A novel study on soft rough rings (ideals) over rings

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Abstract. In this paper, we investigate the relationship among rough sets, soft sets and rings. The concept of soft rough rings (ideals) of rings is introduced, which is an extended concept of rough rings (ideals) of rings. Further, we first put forward the concepts of C-soft sets and CC-soft sets over rings. Moreover, some new soft rough operations over rings are explored. In particular, lower and upper soft rough rings (ideals) over rings with another soft set are investigated.
Keywords: soft set, rough set, soft rough set, soft ideal, soft rough ring (ideal).

1. Introduction

The concept of rough set was introduced by Pawlak [25], a new mathematical approach to deal with uncertain knowledge, has recently received wide attention on the research clues in both of theory and applications. As far as known, an equivalence relation on set into disjoint classes and vice versa. The Pawlak approximation operators are defined by an equivalence relation. However, these equivalence relations in Pawlak rough sets are restrictive for many applied areas. Thus, some more general models have been proposed, such as [30, 31, 32]. Nowadays, rough set theory has been applied to many areas, such as knowledge discovery, machine learning, data analysis, approximate classification, conflict analysis, and so on, see [7, 11, 26]. Kuroki [19] proposed the concept of rough ideals in a semigroup. Davvaz [8, 9] applied this theory to rings. On the other hand, many researchers applied rough set theory to algebraic structures in many papers, such as [8, 10, 14].

Nowadays, the mathematical modelling and manipulating of various types of uncertainties has become an increasingly important issue in solving complicated problems arisings in a wide rang of areas such as economy, engineering,

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environmental science, medicine and social science. As far as known, there are several theories to describe uncertainty, for example, fuzzy set theory [27], rough set theory [25] and other mathematical tools. However, each of them has certain inherent limitations. Based on this reason, in 1999, Molodtsov [24] put forward soft set theory as a new mathematical tool for dealing with uncertainties. Nowadays, the research on soft sets is progressing rapidly. In 2003, Maji et al. [22] proposed some basic operations. Further, Ali et al. [2] revised some operations. In 2011, Ali [3] studied another view on reduction of parameters in soft sets. Afterwards, a wide range of applications of soft sets have been studied in many different fields including game theory, probability theory, smoothness of functions, operation researches, Riemann integrations and measurement theory and so on. Recently, there has been a rapid growth of interest in soft set theory and its applications, such as [4, 5, 6, 23, 18]. In particular, Zhan and Zhu [28] reviewed on decision making methods based on (fuzzy) soft sets and rough soft sets. At the same time, many researchers applied this theory to algebraic structures [15, 16].

Soft set theory and rough set theory are all mathematical tools to deal with uncertainty. Recently, Feng et al. [12, 13] provided a framework to combine rough sets with soft sets, which gives rise to some interesting new concepts such as rough soft sets, soft rough sets and soft rough fuzzy sets. In 2014, Li and Xie [20] investigated the relationship among soft sets, soft rough sets and topologies. In 2015, Zhan and Liu et al. [29] applied rough soft set theory to hemirings. In [21], Ma and Zhan put forth rough soft BCI-algebras by means of an ideal of BCI-algebra. In recent years, Shabir et al. [17] pointed out that there exist some problems on Feng's soft rough set as follows: (1) An upper approximation of a non-empty set may be empty. (2) The upper approximation of a subset X may not contain the set X. In order to solve these problems, Shabir modified the concept of soft rough sets. The underlying concepts are very similar to Pawlak rough sets.

Based on the above idea, it is an interesting work to discuss further on this topic. This paper aims at providing a framework to combine soft sets, rough sets with rings all together, which propose the concept of soft rough rings (ideals). This paper is organized as follows: In Section 2, we recall some concepts and results on soft sets and rough sets. In Section 3, we study some new operations with respect to soft approximation spaces over rings. Further, the lower and upper soft rough rings (ideals) of rings are investigated in Section 4. In particular, in Section 5, we discuss soft rough rings (ideals) based on another soft set.

2. Preliminaries

In this section, we will review some basic notions relevant to soft sets and rough sets. Mododtsov [24] defined the notion of a soft set as follows: Let U be an initial universe and E be a set of parameters. The power set of U is denoted by $\mathscr{P}(U)$. Throughout this paper, unless otherwise stated, R is always a ring.

Definition 2.1 ([24]). A pair $\mathfrak{S} = (F, A)$ is called a soft set over U, where $A \subseteq E$ and $F : A \to \mathscr{P}(U)$ is a set-valued mapping.

For a soft set $\mathfrak{S} = (F, A)$, the set $\text{Supp}(F, A) = \{x \in A | F(x) \neq \emptyset\}$ is called a soft support of the soft set (F, A).

Definition 2.2 ([12]). A soft set $\mathfrak{S} = (F, A)$ over U is called a full soft set if $\bigcup_{a \in A} F(a) = U$.

Definition 2.3 ([1]). Let $\mathfrak{S} = (F, A)$ be a soft set over R. Then

(i) (F, A) is called a soft ring over R if $F(x)$ is a subring of R for all $x \in \text{Supp}(F, A)$.

(ii) (F, A) is called a soft ideal over R if $F(x)$ is an ideal of R for all $x \in \text{Supp}(F, A)$.

