Some results and examples of the bi-f-harmonic maps

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Abstract. In this paper we present some results and examples of the bi-f-harmonic maps. In particular, we study the case of conformal maps between equidimensional manifolds. Examples are constructed when one of the factors is either Euclidean space.
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1. Introduction

Harmonic maps $\phi : (M^m, g) \longrightarrow (N^n, h)$ between Riemannian manifolds are the critical points of the energy functional

$$E(\phi) = \frac{1}{2}\int_D |d\phi|^2 dv_g,$$

for every compact domain $D \subset M$. The Euler-Lagrange equation associated to $E(\phi)$ is $\tau(\phi) = Tr_g \nabla d\phi = 0$, $\tau(\phi)$ is called the tension field of ϕ, one can refer to [5-8] for background on harmonic maps. The stress-energy tensor for a map $\phi : (M^m, g) \longrightarrow (N^n, h)$ defined by (see [2]) $S(\phi) = e(\phi)g - \phi^* h$ and the relation between $S(\phi)$ and $\tau(\phi)$ is given by $\operatorname{div} S(\phi) = -h(\tau(\phi), d\phi)$. As the generalizations of harmonic maps, biharmonic maps are defined as follows. The map $\phi : (M^m, g) \longrightarrow (N^n, h)$ is biharmonic if it is a critical point of the bi-energy functional defined by

$$E_2(\phi) = \frac{1}{2}\int_D |\tau(\phi)|^2 dv_g.$$

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The first variation formula for the bi-energy which is derived in [9] shows that the Euler-Lagrange equation for bi-energy is

$$\tau_2(\phi) = -Tr_g \left(\nabla^{\phi}\right)^2 \tau(\phi) - Tr_g R^N(\tau(\phi), d\phi)d\phi = 0.$$

We will call the operator $\tau_2(\phi)$, the bi-tension field of the map ϕ. In analogy with harmonic maps, Jiang in [9] has constructed for a map ϕ the stress bi-energy tensor defined by

$$S_2(\phi) = \left(\frac{-1}{2}\left|\tau(\phi)\right|^2 + \operatorname{div} h\left(\tau(\phi), d\phi\right)\right) g - 2symh\left(\nabla\tau(\phi), d\phi\right)$$

where $symh\left(\nabla\tau(\phi), d\phi\right)(X,Y) = \frac{1}{2}\left\{h\left(\nabla_X\tau(\phi), d\phi\left(Y\right)\right) + h\left(\nabla_Y\tau(\phi), d\phi\left(X\right)\right)\right\}$ for any $X, Y \in \Gamma\left(TM\right)$. The stress bi-energy tensor of ϕ satisfies the following relationship $\operatorname{div} S_2(\phi) = h\left(\tau_2(\phi), d\phi\right)$. Let $f \in C^{\infty}\left(M\right)$ be a positive function, the map $\phi : (M^m, g) \longrightarrow (N^n, h)$ is said to be f-harmonic if it is a critical point of the f-energy functional:

$$E_f\left(\phi\right) = \frac{1}{2}\int_M f|d\phi|^2 dv_g$$

with respect to compactly supported variations. Equivalently, ϕ is f-harmonic if it satisfies the associated Euler-Lagrange equations $\tau_f\left(\phi\right) = f\tau\left(\phi\right) + d\phi\left(\operatorname{grad} f\right) = f\left(\tau\left(\phi\right) + d\phi\left(\operatorname{grad}\ln f\right)\right) = 0$, $\tau_f\left(\phi\right)$ is called the f-tension field of ϕ. In the context of f-harmonic maps, the stress energy tensor $S_f\left(\phi\right)$ of ϕ associated to the f-energy functional $E_f\left(\phi\right)$ (which we call, the f-stress energy tensor of ϕ) is given by (see [1]) $S_f\left(\phi\right) = f\left(e\left(\phi\right)g - \phi^* h\right)$ and the relation between $S_f\left(\phi\right)$ and $\tau_f\left(\phi\right)$ is given by (see [3]) $\operatorname{div} S_f\left(\phi\right) = -h\left(\tau_f\left(\phi\right), d\phi\right)$. The map ϕ is said to be bi-f-harmonic if it is a critical point of the bi-f-energy functional: $E_{f,2}\left(\phi\right) = \frac{1}{2}\int_M |\tau_f(\phi)|^2 dv_g$. Equivalently, ϕ is bi-f-harmonic if it satisfies the associated Euler-Lagrange equations (see [13]):

$$\tau_{f,2}\left(\phi\right) = -Tr_g\nabla^{\phi} f\nabla^{\phi}\tau_f\left(\phi\right) - fTr_g R^N\left(\tau_f\left(\phi\right), d\phi\right)d\phi = 0, \tag{1}$$

$\tau_{f,2}(\phi)$ is called the bi-f-tension field of ϕ. Note that $Tr_g\nabla^{\phi} f\nabla^{\phi}\tau_f(\phi) = f(Tr_g(\nabla^{\phi})^2\tau_f(\phi) + \nabla_{\operatorname{grad}\ln f}\tau_f(\phi))$, then

$$\tau_{f,2}(\phi) = f(-Tr_g(\nabla^{\phi})^2\tau_f(\phi) - Tr_g R^N(\tau_f(\phi), d\phi)d\phi - \nabla_{gard\ln f}\tau_f(\phi)), \tag{2}$$

and ϕ is bi-f-harmonic if and only if (see [12],[14]),

$$Tr_g(\nabla^{\phi})^2\tau_f(\phi) + Tr_g R^N(\tau_f(\phi), d\phi)d\phi + \nabla_{\operatorname{grad}\ln f}\tau_f(\phi) = 0.$$

Following Jiang's notion, the stress bi-f-bienergy tensor of a ϕ is defined by (see [3])

$$S_{f,2}\left(\phi\right) = \left(\frac{1}{2}\left|\tau_f\left(\phi\right)\right|^2 + fTr_g h\left(\nabla\tau_f\left(\phi\right), d\phi\right)\right) g - 2fsymh\left(\nabla\tau_f\left(\phi\right), d\phi\right),$$

where $Tr_g h(\nabla f\tau(\phi), d\phi) = h(\nabla^{\phi}_{e_i} f\tau(\phi), d\phi(e_i))$ (we sum over repeated indices). The stress bi-f-energy tensor of ϕ satisfies the following relationship div $S_{f,2}(\phi) = h(\tau_{f,2}(\phi), d\phi) + (Tr_g h(\nabla\tau_f(\phi), d\phi))df$. In [10], the author studied the f-harmonicity of some special maps from or into a doubly warped product manifold, he gives the conditions for f-harmonicity of projection maps and some characterizations for non-trivial f-harmonicity of the special product maps, furthermore, he investigates non-trivial f-harmonicity of the product of two harmonic maps. The authors in [15] gave a method to produce biharmonic maps and f-biharmonic maps from given biharmonic maps and they constructed many examples of biharmonic and f-biharmonic maps from the standard sphere S^2 and between two such spheres. In [14], the authors obtain the first variation formula of the bi-f-energy functional, they introduce the bi-f-harmonic maps, which are the natural generalization of biharmonic maps, and they study some properties of the bi-f-harmonic maps. The authors in [4] study a subclass of generalized f-harmonic maps called generalized f-harmonic morphisms which pull back local harmonic functions to local generalized f-harmonic functions. In [12], the authors studied the class of bi-f-harmonic maps and that of f-biharmonic maps from a conformal manifold of dimension ¿ 2 and they gave some results on nonexistence of proper bi-f-harmonic maps and f-biharmonic maps. In this paper, we give other constructions of bi-f-harmonic maps. In the first section, we give the relation between the bi-f-tension field and the bitension field and the relation between the stress bi-f-energy tensor and the stress bi-energy tensor of the smooth map $\phi : (M^m, g) \longrightarrow (N^n, h)$ (Proposition 1 and Proposition 2). In this case we construct some examples of bi-f-harmonic maps and we study the case of the identity map. As a second result, we characterize the bi-f-biharmonicity of the conformal maps $\phi : (M^n, g) \longrightarrow (N^n, h)$ of dilation λ. We begin this section by the particular case where the function f is equal to the dilation λ (Theorem 2 and Theorem 3) and we give an example of bi-λ-harmonic map. We conclude this section by studying the general case (Theorem 4 and Theorem 5).

