde{\varphi})$, we get

$$\widetilde{\zeta} + \phi - \zeta \geq 0 \text{ on } \overline{\Omega}, \tag{2.13}$$

which completes the proof. □

Remark 1. Proposition 1 stays true for the discrete case.

Lemma 2 ([18])**.** *Let $w \in V_h$ satisfy $\mathcal{A}(w,\phi_s) > 0$ for $s = 1,2...m(h)$and $w \geq 0$ on Γ_0.then $w \geq 0$ on $(\overline{\Omega})$.*

Notation 2. (f,φ); $(\widetilde{f},\widetilde{\varphi})$ be a pair of data and $\zeta_h = \partial(f,\varphi); \widetilde{\zeta}_h = \partial(\widetilde{f},\widetilde{\varphi})$ the corresponding solutions to (**??**) .

Proposition 2. *Let DMP hold, we have*

$$\left\|\zeta_h - \widetilde{\zeta}_h\right\|_{L\infty(\Omega)} \leq \max\left\{\left(\frac{1}{\beta}\right)\left\|f-\widetilde{f}\right\|_{L\infty(\Omega)}, \|\varphi-\widetilde{\varphi}\|_{L\infty(\Gamma_0)}\right\} \tag{2.14}$$

Proof. The proof is similar to that of the continuous case. □

3. Schwarz alternating methods for elliptic equation with bilaplace operator

We have the nonlinear elliptic problem with bilaplace operator: find $u \in H^2(\Omega)$ solution of

$$\begin{cases} \Delta^2 u + \alpha u = f, & \text{in } \Omega, \\ \Delta u + \rho \dfrac{\partial u}{\partial \eta} = 0, & \text{in } \Gamma, \\ \dfrac{\partial(\Delta u)}{\partial \eta} = \varphi \text{ in } \Gamma_0, & \text{in } \Omega, \end{cases} \tag{3.1}$$

the weak formulation to be:

$$\mathcal{A}(u,v) = (f,v)_\Omega + (\varphi, v)_{\Gamma_0}. \tag{3.2}$$

We decompose Ω in two overlapping smooth subdomain Ω_1 and Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$, we denote by $\partial\Omega_i$ the boundary of Ω_i and $\Gamma_i = \partial\Omega_i \cap \Omega_j$ and assume that the intersection of $\overline{\Gamma}_i$ and $\overline{\Gamma}_j$;$i \neq j$ is empty. Let

$$V_i^{(w_j)} = v \in \mathcal{H} \text{ such that } \; v = w_j \text{ on } \Gamma_i.$$

We associate with problem (3.2) the following system: find $(u_1, u_2) \in V_1 \times V_2$ solution to

$$\begin{cases} \mathcal{A}_1(u_1, v) + \alpha(u_1, v) = (f_1, v)_{\Omega 1} + (\varphi, v)_{\Gamma_1}, \\ \mathcal{A}_2(u_2, v) + \alpha(u_2, v) = (f_2, v)_{\Omega 2} + (\varphi, v)_{\Gamma_2}, \end{cases} \tag{3.3}$$

where

$$\mathcal{A}_i(u_i, v) = \int_{\Omega_i} \Delta u.\Delta v + \rho \partial_\eta u.\partial_\eta v dx \tag{3.4}$$

and

$$u_i = u/\Omega_i; i = 1, 2$$

3.1 The continuous Schwartz sequences

Let u_0 be an initialization in $C_0\left(\overline{\Omega}\right)$,i.e., continuous functions vanishing on $\partial\Omega$ such that

$$\mathcal{A}(u_0, v) + \alpha(u_0, v) = (f, v). \tag{3.5}$$

Starting from $u_0 = u_0/\Omega_2$, we respectively define the alternating Schwarz sequences (u_1^{n+1}) on Ω_1 such that $u_1^{n+1} \in V_1^{(u_2^n)}$ solves of

$$\mathcal{A}_1(u_1^{n+1}, v) + \alpha(u_1, v) = (f_1, v), \tag{3.6}$$

where

$$f_1 = f(u_1)$$

and (u_2^{n+1}) on Ω_2 such that $u_2^{n+1} \in V_2^{(u_1^{n+1})}$ solves

$$\mathcal{A}_2(u_2^{n+1}, v) + \alpha(u_2, v) = (f_2, v), \quad f_2 = f(u_2) \tag{3.7}$$