Definition 2.4 ([25]). Let R be an equivalence relation on the universe U and (U, R) be a Pawlak approximation space. A subset $X \subseteq U$ is called definable if $R_*X = R^*X$; in the opposite case, i.e., if $R_*X - R^*X \neq \emptyset$, X is said to be a rough set, where two operators are defined as:

$$R_*X = \{x \in U | [x]_R \subseteq X\},$$

$$R^*X = \{x \in U | [x]_R \cap X \neq \emptyset\}.$$

Definition 2.5 ([13]). Let $\mathfrak{S} = (F, A)$ be a soft set over U. Then the pair $P = (U, \mathfrak{S})$ is called a soft approximation space. Based on P, we define the following two operators:

$$\underline{apr}_P(X) = \{x \in U | \exists a \in A[x \in F(a) \subseteq X]\},$$

$$\overline{apr}_P(X) = \{x \in U | \exists a \in A[x \in F(a), F(a) \cap X \neq \emptyset]\},$$

assigning to every subset $X \subseteq U$.

Two sets $\underline{apr}_P(X)$ and $\overline{apr}_P(X)$ are called the lower and upper soft rough approximations of X in P, respectively. If $\underline{apr}_P(X) = \overline{apr}_P(X)$, X is said to be soft definible; otherwise, X is called a soft rough set. In what follows, we call it Feng-soft rough set.

In 2013, Shabir et al. [17] pointed out that there exist some problems on Feng's soft rough set as follows: (1) An upper approximation of a non-empty set may be empty. (2) The upper approximation of a subset X may not contain the set X. In order to solve these problems, Shabir modified the concept of soft rough set as follows.

Definition 2.6 ([17]). Let (F,A) be a soft set over U and $\xi : U \to \mathscr{P}(A)$ be a mapping defined as $\xi(x) = \{a|x \in F(a)\}$. Then the pair (U,ξ) is called MS-approximation space and for any $X \subseteq U$, the lower soft rough approximation and upper soft rough approximation of X are denoted by $\underline{X}_\xi$ and $\overline{X}_\xi$, respectively, which two operators are defined as

$$\underline{X}_\xi = \{x \in X|\xi(x) \neq \xi(y) \text{ for all } y \in X^c\}$$

and

$$\overline{X}_\xi = \{x \in U|\xi(x) = \xi(y) \text{ for some } y \in X\}.$$

If $\underline{X}_\xi = \overline{X}_\xi$, then the X is said to be soft definable, otherwise, X is said to be a soft rough set. In what follows, we call it Shabir-soft rough set.

3. Soft rough sets over rings

In this section, we investigate some new operations and fundamental properties of soft rough sets over rings. Meanwhile, some examples are given. Firstly, we give the concept of soft rough sets over rings as follows.

Definition 3.1. Let (F,A) be a soft set over R and $\xi : R \to \mathscr{P}(A)$ be a mapping defined as $\xi(x) = \{a|x \in F(a)\}$. Then the pair (R,ξ) is called MS-approximation space and for any $X \subseteq R$, the lower soft rough approximation and upper soft rough approximation of X are denoted by $\underline{X}_\xi$ and $\overline{X}_\xi$, respectively, which are two operators are defined as

$$\underline{X}_\xi = \{x \in X|\xi(x) \neq \xi(y) \text{ for all } y \in X^c\}$$

and

$$\overline{X}_\xi = \{x \in S|\xi(x) = \xi(y) \text{ for some } y \in X\}.$$

If $\underline{X}_\xi = \overline{X}_\xi$, then X is said to be soft definable, otherwise, X is said to be soft rough set over R.

Remark 3.2. It follows from Definition 3.1 that for any $X \subseteq R$, we have $\underline{X}_\xi \subseteq X \subseteq \overline{X}_\xi$.

Now, we study some basic properties of lower and upper soft rough approximations of a subset X of R. In order to illustrate the roughness in X w.r.t. MS-approximation spaces over rings, we first introduce two special kinds of soft sets over rings.

Definition 3.3. Let $\mathfrak{S} = (F,A)$ be a soft set over R and $\xi : R \to \mathscr{P}(A)$ be a mapping defined as $\xi(x) = \{a|x \in F(a)\}$. Then $\mathfrak{S}$ is called a C-soft set over R if $\xi(a) = \xi(b)$ and $\xi(c) = \xi(d)$ imply $\xi(a+c) = \xi(b+d)$ and $\xi(ac) = \xi(bd)$ for all $a,b,c,d \in R$.

Definition 3.4. Let $\mathfrak{S} = (F, A)$ be a C-soft set over R and $\xi : R \to \mathscr{P}(A)$ be a mapping defined as $\xi(x) = \{a | x \in F(a)\}$. Then $\mathfrak{S}$ is called a CC-soft set over R if for all $c \in R$,

(i) $\xi(c) = \xi(x + y)$ for $x, y \in R$, there exist $a, b \in R$ such that $\xi(x) = \xi(a)$ and $\xi(y) = \xi(b)$ satisfying $c = a + b$.

(ii) $\xi(c) = \xi(xy)$ for $x, y \in R$, there exist $a, b \in R$ such that $\xi(x) = \xi(a)$ and $\xi(y) = \xi(b)$ satisfying $c = ab$.

Example 3.5. Let $R = \{0, a, b, c\}$ be a set with an addition operation $(+)$ and a multiplication operation $(\cdot)$ as follows:

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	a	a
c	0	0	a	a

Then R is a ring. $\mathfrak{S} = (F, A)$ is a soft set over R which is given by Table 1.