2. The main results of bi-f-harmonic maps

As a first result, if we consider $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between Riemannian manifolds and let $f \in C^\infty(M)$ be a positive function, the relation between the bi-f-tension and the bi-tension field of ϕ is given by the following proposition.

Proposition 1. *Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between Riemannian manifolds and let $f \in C^\infty(M)$ be a positive function. Then*

$$\begin{aligned}\tau_{f,2}(\phi) &= f^2\tau_2(\phi) - 3f^2\nabla^{\phi}_{\operatorname{grad}\ln f}\tau(\phi) - f^2(\Delta\ln f + 2|\operatorname{grad}\ln f|^2)\tau(\phi)\\ &\quad - f^2(Tr_g(\nabla^{\phi})^2 d\phi(\operatorname{grad}\ln f) + Tr_g R^N(d\phi(\operatorname{grad}\ln f), d\phi)d\phi)\\ &\quad - 3f^2\nabla^{\phi}_{\operatorname{grad}\ln f} d\phi(\operatorname{grad}\ln f) - f^2(\Delta\ln f + 2|\operatorname{grad}\ln f|^2)d\phi(\operatorname{grad}\ln f).\end{aligned} \tag{3}$$

Proof of Proposition 1. Note that by a simple calculate, we have for any $V \in \Gamma\left(\phi^{-1}TN\right)$

$$Tr_g(\nabla^\phi)^2 fV = f(Tr_g(\nabla^\phi)^2 V + 2\nabla^\phi_{\operatorname{grad}\ln f} V + (\Delta \ln f + |\operatorname{grad}\ln f|^2)V). \tag{4}$$

By definition, the bi-f-tension field of ϕ is given by

$$\tau_{f,2}(\phi) = -fTr_g\left(\nabla^\phi\right)^2 \tau_f(\phi) - fTr_g R^N(\tau_f(\phi), d\phi)\, d\phi - f\nabla^\phi_{\operatorname{grad}\ln f}\tau_f(\phi), \tag{5}$$

where $\tau_f(\phi) = f(\tau(\phi) + d\phi(\operatorname{grad}\ln f))$.

Let us choose $(e_i)_{1\leq i\leq m}$ to be an orthonormal frame on M, let us start with the calculation of the first term $Tr_g(\nabla^\phi)^2\tau_f(\phi)$ of (5), we have $Tr_g(\nabla^\phi)^2\tau_f(\phi) = Tr_g(\nabla^\phi)^2 f\tau(\phi) + Tr_g(\nabla^\phi)^2 f d\phi(\operatorname{grad}\ln f)$. Using (4), we obtain $Tr_g(\nabla^\phi)^2 f\tau(\phi) = f(Tr_g(\nabla^\phi)^2\tau(\phi) + 2\nabla^\phi_{\operatorname{grad}\ln f}\tau(\phi)) + f(\Delta\ln f + |\operatorname{grad}\ln f|^2)\tau(\phi)$ and

$$\begin{aligned}Tr_g(\nabla^\phi)^2 f d\phi(\operatorname{grad}\ln f) &= f(Tr_g(\nabla^\phi)^2 d\phi(\operatorname{grad}\ln f) + 2\nabla^\phi_{\operatorname{grad}\ln f} d\phi(\operatorname{grad}\ln f))\\ &\quad + f(\Delta\ln f + |\operatorname{grad}\ln f|^2) d\phi(\operatorname{grad}\ln f).\end{aligned}$$

Its follow that

$$\begin{aligned}Tr_g(\nabla^\phi)^2\tau_f(\phi) &= fTr_g(\nabla^\phi)^2\tau(\phi) + fTr_g(\nabla^\phi)^2 d\phi(\operatorname{grad}\ln f)\\ &\quad + 2f\nabla^\phi_{\operatorname{grad}\ln f}\tau(\phi) + 2f\nabla^\phi_{\operatorname{grad}\ln f} d\phi(\operatorname{grad}\ln f)\\ &\quad + f(\Delta\ln f + |\operatorname{grad}\ln f|^2)(\tau(\phi) + d\phi(\operatorname{grad}\ln f)).\end{aligned} \tag{6}$$

For the term $Tr_g R^N(\tau_f(\phi), d\phi)d\phi$, it is easy to see that

$$\begin{aligned}Tr_g R^N(\tau_f(\phi), d\phi)d\phi &= fTr_g R^N(\tau(\phi), d\phi)d\phi\\ &\quad + fTr_g R^N(d\phi(\operatorname{grad}\ln f), d\phi)d\phi.\end{aligned} \tag{7}$$

Finally for the term $\nabla^\phi_{\operatorname{grad}\ln f}\tau_f(\phi)$, we have

$$\begin{aligned}\nabla^\phi_{\operatorname{grad}\ln f}\tau_f(\phi) &= \nabla^\phi_{\operatorname{grad}\ln f} f\tau(\phi) + \nabla^\phi_{\operatorname{grad}\ln f} f d\phi(\operatorname{grad}\ln f)\\ &= f\nabla^\phi_{\operatorname{grad}\ln f}\tau(\phi) + f|\operatorname{grad}\ln f|^2\tau(\phi)\\ &\quad + f\nabla^\phi_{\operatorname{grad}\ln f} d\phi(\operatorname{grad}\ln f) + f|\operatorname{grad}\ln f|^2 d\phi(\operatorname{grad}\ln f).\end{aligned} \tag{8}$$

If we replace (6), (7) and (8) in (5), we deduce that

$$\begin{aligned}\tau_{f,2}(\phi) &= f^2\tau_2(\phi) - 3f^2\nabla^\phi_{\operatorname{grad}\ln f}\tau(\phi) - f^2(\Delta\ln f + 2|\operatorname{grad}\ln f|^2)\tau(\phi)\\ &\quad - f^2(Tr_g(\nabla^\phi)^2 d\phi(\operatorname{grad}\ln f) + Tr_g R^N(d\phi(\operatorname{grad}\ln f), d\phi)d\phi)\\ &\quad - 3f^2\nabla^\phi_{\operatorname{grad}\ln f} d\phi(\operatorname{grad}\ln f) - f^2(\Delta\ln f + 2|\operatorname{grad}\ln f|^2)d\phi(\operatorname{grad}\ln f).\end{aligned}$$

This completes the proof of Proposition 1.