Theorem 2 ([7])**.** *The sequences* $(u_{1h}^{n+1}); (u_{2h}^{n+1})$, $n \geq 0$ *produced by the Schwarz alternating method converge geometrically to a solution* u *of the elliptic obstacle problem. More precisely, there exist* $\lambda_1, \lambda_2 \in (0,1)$ *which depend on* (Ω_1, Γ_2) *and* (Ω_2, Γ_1) *such that for all* $n \geq 0$,

$$\sup_{\overline{\Omega}_1} \left| u_{1h} - u_1^{2n+1} \right| \leq \lambda_1^n \lambda_2^n \sup_{\Gamma_1} \left| u_{1h} - u_{1h}^0 \right| \tag{3.8}$$

and

$$\sup_{\overline{\Omega}_2} \left| u_{2h} - u_2^{2n} \right| \leq \lambda_1^n \lambda_2^{n-1} \sup_{\Gamma_2} \left| u_{2h} - u_{2h}^0 \right|. \tag{3.9}$$

3.2 The discrete Schwartz sequences

As we have defined before, for $i = 1, 2$, let τ^{h_i} be a standard regular and quasiuniform finite element triangulation in $\Omega_i; h_i$, being the mesh size. The two meshes being mutually independent $\Omega_1 \cap \Omega_2$, a triangle belonging to one triangulation does not necessarily belong to the other and for every $w \in C(\Omega_i)$, we set

$$V_{hi}^{(w_j)} = \left\{ \; v \in \mathcal{H} \text{ such that } v = \phi \text{ on } \Gamma_{01} \cap \Gamma_{02}; \; v = \pi_{h_i}(w) \; \text{ on } \Gamma_{0i}, \; \right\}.$$

where π_{h_i}denote an interpolation operator on Γ_{0i}.

Now, we define the discrete counterparts of the continuous Schwarz sequences defined in (3.6) and (3.7) .

Indeed, let u_{0h} be the discrete analog of u_0, defined in (3.5), we respectively, define by $u_{1h}^{n+1} \in V_{h1}^{(u_{2h}^n)}$ such that

$$\mathcal{A}_1(u_{1h}^{n+1}, v) = (f(u_{1h}^{n+1}), v), \forall v \in V_h^{(\varphi)}; \; n \geq 0 \tag{3.10}$$

and $u_{2h}^{n+1} \in V_{h2}^{(u_{1h}^{n+1})}$ such that

$$\mathcal{A}_2(u_{2h}^{n+1}, v) = (f(u_{2h}^{n+1}), v), \forall v \in V_h^{(\varphi)}; \; n \geq 0. \tag{3.11}$$

3.3 Error analysis for the nonlinear elliptic equation

We begin by introducing two discrete auxiliary sequences and prove a fundamental lemma.

3.3.1 Two auxiliary Schwarz sequences

For $w_{2h}^0 = u_{2h}^0$, we define the sequences w_{1h}^{n+1} and w_{2h}^{n+1} such that $u_{1h}^{n+1} \in V_{h1}^{(u_2^n)}$ solves

$$\mathcal{A}_1(w_{1h}^{n+1}, v) = (f(u_{1h}^{n+1}), v), \forall v \in V_{h1}^{(\varphi)}; n \geq 0, \tag{3.12}$$

and $w_{2h}^{n+1} \in V_{2h}^{(u_{1h}^{n+1})}$ solves

$$\mathcal{A}_2(w_{2h}^{n+1}, v) = (f(u_{2h}^{n+1}), v), \forall v \in V_{h2}^{(\varphi)}; n \geq 0, \tag{3.13}$$

