The interval valued fuzzy graph associated with a Crisp graph

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Abstract. We define the interval valued fuzzy graph (IVFG) associated with a crisp graph based on the degrees of the nodes of the crisp graph and study its various properties. The nature of arcs of the IVFG associated with a crisp graph can be determined if the adjacency matrix of the crisp graph is given. We show that the IVFG associated with a regular graph is regular, totally regular, edge regular and totally edge regular, but the IVFG associated with a complete graph is not a complete IVFG. We prove that the IVFG associated with $C_n, n \geq 3$ is an interval valued fuzzy cycle and the IVFG associated with the wheel graph $W_n, n \geq 5$ is an interval valued fuzzy tree.
Keywords: interval valued fuzzy graph, Crisp graph, Regular graph

1. Introduction

Graphs are used to symbolize relations between objects. But uncertainty and vagueness happen to be significant elements in relations. To address this, Rosenfeld [14] developed the theory of fuzzy graphs giving membership degrees to nodes and arcs of crisp graphs. But there may be incidences in which one do not have an exact information about the choice of the membership degrees. In these situations, it is better to represent the membership degree by means of an interval. Replacing the membership degrees of nodes and arcs in fuzzy graphs by interval valued fuzzy sets such that they satisfy some particular condition,

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interval valued fuzzy graphs were defined. Thus IVFG provide a better explanation of vagueness and uncertainty within the particular interval than fuzzy graphs. Hongmei and Lianhua [4] introduced IVFGs generalizing fuzzy graphs.

We can associate any number of IVFGs with a particular crisp graph. This paper is an ernest effort to construct a particular type of IVFG from a crisp graph based on the degrees of the nodes of the crisp graph.Throughout our work, we consider only simple, connected, undirected graphs.

Definition 1.1 ([2]). *Let $G^\star = (V, E)$ be a crisp graph. Then an interval valued fuzzy graph (IVFG) G on $G^\star$ is defined as a pair $G = (A, B)$, where $A = [\mu_A^-(x), \mu_A^+(x)]$ is an interval valued fuzzy set on V and $B = [\mu_B^-(xy), \mu_B^+(xy)]$ is an interval valued fuzzy set on E such that $\mu_B^-(xy) \leq \min(\mu_A^-(x), \mu_A^-(y))$ and $\mu_B^+(xy) \leq \min(\mu_A^+(x), \mu_A^+(y))$ for all $xy \in E$.*

If $\mu_B^-(xy) = min(\mu_A^-(x), \mu_A^-(y))$ and $\mu_{B^+}(xy) = \min(\mu_{A^+}(x), \mu_{A^+}(y))$ for all $x, y \in V$, then G is called a complete IVFG (CIVFG).

Definition 1.2 ([7]). *An IVFG G is said to be connected if any two nodes are joined by a path.*

Definition 1.3 ([5]). *An IVFG G is said to be regular (totally regular) if every node of G has the same degree (total degree).*

Definition 1.4 ([6]). *An IVFG G is said to be edge regular (totally edge regular) if every arc of G has the same edge degree (total edge degree).*

Definition 1.5 ([13]). *An IVFG G is said to be a complete product interval valued fuzzy graph (CPIVFG) if $\mu_B^-(xy) = \mu_A^-(x) \times \mu_A^-(y)$ and $\mu_{B^+}(xy) = \mu_{A^+}(x) \times \mu_{A^+}(y)$ for all $x, y \in V$ where $\times$ denotes ordinary multiplication.*

Definition 1.6 ([12]). *Two arcs e_1 and e_2 are said to be comparable if their membership degrees are such that either $\mu_{B^-}(e_1) > \mu_{B^-}(e_2)$and $\mu_{B^+}(e_1) > \mu_{B^+}(e_2)$ or $\mu_{B^-}(e_1) < \mu_{B^-}(e_2)$ and $\mu_{B^+}(e_1) < \mu_{B^+}(e_2)$. They are said to be equal if their membership degrees are equal.*

Definition 1.7 ([11]). *A connected IVFG G is said to be strongly connected if there exists a strong path between every two nodes of G.*