Table 1 Soft set $\mathfrak{S}$

	0	a	b	c
e_1	1	1	1	0
e_2	0	0	0	1
e_3	1	1	1	0

Then the mapping $\xi : R \to \mathscr{P}(A)$ of MS-approximation space (R, ξ) is given by $\xi(0) = \xi(a) = \xi(b) = \{e_1, e_3\}$, $\xi(c) = \{e_2\}$. Then we can check that $\mathfrak{S}$ is not a C-soft set over R, because $\xi(0) = \xi(b)$ and $\xi(a) = \xi(b)$ but $\xi(0 + a) = \xi(a) \neq \xi(c) = \xi(a + b)$.

Example 3.6. We consider the ring R in Example 3.5. $\mathfrak{S} = (F, A)$ is a soft set over R which is given by Table 2.

Table 2 Soft set $\mathfrak{S}$

	0	a	b	c
e_1	1	1	1	1
e_2	0	0	1	1

Then the mapping $\xi : R \to \mathscr{P}(A)$ of MS-approximation space (R, ξ) is given by $\xi(0) = \xi(a) = \{e_1\}$, $\xi(b) = \xi(c) = \{e_1, e_2\}$. It is easy to check that $\mathfrak{S}$ is a C-soft set over R. Nevertheless, $\mathfrak{S}$ is not a CC-soft set over R.

Example 3.7. Let $S = \{0, a, b, c\}$ be a set with an addition operation $(+)$ and a multiplication operation $(\cdot)$ as follows:

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

Then R is a ring. $\mathfrak{S} = (F, A)$ is a soft set over R which is given by Table 3.

Table 3 Soft set $\mathfrak{S}$

	0	a	b	c
e_1	1	1	1	1
e_2	1	1	0	0
e_3	0	0	1	1

Then the mapping $\xi : R \to \mathscr{P}(A)$ of MS-approximation space (R, ξ) is given by $\xi(0) = \xi(a) = \{e_1, e_2\}$, $\xi(b) = \xi(c) = \{e_1, e_3\}$. Then we can check that $\mathfrak{S}$ is a CC-soft set over R.

Proposition 3.8. Let $\mathfrak{S} = (F, A)$ be a C-soft set over R and X, Y any two non-empty subsets of R. Then

$$\overline{X}_\xi \cdot \overline{Y}_\xi \subseteq \overline{X \cdot Y}_\xi.$$

Proof. Let $c \in \overline{X}_\xi \cdot \overline{Y}_\xi$. Then $c = \sum_{i=1}^n x_i y_i$, where $x_i \in \overline{X}_\xi$ and $y_i \in \overline{Y}_\xi$. It follows from Definition 3.1 that $\xi(x_i) = \xi(x)$ and $\xi(y_i) = \xi(y)$ for some $x \in X$, $y \in Y$. Since $\mathfrak{S}$ is a C-soft set, $\xi(x_i y_i) = \xi(xy)$ and $\xi(\sum_{i=1}^n x_i y_i) = \xi(\sum_{i=1}^n xy)$ for some $\sum_{i=1}^n xy \in X \cdot Y$. Thus $c = \sum_{i=1}^n x_i y_i \in \overline{X \cdot Y}_\xi$, and so, $\overline{X}_\xi \cdot \overline{Y}_\xi \subseteq \overline{X \cdot Y}_\xi$. □

Proposition 3.9. Let $\mathfrak{S} = (F, A)$ be a C-soft set over R and X, Y any two non-empty subsets of R. Then

$$\overline{X}_\xi + \overline{Y}_\xi \subseteq \overline{X + Y}_\xi.$$

Proof. Let $c \in \overline{X}_\xi + \overline{Y}_\xi$, then $c = x_1 + y_1$, where $x_1 \in \overline{X}_\xi$ and $y_1 \in \overline{Y}_\xi$. It follows from Definition 3.1 that $\xi(x_1) = \xi(x)$ and $\xi(y_1) = \xi(y)$ for some $x \in X$, $y \in Y$. Since $\mathfrak{S}$ is a C-soft set, $\xi(x_1 + y_1) = \xi(x + y)$ for $x + y \in X + Y$. Hence $c = x_1 + y_1 \in \overline{X + Y}_\xi$. That is, $\overline{X}_\xi + \overline{Y}_\xi \subseteq \overline{X + Y}_\xi$. □

The following example shows that the containment in Propositions 3.8 and 3.9 is proper.

Example 3.10. Consider the ring R in Example 3.5. $\mathfrak{S} = (F, A)$ is a soft set over R which is given by Table 4.

Table 4 Soft set $\mathfrak{S}$

	0	a	b	c
e_1	1	1	1	1
e_2	0	0	0	0

Then the mapping $\xi : R \to \mathscr{P}(A)$ of MS-approximation space (R, ξ) is given by $\xi(0) = \xi(a) = \xi(b) = \xi(c) = \{e_1\}$. Then we can check that $\mathfrak{S}$ is a C-soft set over R.

If we take $X = \{0, a\}$ and $Y = \{a, c\}$, then $\overline{X}_\xi = \{0, a, b, c\}$ and $\overline{Y}_\xi = \{0, a, b, c\}$, so $\overline{X}_\xi \cdot \overline{Y}_\xi = \{0, a\}$. Also we have, $\overline{X \cdot Y}_\xi = \overline{\{0, a\}}_\xi = \{0, a, b, c\}$. Thus $\overline{X}_\xi \cdot \overline{Y}_\xi \subsetneqq \overline{X \cdot Y}_\xi$. Similarly, we have $\overline{X}_\xi + \overline{Y}_\xi \subsetneqq \overline{X + Y}_\xi$.