Remark 1. Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between riemannian manifolds and let $f \in C^\infty(M)$ be a positive function. Then ϕ is bi-f-biharmonic if and only if

$$\begin{aligned}&\tau_2(\phi) - (Tr_g(\nabla^\phi)^2 d\phi(\operatorname{grad}\ln f) + Tr_g R^N(d\phi(\operatorname{grad}\ln f), d\phi)d\phi)\\&- 3\nabla^\phi_{\operatorname{grad}\ln f}\tau(\phi) - (\Delta \ln f + 2|\operatorname{grad}\ln f|^2)\tau(\phi)\\&- 3\nabla^\phi_{\operatorname{grad}\ln f} d\phi(\operatorname{grad}\ln f) - (\Delta \ln f + 2|\operatorname{grad}\ln f|^2) d\phi(\operatorname{grad}\ln f) = 0.\end{aligned}$$

In the case where ϕ is a harmonic map, we have the following result:

Remark 2. Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a harmonic map and let $f \in C^\infty(M)$ be a positive function. Then ϕ is bi-f-biharmonic if and only if

$$\begin{aligned}&Tr_g(\nabla^\phi)^2 d\phi(\operatorname{grad}\ln f) + Tr_g R^N(d\phi(\operatorname{grad}\ln f), d\phi)d\phi\\&+ 3\nabla^\phi_{\operatorname{grad}\ln f} d\phi(\operatorname{grad}\ln f) + (\Delta \ln f + 2|\operatorname{grad}\ln f|^2) d\phi(\operatorname{grad}\ln f) = 0.\end{aligned}$$

We apply this remark to construct some examples of bi-f-harmonic maps.

Example 1. Let the projection $\phi : (\mathbb{R}^4, g_{\mathbb{R}^4}) \rightarrow (\mathbb{R}^3, g_{\mathbb{R}^3})$ defined by $\phi(t, x_2, x_3, x_4) = (t, x_2, x_3)$. We suppose that $\ln f = \alpha(t)$, then by Remark 2, the projection ϕ is bi-f-harmonic if and only if $\alpha''' + 4\alpha'\alpha'' + 2(\alpha')^3 = 0$. Let $\beta = \alpha'$, then the last expression becomes $\beta'' + 4\beta\beta' + 2\beta^3 = 0$. We deduce the special solutions of the form $\beta = \frac{a}{t}$ where $a \in \mathbb{R}$. Eliminating the trivial solution $a = 0$, we obtain $a = 1$ which gives us $f(t) = C|t|$ $(C > 0)$. In this case the projection ϕ is bi-f-harmonic maps, where $f(t) = C|t|$ $(C > 0)$.

Example 2. Let $\phi : (\mathbb{R}^{2n}, g_{\mathbb{R}^{2n}}) \longrightarrow (\mathbb{R}^{n+1}, g_{\mathbb{R}^{n+1}})$ be the Hopf map defined by $\phi(x, y) = (|x|^2 - |y|^2, 2x\overline{y}) \in \mathbb{R} \oplus K$, where $n = 2, 4$ or 8 and $x, y \in K \simeq \mathbb{R}^n$ with $K = C$ (complex numbers), H (quaternions), or O (octonions), respectively. Let us write a point $x, y \in \mathbb{R}^{2n}$ in the form $(x, y) = r(\cos\theta \cdot p, \sin\theta \cdot q), (r \in [0, +\infty), \theta \in [0, \frac{\pi}{2}], p, q \in S^{n-1})$ and those of $\mathbb{R}^{n+1}$ in the form $s(\cos t, \sin t \cdot w), (s \in [0, +\infty), \theta \in [0, \pi], w \in S^{n-1})$. The map ϕ takes the form $\phi(r\cos\theta \cdot p, r\sin\theta \cdot q) = (s\cos t, s\sin t \cdot p\overline{q})$, where $s = r^2$ and $t = 2\theta$. The metrics on $\mathbb{R}^{2n}$ and $\mathbb{R}^{n+1}$ have respectively the expressions $g_{\mathbb{R}^{2n}} = dr^2 + r^2 d\theta^2 + r^2\cos^2\theta \cdot g_{S^{n-1}} + r^2\sin^2\theta \cdot g_{S^{n-1}}$ and $g_{\mathbb{R}^{n+1}} = ds^2 + s^2 dt^2 + s^2\sin^2 t \cdot g_{S^{n-1}}$. An orthonormal basis of $\mathbb{R}^{2n}$ is given by $e_1 = \frac{\partial}{\partial r}, e_2 = \frac{1}{r}\frac{\partial}{\partial\theta}, e_j = \frac{1}{r\cos\theta}\xi_j, e_k = \frac{1}{r\sin\theta}\xi_k$, where $\xi_j = r(\cos\theta \cdot X_j, 0), j = 3, \ldots, n+1$ and $\xi_k = r(0, \sin\theta \cdot X_k), k = n+2, \ldots, 2n$ where the vectors X_j and X_k are unit vectors tangent to the sphere S^{n-1}. We have $\nabla_{e_i} e_i = \frac{1-2n}{r}\frac{\partial}{\partial r} + \frac{n-1}{r^2}(\tan\theta - \cot\theta)\frac{\partial}{\partial\theta}$. Suppose that the function f depends only on r and set $\ln f = \alpha(r)$. Then by Remark 2, the map ϕ is bi-f-harmonic if and only if the function $\beta = \alpha'$ satisfies the following differential equation $r\beta'' + 4r\beta\beta' + (2n+1)\beta' + \frac{2n-1}{r}\beta + (2n+2)\beta^2 + 2r\beta^3 = 0$. We deduce the special solutions of the form $\beta = \frac{a}{r}$ where $a \in \mathbb{R}$. Eliminating the trivial solution $a = 0$, we obtain $a = -n+1$ which gives us $f(r) = \frac{C}{r^{n-1}}$ $(C > 0)$. In this case the Hopf map $\phi : (\mathbb{R}^{2n}, g_{\mathbb{R}^{2n}}) \longrightarrow (\mathbb{R}^{n+1}, g_{\mathbb{R}^{n+1}})$ is bi-f-harmonic map, where $f(r) = \frac{C}{r^{n-1}}$.

In particular, if we consider the identity map, we obtain:

Corollary 1. *The identity map $Id_M : (M^m, g) \longrightarrow (M^m, g)$ is bi-f-harmonic if and only if the function f satisfies the following equation*

$$\begin{aligned}
&\operatorname{grad}\Delta\ln f + \frac{3}{2}\operatorname{grad}(|\operatorname{grad}\ln f|^2)\\
&+ (\Delta\ln f + 2|\operatorname{grad}\ln f|^2)\operatorname{grad}\ln f + 2Ricci(\operatorname{grad}\ln f) = 0.
\end{aligned} \tag{9}$$

In the following we shall present an example of bi-f-harmonic.