Definition 1.8 ([12]). *An arc (u, v) in an IVFG G is said to be α strong if $\mu_{B^-}(u, v) > NCONN_{G-(u,v)}(u, v)$ and $\mu_{B^+}(u, v) > PCONN_{G-(u,v)}(u, v)$. It is said to be β strong if $\mu_{B^-}(u, v) = NCONN_{G-(u,v)}(u, v)$ and $\mu_{B^+}(u, v) = PCONN_{G-(u,v)}(u, v)$. And it is called a weak arc if*

$$\mu_{B^-}(u, v) < NCONN_{G-(u,v)}(u, v)$$

and $\mu_{B^+}(u, v) < PCONN_{G-(u,v)}(u, v)$

Some other concepts that are used in this work are provided in the Table 1. We use the following theorems in the subsequent discussions.

Notation	Concept
$O(G)$	Order of G [5]
S(G)	Size of G [5]
$d(u)$	Degree of a node u[5]
$td(u)$	Total degree of a node u[5]
$d(uv)$	Degree of an arc uv[5]
$td(uv)$	Total degree of an arc uv[5]
P	Path in an IVFG [7]
	Strong path in an IVFG [12]
$S_{\mu^-}(P)$, $S_{\mu^+}(P)$	Strength of the path P [7]
$(\mu_{B^-})^\infty$, $NCONN_G$	Maximum of the μ^- strength [7]
$(\mu_{B^+})^\infty$, $PCONN_G$	Maximum of the μ^+ strength [7]
	Interval valued fuzzy bridge[7]
	Interval valued fuzzy cutnode[7]
	Interval valued fuzzy tree[9]
	Interval valued fuzzy cycle[9]
	Isomorphism and co-weak isomorphism of IVFGs [2]
	Interval valued fuzzy line graph [1]
	Interval valued fuzzy node arc matrix [10]

Table 1: Major concepts related to interval valued fuzzy graphs

Theorem 1.1 ([11])**.** [A] *Let G be a connected IVFG such that its edges are either comparable or equal. Then G is strongly connected.*

Theorem 1.2 ([12])**.** [B] *Let G be an IVFG whose underlying crisp graph is not a tree. If every two arcs of G are equal,then all the arcs of G will be β strong.*

Theorem 1.3 ([12])**.** [C] *An arc (u, v) in an IVFG $G = (A, B)$ is a weak arc if and only if (u, v) is the unique weakest arc of atleast one cycle in G.*

Theorem 1.4 ([9])**.** [D] *Let $G = (A, B)$ be a connected IVFG on $G^* = (V, E)$, where G^* is not a tree. Then G is an IVFT if and only if G contains only α strong and weak arcs.*

Theorem 1.5 ([9])**.** [E] *Let $G = (A, B)$ be an IVFG on a cycle $G^* = (V, E)$.Then G is an IVFC if and only if G has atleast two β strong arcs.*

Theorem 1.6 ([8])**.** [F] *Let G be a non complete IVFG on $G^* = (V, E)$ with $|V| = n$ and $|E| = m$ Suppose that every two arcs are either comparable or equal and the s weaker arcs where $s = m - n + 1$ lies in s different cycles. Then those s arcs will be weak arcs and all the other $m - s$ arcs are α strong.*

Theorem 1.7 ([8])**.** [G] *Let $G = (A, B)$ be an Interval Valued Fuzzy Graph. Then an arc (u, v) of G is an IVF bridge if and only if it is α strong.*

Theorem 1.8 ([8])**.** [H] *If w is common node of atleast two α strong arcs, then w is an IVF cutnode.*

2. Construction of an interval valued fuzzy graph

Let $G^* = (V, E)$ be a simple connected graph with n nodes and m arcs. Let the nodes be labelled as $v_1, v_2, \ldots, v_n$. We can construct an IVFG from this crisp graph. The construction is as follows: Let v_i be any arbitrary node of G^*. Let $\mu_A^-(v_i) = \frac{d(v_i)-1}{n-1}, \mu_A^+(v_i) = \frac{d(v_i)}{n-1}$.