If we strength the condition, we can obtain the following result:

Proposition 3.11. Let $\mathfrak{S} = (F, A)$ be a CC-soft set over R and X, Y any two non-empty subsets of R. Then

$$\overline{X}_\xi \cdot \overline{Y}_\xi = \overline{X \cdot Y}_\xi.$$

Proof. It follows from Proposition 3.8 that $\overline{X}_\xi \cdot \overline{Y}_\xi \subseteq \overline{X \cdot Y}_\xi$. Now let $c \in \overline{X \cdot Y}_\xi$. Thus $\xi(c) = \xi(\sum_{i=1}^n x_i y_i)$ for some $x_i \in X$ and $y_i \in Y$. Then there exist $a_i, b_i \in R$, such that $\xi(a_i) = \xi(x_i)$ and $\xi(b_i) = \xi(y_i)$ satisfying $c = \sum_{i=1}^n a_i b_i$ since $\mathfrak{S}$ is a CC-soft set over R. Thus $a_i \in \overline{X}_\xi$ and $b_i \in \overline{Y}_\xi$. Hence $c \in \overline{X}_\xi \cdot \overline{Y}_\xi$. Summing up the above arguments, $\overline{X}_\xi \cdot \overline{Y}_\xi = \overline{X \cdot Y}_\xi$. □

Proposition 3.12. Let $\mathfrak{S} = (F, A)$ be a CC-soft set over R and X, Y any two non-empty subsets of R. Then

$$\overline{X}_\xi + \overline{Y}_\xi = \overline{X + Y}_\xi.$$

Proof. It follows from Proposition 3.9 that $\overline{X}_\xi + \overline{Y}_\xi \subseteq \overline{X + Y}_\xi$. Now let $c \in \overline{X + Y}_\xi$. Thus $\xi(c) = \xi(x + y)$ for some $x \in X$ and $y \in Y$. Then there exist $a, b \in R$, such that $\xi(a) = \xi(x)$ and $\xi(b) = \xi(y)$ satisfying $c = a + b$ since $\mathfrak{S}$ is a CC-soft set over R. Thus $a \in \overline{X}_\xi$ and $b \in \overline{Y}_\xi$. Hence $c \in \overline{X}_\xi + \overline{Y}_\xi$. Summing up the above arguments, $\overline{X}_\xi + \overline{Y}_\xi = \overline{X + Y}_\xi$. □

Next, we consider lower soft rough approximations over rings.

Proposition 3.13. Let $\mathfrak{S} = (F, A)$ be a CC-soft set over R and X, Y any two non-empty subsets of R. Then

$$\underline{X}_\xi \cdot \underline{Y}_\xi \subseteq \underline{X \cdot Y}_\xi.$$

Proof. We suppose that $\underline{X}_\xi \cdot \underline{Y}_\xi \subseteq \underline{X \cdot Y}_\xi$ is false, then there exists $c \in \underline{X}_\xi \cdot \underline{Y}_\xi$ but $c \notin \underline{X \cdot Y}_\xi$. Then $c = \sum_{i=1}^n a_i b_i$, where $a_i \in \underline{X}_\xi$ and $b_i \in \underline{Y}_\xi$, and so $\xi(a_i) \neq \xi(x_i)$ and $\xi(b_i) \neq \xi(y_i)$ for all $x_i \in X^c$ and $y_i \in Y^c$.

On the other hand, $c \notin \underline{X \cdot Y}_\xi$, then we may have the following two conditions:

(i) $c \notin X \cdot Y$, which contradicts with $c \in \underline{X}_\xi \cdot \underline{Y}_\xi \subseteq X \cdot Y$;

(ii) $c \in X \cdot Y$ and $\xi(c) = \xi(\sum_{i=1}^n x'_i y'_i)$ for some $\sum_{i=1}^n x'_i y'_i \in (X \cdot Y)^c$. Thus $x'_i \in X^c$ or $y'_i \in Y^c$. In fact, if $x'_i \notin X^c$ and $y'_i \notin Y^c$, we have $\sum_{i=1}^n x'_i y'_i \in X \cdot Y$, a contradiction. Since $\mathfrak{S} = (F, A)$ is a CC-soft set over R, there exist $a'_i, b'_i \in R$ such that $\xi(a'_i) = \xi(x'_i)$ and $\xi(b'_i) = \xi(y'_i)$ satisfying $\sum_{i=1}^n a'_i b'_i = c$, for some $x'_i \in X^c$ or $y'_i \in Y^c$. This is contradiction. Hence, $\underline{X}_\xi \cdot \underline{Y}_\xi \subseteq \underline{X \cdot Y}_\xi$. □

Proposition 3.14. Let $\mathfrak{S} = (F, A)$ be a CC-soft set over R and X, Y any two non-empty subsets of R. Then

$$\underline{X}_\xi + \underline{Y}_\xi \subseteq \underline{X + Y}_\xi.$$

Proof. We suppose that $\underline{X}_\xi + \underline{Y}_\xi \subseteq \underline{X+Y}_\xi$ is false, then there exists $c \in \underline{X}_\xi + \underline{Y}_\xi$ but $c \notin \underline{X+Y}_\xi$. Then $c = a + b$, where $a \in \underline{X}_\xi$ and $b \in \underline{Y}_\xi$. Hence $\xi(a) \neq \xi(x)$ and $\xi(b) \neq \xi(y)$ for all $x \in X^c$ and $y \in Y^c$.