respectively. It is then clear that w_{1h}^{n+1} and w_{2h}^{n+1} are the finite element approximation of u_1^{n+1} and u_2^{n+1} defined in (3.12), (3.13), respectively. Then, as $f(.)$ is continuous, $\left\| f\left(u_i^{n+1}\right)\right\|_\infty \leq C \left\| u_i^{n+1}\right\|_\infty$, (independent i of n). Therefore, making use of standard maximum norm estimates for linear elliptic problems, we have

$$\|u_i^n - u_{ih}^n\|_{L^\infty(\Omega_i)} \leq Ch^2 |\log h| \tag{3.14}$$

where C is a constant independent of both h and n.

Notation 3. From now on, we shall adopt the following notations: $|.|_1 = |.|_{L^\infty(\Gamma_1)}$, $|.|_2 = |.|_{L^\infty(\Gamma_2)}$, $\|.\|_1 = \|.\|_{L^\infty(\Gamma_1)}$, $\|.\|_2 = \|.\|_{L^\infty(\Gamma_2)}$, and we set $\pi_{h_1} = \pi_{h_2} = \pi_h$.

3.4 Iterative discrete algorithm

We give our following discrete algorithm

$$u_{ih}^{n+1} = T_h u_{ih}^{n+1}, u_{ih}^{n+1} \in V_{h_i}^{(u_2^n)}, \tag{3.15}$$

where u_h is the solution of the problem (**??**) and the first iteration u_h^0 is solution of (3.5).

Lemma 3. *Let* $\rho = \dfrac{\alpha}{\beta}$. *Then, under assumption* (1.2), *there exists a constant* C *independent of both* h *and* n *such that*

$$\left\|u_i^{n+1} - u_{ih}^{n+1}\right\|_i \leq \frac{Ch^2 |\log h|}{1-\rho}, \quad i = 1, 2. \tag{3.16}$$

Proof. We know from standard error estimate on uniform norm for linear problem [17] that there exists a constant C independent of h such that

$$\left\|u^0 - u_h^0\right\|_{L=(\Omega)} \leq Ch^2 |\log h| . \tag{3.17}$$

Since $\dfrac{1}{2} < \rho < 1$, then $1 < \rho/(1-\rho)$ and

$$\left\|u_2^0 - u_{2h}^0\right\|_2 \leq Ch^2 |\log h| \leq \frac{\rho C h^2 |\log h|}{1-\rho}. \tag{3.18}$$

Let us now prove (3.16) by induction. Indeed for $n = 1$, using the result of Propsition1, we have in Ω_1

$$\begin{aligned}
\left\|u_1^1 - u_{1h}^1\right\|_1 &\leq \left\|u_1^1 - w_{1h}^1\right\|_1 + \left\|w_1^1 - u_{1h}^1\right\|_1 \\
&\leq Ch^2 \left|\log h\right| + \left\|w_1^1 - u_{1h}^1\right\|_1 \\
&\leq Ch^2 \left|\log h\right| + \max\{\left(\frac{1}{\beta}\right) \left\|f\left(u_1^1\right) - f\left(u_{1h}^1\right)\right\|_1, \left|u_2^0 - u_{2h}^0\right|_1 \\
&\leq Ch^2 \left|\log h\right| + \max\{\left(\frac{1}{\beta}\right) \left\|f\left(u_1^1\right) - f\left(u_{1h}^1\right)\right\|_1, \left\|u_2^0 - u_{2h}^0\right\|_2 \\
&\leq Ch^2 \left|\log h\right| + \max\{\rho \left\|u_1^1 - u_{1h}^1\right\|_1, \left\|u_2^0 - u_{2h}^0\right\|_2\}.
\end{aligned}$$