Now, let $\mu_B^-(v_iv_j) = \frac{[d(v_i)-1][d(v_j)-1]}{(n-1)^2}, \mu_B^+(v_iv_j) = \frac{[d(v_i)][d(v_j)]}{(n-1)^2}$ for all $(v_i, v_j) \in E$. Hence the membership degrees of the arcs are given according to the rule: The lower membership degree of an arc is the product of the lower membership degrees of the corresponding end nodes and the upper membership degree of an arc is the product of the upper membership degrees of the corresponding end nodes. The above construction yields an IVFG as proved below and it is called the IVFG associated with the crisp graph G^*. We use **A** and **B** as in **Definition** 1.1 in the following theorem.

Theorem 2.1. *Let $G^* = (V, E)$ be a simple connected graph with $|V| = n$, $|E| = m$ and let $G = (A, B)$. Then, G is an Interval Valued Fuzzy Graph.*

Proof. Let $G^* = (V, E)$ be a simple connected graph with $|V| = n$, $|E| = m$.

Consider $A = \{(v_i, [\mu_A^-(v_i), \mu_A^+(v_i)]) : v_i \in V\}$ and $B = \{(v_iv_j, [\mu_B^-(v_iv_j), \mu_B^+(v_iv_j)]) : v_iv_j \in E\}$. We prove that A is an IVFS on V and B is an IVFS on E satisfying the requirements of an Theorem.

<u>A is an IVFS on V</u>

From the above construction, we have, $\mu_A^-(v_i) = \frac{d(v_i)-1}{n-1}, \mu_A^+(v_i) = \frac{d(v_i)}{n-1}$.

Since G^* is connected, $d(v_i) \geq 1$.

$\therefore d(v_i) - 1 \geq 0$ and so clearly $\mu_A^-(v_i) = \frac{d(v_i)-1}{n-1} \geq 0$.

Clearly, $\mu_A^-(v_i) < \mu_A^+(v_i)$ since $d(v_i) - 1 < d(v_i)$.

Again, since $d(v_i) \leq n - 1$, $\mu_A^+(v_i) = \frac{d(v_i)}{n-1} \leq 1$

Thus, we have $0 \leq \mu_A^-(v_i) < \mu_A^+(v_i) \leq 1$. Hence A is an IVFS on V.

<u>B is an IVFS on E</u>

We have, $\mu_B^-(v_iv_j) = \frac{[d(v_i)-1][d(v_j)-1]}{(n-1)^2}, \mu_B^+(v_iv_j) = \frac{[d(v_i)][d(v_j)]}{(n-1)^2}$.

Since $d(v_i) - 1 \geq 0$ and $d(v_j) - 1 \geq 0$, $\mu_B^-(v_iv_j) = \frac{[d(v_i)-1][d(v_j)-1]}{(n-1)^2} \geq 0$.

And since, $d(v_i) - 1 < d(v_i)$, $\mu_B^-(v_iv_j) < \mu_B^+(v_iv_j)$.

Also, $d(v_i) \leq n-1$ and $d(v_j) \leq n-1$, implies that $d(v_i)d(v_j) \leq (n-1)^2$ and so $\mu_B^+(v_iv_j) \leq 1$. Thus, we have $0 \leq \mu_B^-(v_iv_j) < \mu_B^+(v_iv_j) \leq 1$. Hence B is an IVFS on E.

Now, we show that A and B satisfy the requirements of an Interval Valued Fuzzy Graph. We know that if x and y are any two real numbers in $[0, 1]$, then, $x.y \leq \min(x, y)$.