On the other hand, $c \notin \underline{X+Y}_\xi$, then we may have the following two conditions:

(i) $c \notin X + Y$, which contradicts with $c \in \underline{X}_\xi + \underline{Y}_\xi \subseteq X + Y$;

(ii) $c \in X + Y$ and $\xi(c) = \xi(x' + y')$ for some $x' + y' \in (X + Y)^c$. Thus $x' \in X^c$ or $y' \in Y^c$. In fact, if $x' \notin X^c$ and $y' \notin Y^c$, we have $x' + y' \in X + Y$, a contradiction. Since $\mathfrak{S} = (F, A)$ is a CC-soft set over R, there exist $a', b' \in R$, such that $\xi(a') = \xi(x')$ and $\xi(b') = \xi(y')$ satisfying $a' + b' = c$, for some $x' \in X^c$ or $y' \in Y^c$. This is contradiction. Hence $\underline{X}_\xi + \underline{Y}_\xi \subseteq \underline{X+Y}_\xi$. □

The following example shows that the containment in Propositions 3.13 and 3.14 are proper.

Example 3.15. Consider the ring S and the soft set $\mathfrak{S} = (F, A)$ in Example 3.7. Then we know that $\mathfrak{S}$ is a CC-soft set over R.

If we take $X = \{a, b, c\}$ and $Y = \{0, b, c\}$, then $\underline{X}_\xi = \{b, c\}$ and $\underline{Y}_\xi = \{b, c\}$, so $\underline{X}_\xi \cdot \underline{Y}_\xi = \{b, c\}$. Also we have $\underline{X \cdot Y}_\xi = \underline{\{0, a\}}_\xi = \{0, a, b, c\}$, that is $\underline{X}_\xi \cdot \underline{Y}_\xi \subsetneqq \underline{X \cdot Y}_\xi$. Similarly, we have $\underline{X}_\xi + \underline{Y}_\xi \subsetneqq \underline{X+Y}_\xi$.

The following example shows that Propositions 3.13 and 3.14 are not true if $\mathfrak{S}$ is not a CC-soft set over R.

Example 3.16. Consider the ring R and the soft set $\mathfrak{S} = (F, A)$ in Example 3.5. Then we know that $\mathfrak{S}$ is not a CC-soft set over R. If we take $X = \{0, a, c\}$ and $Y = \{c\}$, then $\underline{X}_\xi = \{c\}$ and $\underline{Y}_\xi = \{c\}$. So $\underline{X}_\xi \cdot \underline{Y}_\xi = \{a\}$ and $\underline{X}_\xi + \underline{Y}_\xi = \{0\}$. Also we have $\underline{X \cdot Y}_\xi = \underline{\{0, a\}}_\xi = \emptyset$ and $\underline{X+Y}_\xi = \underline{\{0, b, c\}}_\xi = \emptyset$. So $\underline{X}_\xi \cdot \underline{Y}_\xi \nsubseteq \underline{X \cdot Y}_\xi$ and $\underline{X}_\xi + \underline{Y}_\xi \nsubseteq \underline{X+Y}_\xi$.

4. Characterizations of soft rough rings (ideals) of rings

In this section, we characterize soft rough rings (ideals) of rings. First, we give the notion of soft rough rings (ideals) as follows.

Definition 4.1. In Definition 3.1, if $\underline{X}_\xi \neq \overline{X}_\xi$, (i) X is called a lower (upper) soft rough ring (ideal) w.r.t. $\mathfrak{S}$ of R, if $\underline{X}_\xi$ ($\overline{X}_\xi$) is a subring (ideal) of R; (ii) X is called a soft rough ring (ideal) w.r.t. $\mathfrak{S}$ of R, if $\underline{X}_\xi$ and $\overline{X}_\xi$ are subrings (ideals) of R.

Example 4.2. Let $R = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ be a ring of integers modulo 6 and $\mathfrak{S} = (F, A)$ be a soft set over R which is given by Table 5.

Table 5 Soft set $\mathfrak{S}$

	0	1	2	3	4	5
e_1	1	0	1	1	1	0
e_2	0	1	1	0	1	1

Then the mapping $\xi : R \to \mathscr{P}(A)$ of MS-approximation space (R, ξ) is given by $\xi(0) = \xi(3) = \{e_1\}$, $\xi(1) = \xi(5) = \{e_2\}$, $\xi(2) = \xi(4) = \{e_1, e_2\}$. It follows from Definition 4.1 that for $X = \{0, 1, 2, 3\} \subseteq R$, we have

$$\underline{X}_\xi = \{0, 3\} \text{ and } \overline{X}_\xi = \mathbb{Z}_6.$$

This shows that $\underline{X}_\xi$ and $\overline{X}_\xi$ are subrings of R. In other words, X is a soft rough ring of R.

Proposition 4.3. Let (R, ξ) be an MS-approximation space. If X and Y are lower soft rough rings (ideals) of R, then so is $X \cap Y$.

Proof. It follows from Definition 4.1 that $\underline{X}_\xi$ and $\underline{Y}_\xi$ are subrings (ideals) of R, so $\underline{X}_\xi \cap \underline{Y}_\xi$ is a subring (ideal) of R. It is easy to know that $\underline{X}_\xi \cap \underline{Y}_\xi = \underline{X \cap Y}_\xi$, so $\underline{X \cap Y}_\xi$ is also a subring (ideal) of R. Hence $X \cap Y$ is a lower soft rough ring (ideal) of R. □

In general, $X \cap Y$ is not an upper soft rough ring of R, if X and Y are upper soft rough rings of R. Actually we have the following example.