Example 3. Let the identity map $Id_{\mathbb{R}^n\setminus\{0\}} : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$ when we suppose that $\ln f$ is radial ($\ln f = \alpha(r)$). A direct calculation gives $\operatorname{grad}\Delta\ln f = (\alpha''' + \frac{n-1}{r}\alpha'' - \frac{n-1}{r^2}\alpha')\frac{\partial}{\partial r}$, $\operatorname{grad}(|\operatorname{grad}\ln f|^2) = 2\alpha'\alpha''\frac{\partial}{\partial r}$ and $\Delta\ln f = \alpha'' + \frac{n-1}{r}\alpha'$. Then by Corollary 1, we deduce that the map $Id_{\mathbb{R}^n\setminus\{0\}}$ is bi-f-harmonic if and only if the function $\beta = \alpha'$ satisfies the following differential equation $\beta'' + \frac{n-1}{r}\beta' - \frac{n-1}{r^2}\beta + 4\beta\beta' + \frac{n-1}{r}\beta^2 + 2\beta^3 = 0$. Looking for particular solutions of type $\beta = \frac{a}{r}$ ($a \in \mathbb{R}^*$), then $Id_{\mathbb{R}^n\setminus\{0\}} : \mathbb{R}^n\setminus\{0\} \longrightarrow \mathbb{R}^n\setminus\{0\}$ is bi-f-harmonic if and only if $a = \frac{-n+5\pm\sqrt{(n-1)(n+7)}}{4}$. We obtain $f(r) = Cr^{\frac{-n+5\pm\sqrt{(n-1)(n+7)}}{4}}$ $(C > 0)$ and in this case the identity map $Id_{\mathbb{R}^n\setminus\{0\}} : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$ is bi-f-harmonic.

Example 4. We consider $M = S^n$ with parameterization $x = (\cos s, \sin s{\cdot}y)$, $s \in [0, \pi]$, $y \in S^{n-1}$. An orthonormal basis of S^n is given by $e_1 = \frac{\partial}{\partial s}$, $e_i = (0, f_i)$, $i = 2, \dots, n$ where the vectors f_i are tangents to the sphere S^{n-1}. We suppose that $\ln f = \alpha(s)$. A direct calculation gives

$$\begin{aligned}
&\operatorname{grad}\ln f = \operatorname{grad}\alpha = \alpha'\frac{\partial}{\partial s},\\
&|\operatorname{grad}\ln f|^2 = |\operatorname{grad}\alpha|^2 = (\alpha')^2,\\
&\operatorname{grad}(|\operatorname{grad}\ln f|^2) = \operatorname{grad}(|\operatorname{grad}\alpha|^2) = 2\alpha'\alpha'',\\
&\Delta\ln f = \Delta\alpha = \alpha'' + (n-1)(\cot s)\alpha',\\
&\operatorname{grad}\Delta\ln f = \operatorname{grad}\Delta\alpha = (\alpha''' + (n-1)(\cot s)\alpha'' - (n-1)(1+\cot^2 s)\alpha')\frac{\partial}{\partial s}
\end{aligned}$$

and $Ricci^{S^n}(\operatorname{grad}\ln f) = Ricci^{S^n}(\operatorname{grad}\alpha) = (n-1)\alpha'\frac{\partial}{\partial s}$. Then by Corollary 1, we deduce that the map Id_{S^n} is bi-f-harmonic if and only if the function $\beta = \alpha'$ satisfies the following differential equation $\beta'' + (n-1)\cot s\beta' + 4\beta\beta' + (n-1)(1-\cot^2 s)\beta + (n-1)\cot s\beta^2 + 2\beta^3 = 0$. For example, if $n = 1$ the map Id_{S^1} is bi-f-harmonic if and only if $\beta'' + 4\beta\beta' + 2\beta^3 = 0$. We find particular solutions of type $\beta = \frac{a}{s}$, we obtain $a = 1$ which gives us $f(s) = ks$ $(k > 0)$. Then the map Id_{S^1} is bi-f-harmonic, where $f(s) = ks$ $(k > 0)$.

Now, for a smooth map $\phi : (M^m, g) \longrightarrow (N^n, h)$ and a positive function $f \in C^\infty(M)$, we give a relation between the stress bi-f-energy tensor and the stress bi–energy tensor of ϕ.

3. The case of conformal maps

We study a conformal maps between manifolds of the same dimension $n \geq 3$. Note that by a result in [2], any such map can have no critical points and so is a local conformal diffeomorphism. Recall that a mapping $\phi : (M^n, g) \to (N^n, h)$ is called conformal if there exists a C^∞ function $\lambda : M \to \mathbb{R}_+^*$ such that for any $X, Y \in \Gamma(TM)$, $h(d\phi(X), d\phi(Y)) = \lambda^2 g(X, Y)$. The function λ is called the dilation for the map ϕ. The tension field and the stress energy tensor for a conformal map are given by (see [2]):

Proposition 2. *Let $\phi : (M^n, g) \to (N^n, h)$ be a conformal map of dilation λ, we have $\tau(\phi) = (2-n)d\phi(\operatorname{grad} \ln \lambda)$, $S(\phi) = \frac{n-2}{2}\lambda^2 g$ and $div\, S(\phi) = (n-2)\lambda^2 d\ln\lambda$.*

For a conformal map, we have the following theorem (see [12])

Theorem 1 ([12])**.** *Let $\phi : (M^n, g) \to (N^n, h)$, ($n \geq 3$) be a conformal map of dilation λ, then for any function $\gamma \in C^\infty(M)$, we have*

$$\begin{aligned} Tr_g(\nabla^\phi)^2 d\phi(\operatorname{grad}\gamma) &= d\phi(\operatorname{grad}\Delta\gamma) + 4d\phi(\nabla_{\operatorname{grad}\ln\lambda}\operatorname{grad}\gamma) \\ &\quad + d\phi(Ricci^M(\operatorname{grad}\gamma)) \\ &\quad + (\Delta\ln\lambda)d\phi(\operatorname{grad}\gamma) - 2(\Delta\gamma)d\phi(\operatorname{grad}\ln\lambda) \\ &\quad - (n-2)d\ln\lambda(\operatorname{grad}\gamma)d\phi(\operatorname{grad}\ln\lambda) \end{aligned} \tag{10}$$

and

$$\begin{aligned} Tr_g R^N(d\phi(\operatorname{grad}\gamma), d\phi)d\phi &= d\phi(Ricci^M(\operatorname{grad}\gamma)) \\ &\quad - (n-2)d\phi(\nabla_{\operatorname{grad}\gamma}\operatorname{grad}\ln\lambda) \\ &\quad - (\Delta\ln\lambda + (n-2)|\operatorname{grad}\ln\lambda|^2)d\phi(\operatorname{grad}\gamma) \\ &\quad + (n-2)d\ln\lambda(\operatorname{grad}\gamma)d\phi(\operatorname{grad}\ln\lambda) \end{aligned} \tag{11}$$

Remark 3. Let $\phi : (M^n, g) \to (N^n, h)$ be a conformal map of dilation λ. The f-tension field of ϕ is given by $\tau_f(\phi) = f(\tau(\phi) + d\phi(\operatorname{grad}\ln f)) = f((2-n)d\phi(\operatorname{grad}\ln\lambda) + d\phi(\operatorname{grad}\ln f)) = f d\phi(\operatorname{grad}\ln f\lambda^{2-n})$. In particular if $f = \lambda$, the λ-tension field of ϕ is given by $\tau_\lambda(\phi) = (3-n)\lambda d\phi(\operatorname{grad}\ln\lambda)$.

In the first, we calculate bi-λ-tension field for a conformal map ϕ.