$\therefore \mu_B^-(v_iv_j) = \frac{[d(v_i)-1][d(v_j)-1]}{(n-1)^2} \leq \min(\frac{[d(v_i)-1]}{n-1}, \frac{[d(v_j)-1]}{n-1}) = \min(\mu_A^-(v_i), \mu_A^-(v_j))$ and $\mu_B^+(v_iv_j) = \frac{d(v_i)d(v_j)}{(n-1)^2} \leq \min(\frac{d(v_i)}{n-1}, \frac{d(v_j)}{n-1}) = \min(\mu_A^+(v_i), \mu_A^+(v_j))$. Thus, $G = (A, B)$ constructed as above is an IVFG. □

Example 2.1. *Figure 1 shows a crisp graph G^* and its associated Interval Valued Fuzzy Graph G. Using G, we can verify the above procedure.*

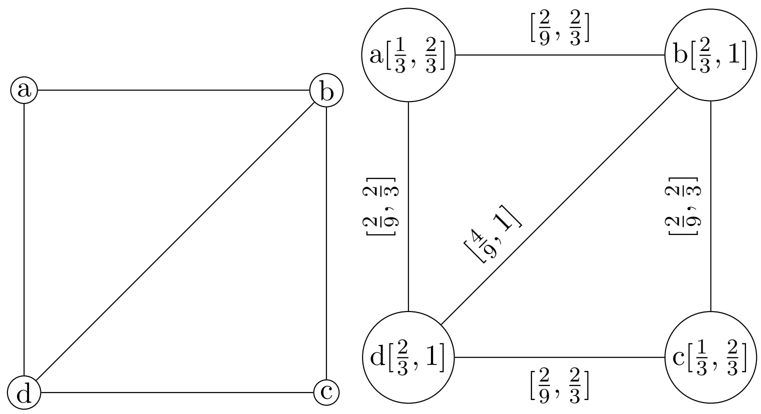

Figure 1: Example to illustrate the construction of the associated IVFG of a crisp graph

3. Main results

Theorem 3.1. *Let $G^* = (V, E)$ be a simple connected graph with $|V| = n$, $|E| = m$ and let $G = (A, B)$ be the Interval Valued Fuzzy Graph associated with G^*. Then $O(G) = [\frac{2m-n}{n-1}, \frac{2m}{n-1}]$*

Proof. Given that $G = (A, B)$ is the IVFG associated with the simple connected graph $G^* = (V, E)$ where $|V| = n$ and $|E| = m$.

$$\begin{aligned} O(G) = [O^-(G), O^+(G)] &= \left[\sum_{i=1}^{n} \mu_{A^-}(v_i), \sum_{i=1}^{n} \mu_{A^+}(v_i)\right] \\ &= \left[\sum_{i=1}^{n} \frac{d(v_i) - 1}{n-1}, \sum_{i=1}^{n} \frac{d(v_i)}{n-1}\right] \\ &= \left[\frac{\sum_{i=1}^{n} d(v_i) - \sum_{i=1}^{n}(1)}{n-1}, \frac{\sum_{i=1}^{n} d(v_i)}{n-1}\right] \\ &= \left[\frac{2m-n}{n-1}, \frac{2m}{n-1}\right] \text{ since } \sum_{i=1}^{n} d(v_i) = 2m \end{aligned}$$

□

Theorem 3.2. *Let $G = (A, B)$ be the Interval Valued Fuzzy Graph associated with the simple connected graph $G^* = (V, E)$. Then $S(G) = [S^-(G), S^+(G)]$ where*

$$S^-(G) = \frac{\sum\limits_{1\le i<j\le n, v_iv_j\in E} d(v_i)d(v_j) - \sum\limits_{1\le i<j\le n, v_iv_j\in E} [d(v_i) + d(v_j)] + m}{(n-1)^2},$$

$$S^+(G) = \frac{\sum\limits_{1\le i<j\le n, v_iv_j\in E} d(v_i)d(v_j)}{(n-1)^2}.$$

Proof.