Example 4.4. Consider the ring R and the soft set $\mathfrak{S} = (F, A)$ in Example 3.7. Now we define $X = \{0, c\}$ and $Y = \{a, c\}$, then $\overline{X}_\xi = \{0, a, b, c\}$ and $\overline{Y}_\xi = \{0, a, b, c\}$ are subrings of R. That is X and Y are upper soft rough rings of R. However, $\overline{X \cap Y}_\xi = \overline{\{c\}}_\xi = \{b, c\}$ is not an upper soft rough ring of R.

Finally, we study the lower and upper soft rough rings (ideals) of rings.

Theorem 4.5. Let $\mathfrak{S} = (F, A)$ be a C-soft set over R and X be a subring of R. Then X is an upper soft rough ring of R.

Proof. Since $X \subseteq \overline{X}_\xi$, $0 \in \overline{X}_\xi$. For all $m, n \in \overline{X}_\xi$, then $\xi(m) = \xi(x)$ and $\xi(n) = \xi(y)$ for some $x, y \in X$. It follows from $\xi(n) = \xi(y)$ that we have $\xi(-n) = \xi(-y)$. Since $\mathfrak{S}$ is a C-soft set and X is a subring of R, $\xi(m - n) = \xi(x - y)$ for $x - y \in X$, thus $m - n \in \overline{X}_\xi$. Similarly, $mn \in \overline{X}_\xi$. Hence, $\overline{X}_\xi$ is a subring of R, that is X is an upper soft rough ring of R. □

Theorem 4.6. Let $\mathfrak{S} = (F, A)$ be a CC-soft set over R and X be a subring of R. Then X is a lower soft rough ring of R if $\underline{X}_\xi \neq \emptyset$.

Proof. Let $\underline{X}_\xi \neq \emptyset$, for all $m, n \in \underline{X}_\xi$. We suppose that $m \cdot n \notin \underline{X}_\xi$. Then we have $\xi(m) \neq \xi(x)$ for all $x \in X^c$ and $\xi(n) \neq \xi(y)$ for all $y \in X^c$.

On the other hand, $m \cdot n \notin \underline{X}_\xi$, then we may have the following two conditions:

(i) $m \cdot n \notin X$, which contradicts with $m \cdot n \in \underline{X}_\xi \cdot \underline{Y}_\xi \subseteq X \cdot X \subseteq X$;

(ii) $m \cdot n \in X$ and $\xi(x') = \xi(m \cdot n)$ for some $x' \in X^c$. Since $\mathfrak{S} = (F, A)$ is a CC-soft set, there exist $x_1, y_1 \in R$ such that $\xi(m) = \xi(x_1)$ and $\xi(n) = \xi(y_1)$ satisfying $x_1 \cdot y_1 = x' \in X^c$. Thus, $x_1 \in X^c$ or $y_1 \in X^c$. In fact, if $x_1 \notin X^c$ and $y_1 \notin X^c$, we have $x_1 \cdot y_1 \in X \cdot X \subseteq X$, a contradiction. That is there exist

$x_1 \in X^c$ such that $\xi(m) = \xi(x_1)$ or $y_1 \in X^c$ such that $\xi(n) = \xi(y_1)$. This is contradiction to $m \cdot n \notin \underline{X}_\xi$. Thus $m \cdot n \in \underline{X}_\xi$. Similarly, we have $m - n \in \underline{X}_\xi$. This implies $\underline{X}_\xi$ is a subring of R, that is X is a lower soft rough ring of R. □

Theorem 4.7. Let $\mathfrak{S} = (F, A)$ be a C-soft set over R and X be an ideal of R. Then X is an upper soft rough ideal of R.

Proof. Let X be an ideal of R. It follows from Theorem 4.5 that $\overline{X}_\xi$ is a subring of R. If $r \in R$ and $s \in \overline{X}_\xi$, then $\xi(s) = \xi(x)$ for some $x \in X$. Since $\xi(r) = \xi(r)$ and $\mathfrak{S}$ is a C-soft set, $\xi(rs) = \xi(rx)$ for some $rx \in R \cdot X \subseteq X$, thus $rs \in \overline{X}_\xi$. Hence $\overline{X}_\xi$ is a left ideal of R. Similarly, we have $\overline{X}_\xi$ is a right ideal of R. So $\overline{X}_\xi$ is an ideal of R, that is X is an upper soft rough ideal of R. □

Theorem 4.8. Let $\mathfrak{S} = (F, A)$ be a CC-soft set over R and X be an ideal of R. Then X is a lower soft rough ideal of R if $\underline{X}_\xi \neq \emptyset$.

Proof. Let X be an ideal of R. It follows from Theorem 4.6 that $\underline{X}_\xi$ is a subring of R. By Proposition 3.13, we have $R \cdot \underline{X}_\xi = \underline{R}_\xi \cdot \underline{R}_\xi \subseteq \underline{R \cdot X}_\xi \subseteq \underline{X}_\xi$. Similarly, $\underline{X}_\xi \cdot R \subseteq \underline{X}_\xi$. Therefore, $\underline{X}_\xi$ is an ideal of R, that is X is a lower soft rough ideal. □

The next example shows that the converse of Theorems 4.7 and 4.8 do not hold in general.

Example 4.9. We consider the ring R and soft set $\mathfrak{S} = (F, A)$ in Example 3.7. Then $\mathfrak{S}$ is a CC-soft set over R.

Now we define $X = \{0, a, b\}$. So $\underline{X}_\xi = \{0, a\}$, $\overline{X}_\xi = \{0, a, b, c\}$. It is easy to know that $\overline{X}_\xi$ and $\underline{X}_\xi$ are ideals of R. However, X is not an ideal of R.