Theorem 2. *Let $\phi : (M^n, g) \to (N^n, h)$, ($n \geq 3$) to be a conformal map of dilation λ, then the bi-λ-tension field of ϕ is given by $\tau_2(\phi) = (n-2)d\phi(H(\lambda))$ where*

$$\begin{aligned} H(\lambda) &= \operatorname{grad}\Delta\ln\lambda - \frac{n-9}{2}\operatorname{grad}(|\operatorname{grad}\ln\lambda|^2) + 2Ricci^M(\operatorname{grad}\ln\lambda) \\ &\quad - ((n-7)|\operatorname{grad}\ln\lambda|^2 + \Delta\ln\lambda)\operatorname{grad}\ln\lambda. \end{aligned} \tag{12}$$

Proof of Theorem 2. By definition, the bi-λ-tension field is given by $\tau_{\lambda,2}(\phi) = \lambda(-Tr_g(\nabla^{\phi})^2\tau_\lambda(\phi) - Tr_g R^N(\tau_\lambda(\phi), d\phi)d\phi - \nabla_{gard \ln\lambda}\tau_\lambda(\phi))$. The λ-tension field of the conformal map ϕ is given by $\tau_\lambda(\phi) = (3-n)\lambda d\phi(\operatorname{grad}\ln\lambda)$, it follows that

$$\begin{aligned}\tau_{\lambda,2}(\phi) &= (n-3)\lambda(Tr_g(\nabla^{\phi})^2\lambda d\phi(\operatorname{grad}\ln\lambda)\\ &\quad + Tr_g R^N(\lambda d\phi(\operatorname{grad}\ln\lambda), d\phi)d\phi + \nabla_{gard\ln\lambda}\lambda d\phi(\operatorname{grad}\ln\lambda)).\end{aligned} \tag{13}$$

It is easy to see that

$$\begin{aligned}Tr_g(\nabla^{\phi})^2\lambda d\phi(\operatorname{grad}\ln\lambda) &= \lambda Tr_g(\nabla^{\phi})^2 d\phi(\operatorname{grad}\ln\lambda)\\ &+ 2\lambda\nabla_{gard\ln\lambda} d\phi(\operatorname{grad}\ln\lambda)\\ &+ \lambda(\Delta\ln\lambda)d\phi(\operatorname{grad}\ln\lambda) + \lambda|\operatorname{grad}\ln\lambda|^2 d\phi(\operatorname{grad}\ln\lambda).\end{aligned} \tag{14}$$

By Theorem 1, we have

$$\begin{aligned}Tr_g(\nabla^{\phi})^2 d\phi(\operatorname{grad}\ln\lambda) &= d\phi(\operatorname{grad}\Delta\ln\lambda)\\ &+ 2d\phi(\operatorname{grad}(|\operatorname{grad}\ln\lambda|^2)) + d\phi(Ricci^M(\operatorname{grad}\ln\lambda))\\ &- (\Delta\ln\lambda)d\phi(\operatorname{grad}\ln\lambda) - (n-2)|\operatorname{grad}\ln\lambda|^2 d\phi(\operatorname{grad}\ln\lambda).\end{aligned} \tag{15}$$

For the term $\nabla_{gard\ln\lambda} d\phi(\operatorname{grad}\ln\lambda)$, we have (see [2])

$$\begin{aligned}\nabla_{\operatorname{grad}\ln\lambda} d\phi(\operatorname{grad}\ln\lambda) &= |\operatorname{grad}\ln\lambda|^2 d\phi(\operatorname{grad}\ln\lambda)\\ &\quad + \frac{1}{2}d\phi(\operatorname{grad}(|\operatorname{grad}\ln\lambda|^2)).\end{aligned} \tag{16}$$

If we replace (15) and (16) in (14), we deduce that

$$\begin{aligned}Tr_g(\nabla^{\phi})^2\lambda d\phi(\operatorname{grad}\ln\lambda) &= \lambda d\phi(\operatorname{grad}\Delta\ln\lambda) + 3\lambda d\phi(\operatorname{grad}(|\operatorname{grad}\ln\lambda|^2))\\ &- (n-5)\lambda|\operatorname{grad}\ln\lambda|^2 d\phi(\operatorname{grad}\ln\lambda) + \lambda d\phi(Ricci^M(\operatorname{grad}\ln\lambda)).\end{aligned} \tag{17}$$

Now, we will calculate $Tr_g R^N(d\phi(\operatorname{grad}\ln\lambda), d\phi)d\phi$. By Theorem 1, we obtain

$$\begin{aligned}Tr_g R^N(d\phi(\operatorname{grad}\ln\lambda), d\phi)d\phi &= d\phi(Ricci^M(\operatorname{grad}\ln\lambda))\\ &- \frac{n-2}{2}d\phi(\operatorname{grad}(|\operatorname{grad}\ln\lambda|^2)) - (\Delta\ln\lambda)d\phi(\operatorname{grad}\ln\lambda).\end{aligned} \tag{18}$$

To complete the proof, we will simplify the term $\nabla_{gard\ln\lambda}\lambda d\phi(\operatorname{grad}\ln\lambda)$, a simple calculation gives

$$\begin{aligned}\nabla_{gard\ln\lambda}\lambda d\phi(\operatorname{grad}\ln\lambda) &= \frac{1}{2}\lambda d\phi(\operatorname{grad}(|\operatorname{grad}\ln\lambda|^2))\\ &+ 2\lambda|\operatorname{grad}\ln\lambda|^2 d\phi(\operatorname{grad}\ln\lambda).\end{aligned} \tag{19}$$

If we substitute (17), (18) and (19) in (13), we conclude that $\tau_{\lambda,2}(\phi) = (n-3)\lambda^2 d\phi(H(\lambda))$ where

$$\begin{aligned}H(\lambda) &= \operatorname{grad}\Delta\ln\lambda - \frac{n-9}{2}\operatorname{grad}(|\operatorname{grad}\ln\lambda|^2) + 2Ricci^M(\operatorname{grad}\ln\lambda)\\ &- ((n-7)|\operatorname{grad}\ln\lambda|^2 + \Delta\ln\lambda)\operatorname{grad}\ln\lambda.\end{aligned}$$

Corollary 2. *Let $\phi : (M^n, g) \to (N^n, h)$, $(n \geq 4)$ to be a conformal map of dilation λ, then ϕ is bi-λ-harmonic if and only if*

$$\begin{aligned} \text{grad}\,\Delta \ln \lambda - \frac{n-9}{2}\,\text{grad}(|\,\text{grad} \ln \lambda|^2) - (\Delta \ln \lambda)\,\text{grad} \ln \lambda \\ - (n-7)|\,\text{grad} \ln \lambda|^2\,\text{grad} \ln \lambda + 2Ricci^M(\text{grad} \ln \lambda) = 0. \end{aligned} \tag{20}$$

In particular, we prove that the bi-λ-harmonicity of the conformal map $\phi : (\mathbb{R}^n, g) \to (N^n, h)$ $(n \geq 3)$ where the dilation λ is radial $(\ln \lambda = \alpha(r), r = |x|\text{and } \alpha \in C^\infty(\mathbb{R}, \mathbb{R}))$ is equivalent to an ordinary differential equation of the second order. More precisely, we have