$$\begin{aligned} S^-(G) &= \sum_{v_iv_j\in E, v_i\ne v_j} \mu_{B^-}(v_iv_j) \\ &= \sum_{v_iv_j\in E, v_i\ne v_j} \left[\frac{d(v_i)-1}{n-1}\right]\left[\frac{d(v_j)-1}{n-1}\right] \\ &= \frac{\sum\limits_{1\le i<j\le n, v_iv_j\in E} d(v_i)d(v_j) - \sum\limits_{1\le i<j\le n, v_iv_j\in E} [d(v_i) + d(v_j)] + \sum\limits_{1\le i<j\le n, v_iv_j\in E} (1)}{(n-1)^2} \\ &= \frac{\sum\limits_{1\le i<j\le n, v_iv_j\in E} d(v_i)d(v_j) - \sum\limits_{1\le i<j\le n, v_iv_j\in E} [d(v_i) + d(v_j)] + m}{(n-1)^2}. \end{aligned}$$

Similarly, we can prove that $S^+(G) = \frac{\sum_{1\le i<j\le n, v_iv_j\in E} d(v_i)d(v_j)}{(n-1)^2}$. □

Theorem 3.3. *The Interval Valued Fuzzy Graph associated with a crisp graph is strongly connected.*

Proof. Let $G^* = (V, E)$ be a simple connected graph and let $G = (A, B)$ be the IVFG associated with G^*. It is obvious from the construction of G that any two arcs of G are either comparable or equal. Then, by Theorem (A), G is strongly connected. □

Theorem 3.4. *The nature of the arcs of the Interval Valued Fuzzy Graph G associated with G^* can be determined if the adjacency matrix of G^* is given.*

Proof. Let G^* be a crisp graph with vertices $v_1, v_2, \ldots, v_n$ and suppose $[A]$ be its adjacency matrix . The sum of the i^{th} row or column of matrix $[A]$ gives the degree of v_i. Once we know $d(v_i)$, we can write the interval valued fuzzy node arc matrix of G. Then by an algorithm in [10], we can determine the nature of arcs of G. □

Theorem 3.5. *The Interval Valued Fuzzy Graph G associated with a regular graph G^* of degree k is a regular IVFG with degree $[\frac{k(k-1)^2}{(n-1)^2}, \frac{k^3}{(n-1)^2}]$.*

Proof.

$$
\begin{aligned}
d_G(v_i) &= \left[d_G^-(v_i), d_G^+(v_i)\right] \\
&= \left[\sum_{v_iv_j \in E} \mu_{B^-}(v_iv_j), \sum_{v_iv_j \in E} \mu_{B^+}(v_iv_j)\right] \\
&= \left[\sum_{v_iv_j \in E} \left[\frac{d(v_i)-1}{n-1}\right]\left[\frac{d(v_j)-1}{n-1}\right], \sum_{v_iv_j \in E} \left[\frac{d(v_i)}{n-1}\right]\left[\frac{d(v_j)}{n-1}\right]\right] \\
&= \left[\sum_{v_iv_j \in E} \left[\frac{k-1}{n-1}\right]\left[\frac{k-1}{n-1}\right], \sum_{v_iv_j \in E} \left[\frac{k}{n-1}\right]\left[\frac{k}{n-1}\right]\right] \\
&= \left[\frac{d(v_i)(k-1)^2}{(n-1)^2}, \frac{d(v_i)k^2}{(n-1)^2}\right] = \left[\frac{k(k-1)^2}{(n-1)^2}, \frac{k^3}{(n-1)^2}\right].
\end{aligned}
$$

□

Theorem 3.6. *The Interval Valued Fuzzy Graph G associated with a regular graph G^* of degree k is a totally regular Interval Valued Fuzzy Graph with total degree* $[\frac{(k-1)[k^2-k+(n-1)]}{(n-1)^2}, \frac{k[k^2+(n-1)]}{(n-1)^2}]$

Proof.