Theorem 4.10. Let $\mathfrak{S} = (F, A)$ be a C-soft set over R and X be a *bi*-ideal of R. Then X is an upper soft rough *bi*-ideal of R.

Proof. Let X be a *bi*-ideal of R. It follows from Theorem 4.7 that $\overline{X}_\xi$ is an ideal of R. It follows from Proposition 3.8 that $\overline{X}_\xi \cdot R \cdot \overline{X}_\xi = \overline{X}_\xi \cdot \overline{R}_\xi \cdot \overline{X}_\xi \subseteq \overline{X \cdot R \cdot X}_\xi \subseteq \overline{X}_\xi$. Hence $\overline{X}_\xi$ is a *bi*-ideal of R, that is X is an upper soft rough *bi*-ideal of R. □

Theorem 4.11. Let $\mathfrak{S} = (F, A)$ be a CC-soft set over R and X be a *bi*-ideal of R. Then X is a lower soft rough *bi*-ideal of R if $\underline{X}_\xi \neq \emptyset$.

Proof. Let X be a *bi*-ideal of R. It follows from Theorem 4.8 that $\underline{X}_\xi$ is an ideal of R. It follows from Proposition 3.13, that $\underline{X}_\xi \cdot R \cdot \underline{X}_\xi = \underline{X}_\xi \cdot \underline{R}_\xi \cdot \underline{X}_\xi \subseteq \underline{X \cdot R \cdot X}_\xi \subseteq \underline{X}_\xi$. Hence $\underline{X}_\xi$ is a *bi*-ideal of R, that is X is a lower soft rough *bi*-ideal of R. □

5. Soft rough rings (ideals) with respect to another soft set

In this section, we investigate soft rough rings (ideals) based on another soft set.

Definition 5.1. Let $\mathfrak{S} = (F, A)$ be a soft set over R and $\xi : R \to \mathscr{P}(A)$ be a mapping defined as $\xi(x) = \{a : x \in F(a)\}$. Let $\mathfrak{T} = (G, B)$ be another soft set defined over R. The lower and upper soft rough approximations of $\mathfrak{T}$ with respect to $\mathfrak{S}$ are denoted by $\underline{(G,B)}_\xi = (\underline{G}_\xi, B)$ and $\overline{(G,B)}_\xi = (\overline{G}_\xi, B)$, respectively, which are two operators defined as

$$\underline{G(e)}_\xi = \{x \in G(e) | \xi(x) \neq \xi(y) \text{ for all } y \in S - G(e)\}$$

and

$$\overline{G(e)}_\xi = \{x \in S | \xi(x) = \xi(y) \text{ for some } y \in G(e)\}$$

for all $e \in B, x \in X$.
(i) If $\underline{(G,B)}_\xi = \overline{(G,B)}_\xi$, then $\mathfrak{T}$ is called definable.
(ii) If $\underline{(G,B)}_\xi \neq \overline{(G,B)}_\xi$ and $\underline{G(e)}_\xi$ ($\overline{G(e)}_\xi$) is a subring (ideal) of R for all $e \in B$, then $\mathfrak{T}$ is called a lower (upper) soft rough ring (ideal) with respect to $\mathfrak{S}$ over R. Moreover, $\mathfrak{T}$ is called a lower (upper) soft rough ring (ideal) with respect to $\mathfrak{S}$ over R if $\underline{G(e)}_\xi$ and $\overline{G(e)}_\xi$ are subrings (ideals) with respect to $\mathfrak{S}$ over R for all $e \in B$.

Example 5.2. We consider the ring R in Example 4.2, soft set $\mathfrak{S} = (F, A)$ over R which is given by Table 6.

Table 6 Soft set $\mathfrak{S}$

	0	1	2	3	4	5
e_1	1	0	1	1	1	0
e_2	0	1	0	0	0	1
e_3	1	1	0	1	0	1

Then the mapping $\xi : R \to \mathscr{P}(A)$ of MS-approximation space (R, ξ) is given by $\xi(0) = \xi(3) = \{e_1, e_3\}$, $\xi(1) = \xi(5) = \{e_2, e_3\}$, $\xi(2) = \xi(4) = \{e_1\}$. Define a soft set $\mathfrak{T} = (G, B)$ as the following Table 7.

Table 7 Soft set $\mathfrak{S}$

	0	1	2	3	4	5
e_1	1	1	1	1	0	0
e_2	1	0	0	1	0	0
e_3	1	1	1	0	0	0

By calculating, $\underline{G(e_1)}_\xi = \{0,3\}$, $\overline{G(e_1)}_\xi = \{0,1,2,3,4,5\}$, $\underline{G(e_2)}_\xi = \{0,3\}$, $\overline{G(e_1)}_\xi = \{0,3\}$, $\underline{G(e_3)}_\xi = \emptyset$, $\overline{G(e_3)}_\xi = \{0,1,2,3,4,5\}$ It is easy to check that $\underline{(G,B)}_\xi$ and $\overline{(G,B)}_\xi$ are subrings of R for all $e \in B$. In other words, $\mathfrak{T} = (G, B)$ is a soft rough ring with respect to $\mathfrak{S}$ of R.