Corollary 3. *Let $\phi : (\mathbb{R}^n, g) \to (N^n, h)$ $(n \geq 4)$ to be a conformal map of dilation λ when we suppose that $\ln \lambda$ is radial $(\ln \lambda = \alpha(r), r = |x|$and $\alpha \in C^\infty(\mathbb{R}, \mathbb{R}))$. Then ϕ is bi-λ-harmonic if and only if $\beta = \alpha'$ satisfies the following ordinary differential equation:*

$$\beta'' - (n-8)\beta\beta' + \frac{n-1}{r}\beta' - \frac{n-1}{r^2}\beta - \frac{n-1}{r}\beta^2 - (n-7)\beta^3 = 0. \tag{21}$$

Proof of Corollary 3. Let $\phi : (\mathbb{R}^n, g) \to (N^n, h)$ $(n \geq 4)$ to be a conformal map of dilation λ when we suppose that $\ln \lambda$ is radial $(\ln \lambda = \alpha(r), r = |x|\text{and } \alpha \in C^\infty(\mathbb{R}, \mathbb{R}))$. By Corollary 2, ϕ is bi-λ-harmonic if and only if

$$\begin{aligned} \text{grad}\,\Delta \ln \lambda - \frac{n-9}{2}\,\text{grad}(|\,\text{grad} \ln \lambda|^2) - (\Delta \ln \lambda)\,\text{grad} \ln \lambda \\ - (n-7)|\,\text{grad} \ln \lambda|^2\,\text{grad} \ln \lambda = 0. \end{aligned}$$

A direct calculation shows that

$$\begin{aligned} &\text{grad} \ln \lambda = \alpha' \frac{\partial}{\partial r}, |\,\text{grad} \ln \lambda|^2 = (\alpha')^2, \text{grad}(|\,\text{grad} \ln \lambda|^2) = 2\alpha'\alpha" \frac{\partial}{\partial r}, \\ &\Delta \ln \lambda = \alpha" + \frac{n-1}{r}\alpha', \end{aligned}$$

and $\text{grad}\,\Delta \ln \lambda = (\alpha''' + \frac{n-1}{r}\alpha'' - \frac{n-1}{r^2}\alpha')\frac{\partial}{\partial r}$. Hence, ϕ is bi-λ-harmonic if and only if the function α satisfies the following differential equation

$$\alpha''' - (n-8)\alpha'\alpha'' + \frac{n-1}{r}\alpha'' - \frac{n-1}{r^2}\alpha' - \frac{n-1}{r}(\alpha')^2 - (n-7)(\alpha')^3 = 0.$$

Let $\beta = \alpha'$, the last equation is equivalent to the second-order differential equation $\beta'' - (n-8)\beta\beta' + \frac{n-1}{r}\beta' - \frac{n-1}{r^2}\beta - \frac{n-1}{r}\beta^2 - (n-7)\beta^3 = 0$. As a consequence of the Corollary 3, we will present some remarks which we give a particular solutions of the equation (18) that allows us to construct a bi-λ-harmonic maps.

Remark 4. Looking for particular solutions of type $\beta = \frac{a}{r}$ $(a \in \mathbb{R}^*)$. By (18), we deduce that $\phi : (\mathbb{R}^n, g) \to (N^n, h)$ $(n \geq 4)$ is bi-λ-harmonic if and only if a is a solution of the algebraic equation $(n-7)a^2 + 7a + 2n - 4 = 0$. This equation has real solutions if and only if $n \in \{4, 5, 6, 7, 8\}$.

1. If $n=4$, we find $a=\frac{7+\sqrt{97}}{6}$ or $a=\frac{7-\sqrt{97}}{6}$, so $\lambda=Cr^{\frac{7+\sqrt{97}}{6}}$ or $\lambda=Cr^{\frac{7-\sqrt{97}}{6}}$ $(C>0)$. Then, in this case any conformal map $\phi:(\mathbb{R}^4,g)\to(N^4,h)$ of dilation $\lambda=Cr^{\frac{7+\sqrt{97}}{6}}$ or $\lambda=Cr^{\frac{7-\sqrt{97}}{6}}$ is bi-λ-harmonic.
2. If $n=5$, we find $a=\frac{7+\sqrt{97}}{4}$ or $a=\frac{7-\sqrt{97}}{4}$, so $\lambda=Cr^{\frac{7+\sqrt{97}}{4}}$ or $\lambda=Cr^{\frac{7-\sqrt{97}}{4}}$ $(C>0)$. It follows that any conformal map $\phi:(\mathbb{R}^5,g)\to(N^5,h)$ of dilation $\lambda=Cr^{\frac{7+\sqrt{97}}{4}}$ or $\lambda=Cr^{\frac{7-\sqrt{97}}{4}}$ is bi-λ-harmonic.
3. If $n=6$, we find $a=8$ or $a=-1$, so $\lambda=Cr^8$ or $\lambda=Cr^{-1}$ $(C>0)$. It follows that any conformal map $\phi:(\mathbb{R}^6,g)\to(N^6,h)$ of dilation $\lambda=Cr^8$ or $\lambda=Cr^{-1}$ is bi-λ-harmonic.
4. If $n=7$, a=$\frac{-10}{7}$, so $\lambda=Cr^{\frac{-10}{7}}$ $(C>0)$. Then, in this case any conformal map $\phi:(\mathbb{R}^7,g)\to(N^7,h)$ of dilation $\lambda=Cr^{\frac{-10}{7}}$ is bi-λ-harmonic.
5. If $n=8$, we find $a=-3$ or $a=-4$, so $\lambda=Cr^{-3}$ or $\lambda=Cr^{-4}$ $(C>0)$. It follows that any conformal map $\phi:(\mathbb{R}^8,g)\to(N^8,h)$ of dilation $\lambda=Cr^{-3}$ or $\lambda=Cr^{-4}$ is bi-λ-harmonic.

In a second result, wa calculate the stress bi-λ-energy tensor for a conformal map ϕ when we prove that $S_{\lambda,2}(\phi)$ depend only on the dilation.

Theorem 3. *Let $\phi:(M^n,g)\to(N^n,h)$ be a conformal map with dilation λ, then we have*

$$S_{\lambda,2}(\phi)=(3-n)\lambda^4(\frac{n+1}{2}|\operatorname{grad}\ln\lambda|^2+\Delta\ln\lambda)g-2(3-n)\lambda^4(\nabla d\ln\lambda + (d\ln\lambda\otimes d\ln\lambda)), \tag{22}$$

and the trace of $S_{\lambda,2}(\phi)$ is given by

$$Tr_gS_{\lambda,2}(\phi)=(3-n)\lambda^4(\frac{n^2+n-4}{2}|\operatorname{grad}\ln\lambda|^2+(n-2)\Delta\ln\lambda). \tag{23}$$

Proof of Theorem 3. By definition, the bi-λ-energy tensor of ϕ is given by:

$$S_{\lambda,2}(\phi)(X,Y)=(\frac{1}{2}|\tau_\lambda(\phi)|^2+\lambda Tr_gh(\nabla\tau_\lambda(\phi),d\phi))g(X,Y) - 2\lambda symh(\nabla\tau_\lambda(\phi),d\phi)(X,Y), \tag{24}$$

where $\tau_\lambda(\phi)=(3-n)\lambda d\phi(\operatorname{grad}\ln\lambda)$. Using the equations (2) et (4) of the Proposition 1, we have $\frac{1}{2}|\tau_\lambda(\phi)|^2+\lambda Tr_gh(\nabla\tau_\lambda(\phi),d\phi)=\frac{1}{2}|(3-n)\lambda d\phi(\operatorname{grad}\ln\lambda)|^2+(3-n)\lambda Tr_gh(\nabla\lambda d\phi(\operatorname{grad}\ln\lambda),d\phi)$.