$$
\begin{aligned}
&td_G(v_i) = [td_G^-(v_i), td_G^+(v_i)] \\
&= \left[\sum_{v_iv_j \in E} \mu_{B^-}(v_iv_j) + \mu_{A^-}(v_i), \sum_{v_iv_j \in E} \mu_{B^+}(v_iv_j) + \mu_{A^+}(v_i)\right] \\
&= \left[\sum_{v_iv_j \in E} \left[\frac{d(v_i)-1}{n-1}\right]\left[\frac{d(v_j)-1}{n-1}\right] + \frac{d(v_i)-1}{n-1}, \sum_{v_iv_j \in E} \left[\frac{d(v_i)}{n-1}\right]\left[\frac{d(v_j)}{n-1}\right] + \left[\frac{d(v_i)}{n-1}\right]\right] \\
&= \left[\sum_{v_iv_j \in E} [\frac{k-1}{n-1}][\frac{k-1}{n-1}\right] + \frac{k-1}{n-1}, \sum_{v_iv_j \in E} \left[\frac{k}{n-1}\right]\left[\frac{k}{n-1}\right] + \frac{k}{n-1}\right] \\
&= \left[\frac{d(v_i)(k-1)^2}{(n-1)^2} + \frac{k-1}{n-1}, \frac{d(v_i)k^2}{(n-1)^2} + \frac{k}{n-1}\right] \\
&= \left[\frac{k(k-1)^2}{(n-1)^2} + \frac{k-1}{n-1}, \frac{k^3}{(n-1)^2} + \frac{k}{n-1}\right] \\
&= \left[\frac{(k-1)[k(k-1)+n-1]}{(n-1)^2}, \frac{k[k^2+(n-1)]}{(n-1)^2}\right] \\
&= \left[\frac{(k-1)[k^2-k+(n-1)]}{(n-1)^2}, \frac{k[k^2+(n-1)]}{(n-1)^2}\right].
\end{aligned}
$$

□

Theorem 3.7. *The Interval Valued Fuzzy Graph G associated with a regular graph G^* of degree k is an edge regular Interval Valued Fuzzy Graph with edge degree* $[\frac{2(k-1)^3}{(n-1)^2}, \frac{2k^2(k-1)}{(n-1)^2}]$

Proof.

$$\begin{aligned}
&d_G(v_iv_j) = [d^-(v_iv_j), d^+(v_iv_j)] \\
&= [d_G^-(v_i) + d_G^-(v_j) - 2\mu_{B^-}(v_iv_j), d_G^+(v_i) + d_G^+(v_j) - 2\mu_{B^+}(v_iv_j)] \\
&= \left[\frac{k(k-1)^2}{(n-1)^2} + \frac{k(k-1)^2}{(n-1)^2} - \frac{2(k-1)^2}{(n-1)^2}, \frac{k^3}{(n-1)^2} + \frac{k^3}{(n-1)^2} - \frac{2k^2}{(n-1)^2}\right] \\
&= \left[\frac{2(k-1)^2(k-1)}{(n-1)^2}, \frac{2k^3}{(n-1)^2} - \frac{2k^2}{(n-1)^2}\right] \\
&= \left[\frac{2(k-1)^3}{(n-1)^2}, \frac{2k^2(k-1)}{(n-1)^2}\right].
\end{aligned}$$

□

Theorem 3.8. *The Interval Valued Fuzzy Graph G associated with a regular graph G^* of degree k is a totally edge regular Interval Valued Fuzzy Graph with total edge degree* $[\frac{(k-1)^2(2k-1)}{(n-1)^2}, \frac{k^2(2k-1)}{(n-1)^2}]$

Proof.

$$\begin{aligned}
&td_G(v_iv_j) = [td^-(v_iv_j), td^+(v_iv_j)] \\
&= [d_G^-(v_i) + d_G^-(v_j) - \mu_{B^-}(v_iv_j), d_G^+(v_i) + d_G^+(v_j) - \mu_{B^+}(v_iv_j)] \\
&= \left[\frac{k(k-1)^2}{(n-1)^2} + \frac{k(k-1)^2}{(n-1)^2} - \frac{(k-1)^2}{(n-1)^2}, \frac{k^3}{(n-1)^2} + \frac{k^3}{(n-1)^2} - \frac{k^2}{(n-1)^2}\right] \\
&= \left[\frac{2k(k-1)^2}{(n-1)^2} - \frac{(k-1)^2}{(n-1)^2}, \frac{2k^3}{(n-1)^2} - \frac{k^2}{(n-1)^2}\right] \\
&= \left[\frac{(k-1)^2(2k-1)}{(n-1)^2}, \frac{k^2(2k-1)}{(n-1)^2}\right].
\end{aligned}$$