Definition 5.3. Let $\mathfrak{T} = (G, B)$ and $\mathfrak{I} = (H, C)$ be two soft sets over R with $D = B \cap C \neq \emptyset$. The addition operation $+$ and a multiplication operation $\cdot$ of $\mathfrak{T} + \mathfrak{I}$ and $\mathfrak{T} \cdot \mathfrak{I}$ are defined as $\mathfrak{T} + \mathfrak{I} = (G, B) + (H, C) = (K, D)$ and $\mathfrak{T} \cdot \mathfrak{I} = (G, B) \cdot (H, C) = (L, D)$, where $K(a) = G(a) + H(a)$ and $L(a) = G(a) \cdot H(a)$ for all $a \in D$.

Proposition 5.4. Let $\mathfrak{S} = (F, A)$ be a C-soft set over R and (R, ξ) be an MS-approximation space. Let $\mathfrak{T}_1 = (G_1, B)$ and $\mathfrak{T}_2 = (G_2, C)$ be two soft sets over R with $D = B \cap C \neq \emptyset$. Then

(1) $\overline{(G_1, B)}_\xi + \overline{(G_2, C)}_\xi \subseteq \overline{(G_1 + G_2, D)}_\xi$.

(2) $\overline{(G_1, B)}_\xi \cdot \overline{(G_2, C)}_\xi \subseteq \overline{(G_1 \cdot G_2, D)}_\xi$.

Proof. The proof is similar to that of Propositions 3.8 and 3.9. □

If we strength the condition, we can obtain the following result.

Proposition 5.5. Let $\mathfrak{S} = (F, A)$ be a C-soft set over R and (R, ξ) be an MS-approximation space. Let $\mathfrak{T}_1 = (G_1, B)$ and $\mathfrak{T}_2 = (G_2, C)$ be two soft sets over R with $D = B \cap C \neq \emptyset$. Then

(1) $\overline{(G_1, B)}_\xi + \overline{(G_2, C)}_\xi = \overline{(G_1 + G_2, D)}_\xi$.

(2) $\overline{(G_1, B)}_\xi \cdot \overline{(G_2, C)}_\xi = \overline{(G_1 \cdot G_2, D)}_\xi$.

Proof. The proof is similar to that of Propositions 3.11 and 3.12. □

Next, we consider the lower soft rough approximations over rings.

Proposition 5.6. Let $\mathfrak{S} = (F, A)$ be a C-soft set over R and (R, ξ) be an MS-approximation space. Let $\mathfrak{T}_1 = (G_1, B)$ and $\mathfrak{T}_2 = (G_2, C)$ be two soft sets over R with $D = B \cap C \neq \emptyset$. Then

(1) $\underline{(G_1, B)}_\xi + \underline{(G_2, C)}_\xi \subseteq \underline{(G_1 + G_2, D)}_\xi$.

(2) $\underline{(G_1, B)}_\xi \cdot \underline{(G_2, C)}_\xi \subseteq \underline{(G_1 \cdot G_2, D)}_\xi$.

Proof. The proof is similar to that of Propositions 3.13 and 3.14. □

Finally, we investigate the upper and lower soft rough rings (ideals) with respect to another soft set.

Theorem 5.7. Let $\mathfrak{S} = (F, A)$ be a C-soft set over R and $\mathfrak{T} = (G, B)$ be a soft ring of R. Then $\mathfrak{T}$ is an upper soft rough ring of R.

Proof. The proof is similar to that of Theorem 4.5. □

Theorem 5.8. Let $\mathfrak{S} = (F, A)$ be a CC-soft set over R and $\mathfrak{T} = (G, B)$ be a soft ring of R. Then $\mathfrak{T}$ is a lower soft rough ring of R if $\underline{\mathfrak{T}}_\xi \neq \emptyset$.

Proof. The proof is similar to that of Theorem 4.6. □

Theorem 5.9. Let $\mathfrak{S} = (F, A)$ be a C-soft set over R and $\mathfrak{T} = (G, B)$ be a soft ideal of R. Then $\mathfrak{T}$ is an upper soft rough ideal of R.

Proof. The proof is similar to that of Theorem 4.7. □

Theorem 5.10. Let $\mathfrak{S} = (F, A)$ be a CC-soft set over R and $\mathfrak{T} = (G, B)$ be a soft ideal of R. Then $\mathfrak{T}$ is a lower soft rough ideal of S if $\underline{\mathfrak{T}}_\xi \neq \emptyset$.

Proof. The proof is similar to that of Theorem 4.8. □

Theorem 5.11. Let $\mathfrak{S} = (F, A)$ be a C-soft set over R and $\mathfrak{T} = (G, B)$ be a soft *bi*-ideal of R. Then $\mathfrak{T}$ is an upper soft rough *bi*-ideal of R.

Proof. The proof is similar to that of Theorem 4.10. □

Theorem 5.12. Let $\mathfrak{S} = (F, A)$ be a CC-soft set over R and $\mathfrak{T} = (G, B)$ be a soft *bi*-ideal of R. Then $\mathfrak{T}$ is a lower soft rough *bi*-ideal of R if $\underline{\mathfrak{T}}_\xi \neq \emptyset$.

Proof. The proof is similar to that of Theorem 4.11. □

6. Conclusion

In this paper, we initiate a novel soft rough set theory to rings and propose soft rough rings (ideals) over rings. We discuss some operational properties and algebraic structures of lower and upper soft rough approximations over rings. Besides, several examples are presented in order to investigate their characterizations. In particular, we discuss soft rough rings (ideals) based on another soft set.

As an extension of this work, the following problems maybe considered:

(1) Constructing soft rough sets to other algebras, such as hyperrings, hyperhemirings, BL-algebras and so on;

(2) Investigating decision making methods based on soft rough sets;

(4) Establishing soft rough sets to some applied some areas of applications, such as information sciences, intelligent systems and so on.

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