We distinguish between two cases

$$\max\{\rho \left\|u_1^1 - u_{1h}^1\right\|_1, \left\|u_2^0 - u_{2h}^0\right\|_2\} = \rho \left\|u_1^1 - u_{1h}^1\right\|_1 \tag{3.19}$$

or

$$\max\{\rho \left\|u_1^1 - u_{1h}^1\right\|_1, \left\|u_2^0 - u_{2h}^0\right\|_2\} = \left\|u_2^0 - u_{2h}^0\right\|_2. \tag{3.20}$$

(3.19) implies

$$\begin{cases} \left\|u_1^1 - u_{1h}1\right\|_1 \leq Ch^2 \left|\log h\right| + \rho \left\|u_1^1 - u_{1h}^1\right\|_1, \\ \left\|u_2^0 - u_{2h}^0\right\|_2 \leq \rho \left\|u_1^1 - u_{1h}^1\right\|_1, \end{cases}$$

then

$$\begin{cases} \left\|u_1^1 - u_{1h}^1\right\|_1 \leq \dfrac{Ch^2 \left|\log h\right|}{1-\rho}. \\ \left\|u_2^0 - u_{2h}^0\right\|_2 \leq \rho \left\|u_1^1 - u_{1h}^1\right\|_1 \leq \frac{\rho Ch^2 \left|\log h\right|}{1-\rho}. \end{cases}$$

(3.20) implies

$$\left\|u_1^1 - u_{1h}^1\right\|_1 \leq Ch^2 \left|\log h\right| + \left\|u_2^0 - u_{2h}^0\right\|_2 \leq \left\|u_2^0 - u_{2h}^0\right\|_2,$$

so, by multiplying (3.20) by ρ we get

$$\rho \left\|u_1^1 - u_{1h}^1\right\|_1 \leq \rho Ch^2 \left|\log h\right| + \rho \left\|u_2^0 - u_{2h}^0\right\|_2. \tag{3.21}$$

So, $\rho\|u_1^1 - u_{1h}^1\|_1$ is bounded by both $\rho Ch^2|\log h| + \rho\|u_2^0 - u_{2h}^0\|_2$ and $\|u_2^0 - u_{2h}^0\|_2$, this implies that

$$\rho \left\|u_2^0 - u_{2h}^0\right\|_2 \leq \rho Ch^2 \left|\log h\right| + \rho \left\|u_2^0 - u_{2h}^0\right\|_2, \tag{3.22}$$

or

$$\rho Ch^2 \left|\log h\right| + \rho \left\|u_2^0 - u_{2h}^0\right\|_2 \leq \left\|u_2^0 - u_{2h}^0\right\|_2, \tag{3.23}$$

that is (3.22) implies

$$\left\|u_2^0 - u_{2h}^0\right\|_2 \leq \frac{\rho C h^2 \left|\log h\right|}{1-\rho} \tag{3.24}$$

and (3.23) implies

$$\left\|u_2^0 - u_{2h}^0\right\|_2 \geq \frac{\rho C h^2 \left|\log h\right|}{1-\rho}. \tag{3.25}$$

It follows that only the case (3.22) is true, that is,

$$\left\|u_2^0 - u_{2h}^0\right\|_2 \leq \frac{\rho C h^2 \left|\log h\right|}{1-\rho}, \tag{3.26}$$

then

$$\begin{aligned}\rho \left\|u_1^1 - u_{1h}^1\right\|_1 &\leq C h^2 \left|\log h\right| + \left\|u_2^0 - u_{2h}^0\right\|_2 \\ &\leq C h^2 \left|\log h\right| + \frac{\rho C h^2 \left|\log h\right|}{1-\rho} \leq \frac{C h^2 \left|\log h\right|}{1-\rho}.\end{aligned}$$

So, in both cases (3.19) and (3.20), we have

$$\left\|u_1^1 - u_{1h}^1\right\|_1 \leq \frac{C h^2 \left|\log h\right|}{1-\rho}. \tag{3.27}$$