□

Theorem 3.9. *If G^* is a k regular graph, then all the arcs of the associated Interval Valued Fuzzy Graph G are β strong arcs.*

Proof. Let G^* be a k regular graph and G be the associated IVFG. Then from the definition of G, $[\mu_B^-, \mu_B^+](v_iv_j) = [\frac{(k-1)^2}{(n-1)^2}, \frac{k^2}{(n-1)^2}]$ which is a constant for all arcs (v_iv_j). Hence by Theorem (B), all the arcs of G are β strong arcs. □

The following corollary is immediate if we use Theorem (D).

Corollary 3.1. *The Interval Valued Fuzzy Graph associated with a regular graph cannot be an IVF tree.*

Theorem 3.10. *The Interval Valued Fuzzy Graph associated with the complete graph $K_n, n \geq 2$ is not a CIVFG.*

Proof. Consider the complete graph $K_n, n{\geq}2$.

Label the nodes as $v_1, v_2, v_3, \ldots, v_n$. Let $G = (A, B)$ be the IVFG associated with K_n.

Clearly, $\mu_A^-(v_i) = \frac{n-2}{n-1}, \mu_A^+(v_i) = \frac{n-1}{n-1} = 1$. Hence by the construction of G, $\mu_B^-(v_iv_j) = \frac{(n-2)^2}{(n-1)^2}, \mu_B^+(v_1v_i) = 1$. Clearly,$\mu_B^-(v_iv_j) \neq \min(\mu_A^-(v_i), \mu_A^-(v_j))$. Hence G cannot be a CIVFG. □

Theorem 3.11. *The Interval Valued Fuzzy Graph associated with the complete graph $K_n, n \geq 2$ is a complete PIVFG.*

Proof. Consider the complete graph $K_n = (V, E), n \geq 2$. Label the nodes as $v_1, v_2, v_3, \ldots, v_n$. Let $G = (A, B)$ be the IVFG associated with K_n. From the construction of G, it is clear that $\mu_B^-(v_iv_j) = \mu_A^-(v_i) \times \mu_A^-(v_j)$ and $\mu_B^+(v_iv_j) = \mu_A^+(v_i) \times \mu_A^+(v_j)$ for all $v_i, v_j \in V$ where $\times$ denotes ordinary multiplication. Hence by definition 1.5, G is a complete PIVFG. □

Theorem 3.12. *The Interval Valued Fuzzy Graph associated with $C_n, n \geq 3$ is an IVF cycle.*

Proof. $C_n, n \geq 3$ is n-1 regular. Hence by Theorem 3.9, all arcs of C_n are β strong. Since $n \geq 3$, C_n contains at least two β strong arcs and hence by Theorem (E), $C_n, n \geq 3$ is an IVF cycle. □

In crisp graph theory, W_n does not contain any cutnode. But the IVFG associated with $W_n, n \geq 5$ contains an IVF cutnode which can be concluded from the following theorem.

Theorem 3.13. *The Interval Valued Fuzzy Graph associated with the wheel graph $W_n, n \geq 5$ is an IVFT.*

Proof. Consider the wheel graph $W_n, n \geq 5$.Label the central node as v_1 and the remaining nodes as $v_2, v_3, \ldots, v_n$. Let $G = (A, B)$ be the IVFG associated with W_n.

Clearly, $\mu_A^-(v_1) = \frac{n-2}{n-1}, \mu_A^+(v_1) = \frac{n-1}{n-1} = 1$ and $\mu_A^-(v_i) = \frac{2}{n-1}, \mu_A^+(v_i) = \frac{3}{n-1}$ where $i = 2, 3, \ldots, n$. Also, $\mu_B^-(v_1v_i) = \frac{2(n-2)}{(n-1)^2}, \mu_B^+(v_1v_i) = \frac{3}{n-1}$ where $i = 2, 3, \ldots, n$ and $\mu_B^-(v_nv_2) = \mu_B^-(v_iv_{i+1}) = \frac{4}{(n-1)^2}, \mu_B^+(v_nv_2) = \mu_B^+(v_iv_{i+1}) = \frac{9}{(n-1)^2}$ where $i = 2, 3, \ldots, n-1$.