A simple calculation gives $\frac{1}{2}|(3-n)\lambda d\phi(\operatorname{grad}\ln\lambda)|^2=\frac{(3-n)^2}{2}\lambda^4|\operatorname{grad}\ln\lambda|^2$ and $Tr_gh(\nabla\lambda d\phi(\operatorname{grad}\ln\lambda),d\phi)=(n+1)\lambda^3|\operatorname{grad}\ln\lambda|^2+\lambda^3\Delta\ln\lambda$, then

$$\frac{1}{2}|\tau_\lambda(\phi)|^2+\lambda Tr_gh(\nabla\tau_\lambda(\phi),d\phi)=(3-n)\lambda^4(\frac{n+5}{2}|\operatorname{grad}\ln\lambda|^2+\Delta\ln\lambda). \tag{25}$$

Calculate now $symh(\nabla\tau_\lambda(\phi), d\phi)$, we have by definition for any $X, Y \in \Gamma(TM)$

$$\begin{aligned}
&symh(\nabla\tau_\lambda(\phi), d\phi)(X,Y) = \frac{1}{2}(h(\nabla_X\tau_\lambda(\phi), d\phi(Y)) + h(\nabla_Y\tau_\lambda(\phi), d\phi(X)))\\
&= \frac{3-n}{2}(h(\nabla_X\lambda d\phi(\operatorname{grad}\ln\lambda), d\phi(Y)) + h(\nabla_Y\lambda d\phi(\operatorname{grad}\ln\lambda), d\phi(X)))\\
&= \frac{3-n}{2}\lambda(h(\nabla_X d\phi(\operatorname{grad}\ln\lambda), d\phi(Y)) + h(\nabla_Y d\phi(\operatorname{grad}\ln\lambda), d\phi(X)))\\
&+ \frac{3-n}{2}\lambda(X(\ln\lambda)h(d\phi(\operatorname{grad}\ln\lambda), d\phi(Y)) + Y(\ln\lambda)h(d\phi(\operatorname{grad}\ln\lambda), d\phi(X))).
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
&h(\nabla_X d\phi(\operatorname{grad}\ln\lambda), d\phi(Y)) = \lambda^2(\nabla d\ln\lambda(X,Y) + |\operatorname{grad}\ln\lambda|^2 g(X,Y)),\\
&h(\nabla_Y d\phi(\operatorname{grad}\ln\lambda), d\phi(X)) = \lambda^2(\nabla d\ln\lambda(X,Y) + |\operatorname{grad}\ln\lambda|^2 g(X,Y)),\\
&X(\ln\lambda)h(d\phi(\operatorname{grad}\ln\lambda), d\phi(Y)) = \lambda^2 X(\ln\lambda)Y(\ln\lambda) = \lambda^2(d\ln\lambda\otimes d\ln\lambda)(X,Y)
\end{aligned}$$

and $Y(\ln\lambda)h(d\phi(\operatorname{grad}\ln\lambda), d\phi(X)) = \lambda^2 X(\ln\lambda)Y(\ln\lambda) = \lambda^2(d\ln\lambda\otimes d\ln\lambda)(X,Y)$. It follows that

$$symh(\nabla\tau_\lambda(\phi), d\phi) = (3-n)\lambda^3(\nabla d\ln\lambda + (d\ln\lambda\otimes d\ln\lambda) + |\operatorname{grad}\ln\lambda|^2 g) \tag{26}$$

If we substitute (25) and (26) in (24), we conclude that

$$S_{\lambda,2}(\phi) = (3-n)\lambda^4(\frac{n+1}{2}|\operatorname{grad}\ln\lambda|^2 + \Delta\ln\lambda)g - 2(3-n)\lambda^4(\nabla d\ln\lambda + (d\ln\lambda\otimes d\ln\lambda)).$$

Calculate now the trace of bi-λ-energy tensor of ϕ. Let $(e_i)_{1\le i\le n}$ be an orthonormal frame on M, we have

$$\begin{aligned}
Tr_g S_{\lambda,2}(\phi) &= S_{\lambda,2}(\phi)(e_i, e_i)\\
&= (3-n)\lambda^4(\frac{n+1}{2}|\operatorname{grad}\ln\lambda|^2 + \Delta\ln\lambda)g(e_i,e_i)\\
&- 2(3-n)\lambda^4(\nabla d\ln\lambda(e_i,e_i) + (d\ln\lambda\otimes d\ln\lambda)(e_i,e_i))\\
&= (3-n)n\lambda^4(\frac{n+1}{2}|\operatorname{grad}\ln\lambda|^2 + \Delta\ln\lambda)\\
&- 2(3-n)\lambda^4(\Delta\ln\lambda + |\operatorname{grad}\ln\lambda|^2).
\end{aligned}$$

Then $Tr_g S_{\lambda,2}(\phi) = (3-n)\lambda^4(\frac{n^2+n-4}{2}|\operatorname{grad}\ln\lambda|^2 + (n-2)\Delta\ln\lambda)$. By calculating the Laplacian of the function $\lambda^{\frac{n^2+n-4}{2(n-2)}}$ $((n\ge 4))$, we obtain immediately the following corollary

Corollary 4. *Let $\phi : (M^n, g) \to (N^n, h)$, $(n \ge 4)$ to be a conformal map of dilation λ, then the trace of $Tr_g S_{\lambda,2}(\phi)$ is zero if and only if the function $\lambda^{\frac{n^2+n-4}{2(n-2)}}$ is harmonic.*

Remark 5. Let $\phi : (M^n, g) \to (N^n, h)$, $(n \geq 4)$ to be a conformal map of dilation λ, we have $\operatorname{div} S_{\lambda,2}(\phi) = h(\tau_{\lambda,2}(\phi), d\phi) + (Tr_g h(\nabla\tau_\lambda(\phi), d\phi))d\lambda$. A simple calculation gives us $Tr_g h(\nabla\tau_\lambda(\phi), d\phi) = (3-n)\lambda^3(\Delta \ln\lambda + (n+1)|\operatorname{grad}\ln\lambda|^2)$. If we suppose that the map ϕ is bi-λ-harmonic, then $\operatorname{div} S_{\lambda,2}(\phi)$ is zero if and only if the function λ^{n+1} is harmonic.