Similarly, we have in Ω_2

$$\begin{aligned}\left\|u_2^1 - u_{2h}^1\right\|_2 &\leq C h^2 \left|\log h\right| + \left\|w_2^1 - u_{2h}^1\right\|_2 \\ &\leq C h^2 \left|\log h\right| + \max\{\left(\frac{1}{\beta}\right) \left\|f\left(u_2^1\right) - f\left(u_{2h}^1\right)\right\|_2, \left|u_1^1 - u_{1h}^1\right|_2\} \\ &\leq C h^2 \left|\log h\right| + \max\{\left(\frac{1}{\beta}\right) \left\|f\left(u_2^1\right) - f\left(u_{2h}^1\right)\right\|_2, \left\|u_1^1 - u_{1h}^1\right\|_1\} \\ &\leq C h^2 \left|\log h\right| + \max\{\rho \left\|u_2^1 - u_{2h}^1\right\|_2, \left\|u_1^1 - u_{1h}^1\right\|_1\}.\end{aligned}$$

So,

$$\max\{\rho \left\|u_2^1 - u_{2h}^1\right\|_2, \left\|u_1^1 - u_{1h}^1\right\|_1\} = \rho \left\|u_2^1 - u_{2h}^1\right\|_2 \tag{3.28}$$

or

$$\max\{\rho \left\|u_2^1 - u_{2h}^1\right\|_2, \left\|u_1^1 - u_{1h}^1\right\|_1\} = \left\|u_1^1 - u_{1h}^1\right\|_1. \tag{3.29}$$

cases (3.28) implies

$$\begin{aligned}&\left\|u_2^1 - u_{2h}^1\right\|_2 \leq C h^2 \left|\log h\right| + \rho \left\|u_2^1 - u_{2h}^1\right\|_2, \\ &\left\|u_1^1 - u_{1h}^1\right\|_1 \leq \rho \left\|u_2^1 - u_{2h}^1\right\|_2\end{aligned}$$

so,

$$\begin{cases} \left\|u_2^1 - u_{2h}^1\right\|_2 \leq \dfrac{Ch^2 \left|\log h\right|}{1-\rho}, \left\|u_1^1 - u_{1h}^1\right\|_1 \leq \rho \left\|u_2^1 - u_{2h}^1\right\|_2 \\ \qquad\qquad \leq \dfrac{\rho Ch^2 \left|\log h\right|}{1-\rho} \leq \dfrac{Ch^2 \left|\log h\right|}{1-\rho}, \end{cases}$$

while case (3.29) implies

$$\begin{cases} \left\|u_2^1 - u_{2h}^1\right\|_2 \leq Ch^2 \left|\log h\right| + \left\|u_1^1 - u_{1h}^1\right\|_1 \\ \rho \left\|u_2^1 - u_{2h}^1\right\|_2 \leq \left\|u_1^1 - u_{1h}^1\right\|_1 . \end{cases} , \tag{3.30}$$

So, by multiplying (3.30) by ρ we get

$$\rho \left\|u_2^1 - u_{2h}^1\right\|_2 \leq \rho Ch^2 \left|\log h\right| + \rho \left\|u_1^1 - u_{1h}^1\right\|_1 . \tag{3.31}$$

Hence, $\rho\|u_2^1 - u_{2h}^1\|_2$ is bounded by both $\rho Ch^2|\log h| + \rho\|u_1^1 - u_{1h}^1\|_1$ and $\|u_1^1 - u_{1h}^1\|_1$, then

$$\left\|u_1^1 - u_{1h}^1\right\|_1 \leq \rho Ch^2 \left|\log h\right| + \rho \left\|u_1^1 - u_{1h}^1\right\|_1 \tag{3.32}$$

or

$$Ch^2 \left|\log h\right| + \rho \left\|u_1^1 - u_{1h}^1\right\|_1 \leq \left\|u_1^1 - u_{1h}^1\right\|_1 , \tag{3.33}$$