Since $n \geq 5$, $\mu_B^-(v_iv_{i+1}) < \mu_B^-(v_1v_i) = \mu_B^-(v_1v_{i+1})$ and $\mu_B^+(v_iv_{i+1}) < \mu_B^+(v_1v_i) = \mu_B^+(v_1v_{i+1})$, where $i = 2, 3, \ldots, n-1$. Thus obviously, arc (v_i, v_{i+1}) is the unique weakest arc of the cycle v_i, v_{i+1}, v_1, v_i. Hence by Theorem (C), arc (v_i, v_{i+1}) is a weak arc where $i = 2, 3, \ldots, n-1$. Also $\mu_B^-(v_n v_2) < \mu_B^-(v_1 v_2) = \mu_B^-(v_1 v_n)$ and $\mu_B^+(v_n v_2) < \mu_B^+(v_1 v_2) = \mu_B^+(v_1 v_n)$. So, arc (v_n, v_2) is the unique weakest arc of the cycle v_n, v_2, v_1, v_n. Hence again by Theorem (C) (v_n, v_2) is also a weak arc. Hence the arcs forming the outer cycle are all *weak* arcs. Thus, G satisfies all the requirements of theorem(F) and so by Theorem (F), all the remaining arcs are α strong. Hence by Theorem (D), the wheel graph $W_n, n \geq 5$ is an IVFT. □

Remark 3.1. By Theorem (G), all those α strong arcs are IVF bridges and by Theorem (H), the central node is an IVF cutnode.

In general,the IVFG associated with the line graph of a crisp graph G^* is not isomorphic to the IVF line graph of the IVFG associated with G^*. But as the following theorem shows, they are co-weak isomorphic in the case of $C_n, n \geq 3$.

Theorem 3.14. *The Interval Valued Fuzzy Graph associated with the line graph of $C_n, n \geq 3$ is co-weak isomorphic to the IVF line graph of the Interval Valued Fuzzy Graph associated with $C_n, n \geq 3$.*

Proof. We have $C_n \cong L(C_n)$, the line graph of C_n.Let G be the IVFG associated with the line graph of $C_n, n \geq 3$. Clearly $\mu_A^-(v_i) = \frac{1}{n-1}, \mu_A^+(v_i) = \frac{2}{n-1}$ for every node v_i of G and $\mu_B^-(v_i v_j) = \frac{1}{(n-1)^2}, \mu_B^+(v_i v_j) = \frac{4}{(n-1)^2}$ for every arc (v_i, v_j) of G. Clearly G is isomorphic with the IVFG associated with C_n. Let G' be IVF line graph of the IVFG associated with C_n. Clearly, $\mu_A^-(u_i) = \frac{1}{(n-1)^2}, \mu_A^+(u_i) = \frac{4}{(n-1)^2}$ for every node u_i of G' and $\mu_B^-(u_i u_j) = \frac{1}{(n-1)^2}, \mu_B^+(u_i u_j) = \frac{4}{(n-1)^2}$ for every arc (u_i, u_j) of G'. Hence G is co-weak isomorphic to G'. Hence the theorem. □

4. Conclusion

In this paper, we have constructed an interval valued fuzzy graph from a crisp graph, G^* called the Interval Valued Fuzzy Graph associated with G^* based on the degrees of the nodes of the Interval Valued Fuzzy Graph. We proved that regularity is carried over to the so constructed Interval Valued Fuzzy Graph, but completeness is not carried over to. We have shown that every pair of nodes in the associated Interval Valued Fuzzy Graph is connected by a strong path. We also proved that the Interval Valued Fuzzy Graph associated with $C_n, n \geq 3$ is an IVF cycle and the Interval Valued Fuzzy Graph associated with the wheel graph $W_n, n \geq 5$ is an IVFT.

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