which (3.32) implies

$$\left\|u_1^1 - u_{1h}^1\right\|_1 \leq \frac{\rho Ch^2 \left|\log h\right|}{1-\rho} < \frac{Ch^2 \left|\log h\right|}{1-\rho} \tag{3.34}$$

or (3.33) implies

$$\frac{\rho Ch^2 \left|\log h\right|}{1-\rho} \leq \left\|u_1^1 - u_{1h}^1\right\|_1 < \frac{Ch^2 \left|\log h\right|}{1-\rho}. \tag{3.35}$$

Hence, (3.32) and (3.33) are true because they both coincide with (3.27). So, there is either a contradiction and thus case (3.28) is impossible or case (3.29) is possible only if

$$\left\|u_1^1 - u_{1h}^1\right\|_1 = \rho Ch^2 \left|\log h\right| + \rho \left\|u_1^1 - u_{1h}^1\right\|_1 , \tag{3.36}$$

that is

$$\left\|u_1^1 - u_{1h}^1\right\|_1 = \frac{\rho Ch^2 \left|\log h\right|}{1-\rho}, \tag{3.37}$$

thus

$$\left\|u_2^1 - u_{2h}^1\right\|_2 \leq Ch^2 \left|\log h\right| + \left\|u_1^1 - u_{1h}^1\right\|_1 \leq Ch^2 \left|\log h\right| + \frac{\rho Ch^2 \left|\log h\right|}{1-\rho} \leq \frac{Ch^2 \left|\log h\right|}{1-\rho},$$

that is, both cases (3.28) and (3.29) imply

$$\left\|u_2^1 - u_{2h}^1\right\|_2 \leq \frac{Ch^2 \left|\log h\right|}{1-\rho}. \tag{3.38}$$

Now, let us assume that

$$\|u_2^n - u_{2h}^n\|_2 \leq \frac{Ch^2 \left|\log h\right|}{1-\rho} \tag{3.39}$$

and prove that

$$\begin{cases} \left\|u_1^{n+1} - u_{1h}^{n+1}\right\|_1 \leq \dfrac{Ch^2 \left|\log h\right|}{1-\rho} \\ \left\|u_2^{n+1} - u_{2h}^{n+1}\right\|_2 \leq \dfrac{Ch^2 \left|\log h\right|}{1-\rho} \end{cases}$$

□

Theorem 3. *Let $h = \max(h_1, h_2)$. Then, for n large enough, there exists a constant C independent of both h and n such that*

$$\left\|u_i^{n+1} - u_{ih}^{n+1}\right\|_1 \leq \frac{Ch^2 \left|\log h\right|}{1-\rho}, \quad \forall i = 1, 2. \tag{3.40}$$

Proof. Let us give the proof for $i = 1$. The one for $i = 2$ is similar and so will be omitted. Indeed, Let $\lambda = \lambda_1\lambda_2$, then making use of Theorem 2 and Lemma 3, we get

$$\begin{aligned} \left\|u_1 - u_{1h}^{n+1}\right\|_1 &\leq \left\|u_1 - u_1^{n+1}\right\|_1 + \left\|u_1^{n+1} - u_{1h}^{n+1}\right\|_1 \\ &\leq \lambda_1^n \lambda_2^n \left|u^0 - u\right|_1 + \frac{Ch^2 \left|\log h\right|}{1-\rho} \\ &\leq \lambda^{2n} \left|u^0 - u\right|_1 + \frac{Ch^2 \left|\log h\right|}{1-\rho}. \end{aligned}$$

So, for n large enough, we have

$$\delta^{2n} \leq h^2 \tag{3.41}$$

and thus

$$\left\|u_1 - u_{1h}^{n+1}\right\|_1 \leq Ch^2 + Ch^2 \left|\log h\right| \leq Ch^2 \left|\log h\right|,$$

which is the desired result. □

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