Theorem 4. *Let $\phi : (M^n, g) \to (N^n, h)$, $(n \geq 3)$ to be a conformal map of dilation λ, then bi-f-tension field of ϕ is given by $\tau_{f,2}(\phi) = f^2 d\phi(H(\lambda, f))$ where*

$$\begin{aligned}
&H(\lambda, f) = (n-2)\operatorname{grad}\Delta\ln\lambda - \frac{(n-2)(n-6)}{2}\operatorname{grad}(|\operatorname{grad}\ln\lambda|^2)\\
&-(n-2)(2(\Delta\ln\lambda) + (n-2)|\operatorname{grad}\ln\lambda|^2)\operatorname{grad}\ln\lambda\\
&+2(n-2)Ricci^M(\operatorname{grad}\ln\lambda) + (n(\Delta\ln f) + (2n-1)|\operatorname{grad}\ln f|^2)\operatorname{grad}\ln\lambda\\
&-\operatorname{grad}\Delta\ln f - 4\nabla_{\operatorname{grad}\ln\lambda}\operatorname{grad}\ln f + 4(n-2)|\operatorname{grad}\ln\lambda|^2\operatorname{grad}\ln f\\
&-(\Delta\ln f + 2|\operatorname{grad}\ln f|^2)\operatorname{grad}\ln f + 4(n-2)\nabla_{gard\ln f}\operatorname{grad}\ln\lambda\\
&-6d\ln\lambda(\operatorname{grad}\ln f)\operatorname{grad}\ln f - \frac{3}{2}\operatorname{grad}(|\operatorname{grad}\ln f|^2) - 2Ricci^M(\operatorname{grad}\ln f).
\end{aligned} \tag{27}$$

Proof of Theorem 4. The f-tension field of ϕ is defined by

$$\tau_f(\phi) = f d\phi(\operatorname{grad}\ln f\lambda^{2-n}),$$

then, the bi-f-tension field of ϕ is given by

$$\begin{aligned}
\tau_{f,2}(\phi) = &-f(Tr_g(\nabla^\phi)^2 f d\phi(\operatorname{grad}\ln(\lambda^{2-n}f))\\
&+ fTr_g R^N(d\phi(\operatorname{grad}\ln(\lambda^{2-n}f)), d\phi)d\phi) - f\nabla^\phi_{gard\ln f} f d\phi(\operatorname{grad}\ln(\lambda^{2-n}f)).
\end{aligned}$$

For the term $Tr_g(\nabla^\phi)^2 f d\phi(\operatorname{grad}\ln(\lambda^{2-n}f))$, we have by (4)

$$\begin{aligned}
Tr_g(\nabla^\phi)^2 f d\phi(\operatorname{grad}\ln(\lambda^{2-n}f)) = &\, fTr_g(\nabla^\phi)^2 d\phi(\operatorname{grad}\ln(\lambda^{2-n}f))\\
&+ 2f\nabla^\phi_{\operatorname{grad}\ln f} d\phi(\operatorname{grad}\ln(\lambda^{2-n}f))\\
&+ f(\Delta\ln f + |\operatorname{grad}\ln f|^2)d\phi(\operatorname{grad}\ln(\lambda^{2-n}f))
\end{aligned}$$

and it is easy to see that

$$\begin{aligned}
\nabla^\phi_{gard\ln f} f d\phi(\operatorname{grad}\ln(\lambda^{2-n}f)) = &\, f\nabla^\phi_{gard\ln f} d\phi(\operatorname{grad}\ln(\lambda^{2-n}f))\\
&+ f|\operatorname{grad}\ln f|^2 d\phi(\operatorname{grad}\ln(\lambda^{2-n}f)),
\end{aligned}$$

it follows that

$$\begin{aligned}
\tau_{f,2}(\phi) = &-f^2(Tr_g(\nabla^\phi)^2 d\phi(\operatorname{grad}\ln(\lambda^{2-n}f))\\
&+ Tr_g R^N(d\phi(\operatorname{grad}\ln(\lambda^{2-n}f)), d\phi)d\phi)\\
&- f^2(\Delta\ln f + 2|\operatorname{grad}\ln f|^2)d\phi(\operatorname{grad}\ln(\lambda^{2-n}f))\\
&- 3f^2\nabla^\phi_{gard\ln f} d\phi(\operatorname{grad}\ln(\lambda^{2-n}f))
\end{aligned} \tag{28}$$

We will study term by term the right-hand of (13). For the first term $Tr_g(\nabla^{\phi})^2 d\phi(\operatorname{grad}\ln(\lambda^{2-n}f)) + Tr_g R^N(d\phi(\operatorname{grad}\ln(\lambda^{2-n}f)), d\phi)d\phi$, we have by (10) and (11)

$$\begin{aligned}
&Tr_g(\nabla^{\phi})^2 d\phi(\operatorname{grad}\ln(\lambda^{2-n}f)) + Tr_g R^N(d\phi(\operatorname{grad}\ln(\lambda^{2-n}f)), d\phi)d\phi\\
&= d\phi(\operatorname{grad}\Delta\ln(\lambda^{2-n}f)) + 4d\phi(\nabla_{\operatorname{grad}\ln\lambda}\operatorname{grad}\ln(\lambda^{2-n}f))\\
&- 2(\Delta\ln(\lambda^{2-n}f))d\phi(\operatorname{grad}\ln\lambda) - (n-2)d\phi(\nabla_{\operatorname{grad}\ln(\lambda^{2-n}f)}\operatorname{grad}\ln\lambda)\\
&- (n-2)|\operatorname{grad}\ln\lambda|^2 d\phi(\operatorname{grad}\ln(\lambda^{2-n}f)) + 2d\phi(Ricci^M(\operatorname{grad}\ln(\lambda^{2-n}f))).
\end{aligned}$$

By using the fact that $\ln(\lambda^{2-n}f) = (2-n)\ln\lambda + \ln f$, we obtain

$$\begin{aligned}
&Tr_g(\nabla^{\phi})^2 d\phi(\operatorname{grad}\ln(\lambda^{2-n}f)) + Tr_g R^N(d\phi(\operatorname{grad}\ln(\lambda^{2-n}f)), d\phi)d\phi\\
&= (2-n)d\phi(\operatorname{grad}\Delta\ln\lambda) + \frac{(n-2)(n-6)}{2}d\phi(\operatorname{grad}(|\operatorname{grad}\ln\lambda|^2))\\
&- 2(2-n)(\Delta\ln\lambda)d\phi(\operatorname{grad}\ln\lambda) + (n-2)^2|\operatorname{grad}\ln\lambda|^2 d\phi(\operatorname{grad}\ln\lambda)\\
&+ 2(2-n)d\phi(Ricci^M(\operatorname{grad}\ln\lambda))\\
&+ d\phi(\operatorname{grad}\Delta\ln f) + 4d\phi(\nabla_{\operatorname{grad}\ln\lambda}\operatorname{grad}\ln f)\\
&- 2(\Delta\ln f)d\phi(\operatorname{grad}\ln\lambda) - (n-2)d\phi(\nabla_{\operatorname{grad}\ln f}\operatorname{grad}\ln\lambda)\\
&- (n-2)|\operatorname{grad}\ln\lambda|^2 d\phi(\operatorname{grad}\ln f) + 2d\phi(Ricci^M(\operatorname{grad}\ln f)).
\end{aligned} \tag{29}$$

Now, let us look at the last term $\nabla^{\phi}_{gard\ln f}d\phi(\operatorname{grad}\ln(\lambda^{2-n}f))$; a simple calculation gives $\nabla^{\phi}_{gard\ln f}d\phi(\operatorname{grad}\ln(\lambda^{2-n}f)) = (2-n)\nabla^{\phi}_{gard\ln f}d\phi(\operatorname{grad}\ln\lambda) + \nabla^{\phi}_{gard\ln f}d\phi(\operatorname{grad}\ln f) = (2-n)(|\operatorname{grad}\ln\lambda|^2 d\phi(\operatorname{grad}\ln f) + d\phi(\nabla_{gard\ln f}\operatorname{grad}\ln\lambda))$ $+2d\ln\lambda(\operatorname{grad}\ln f)d\phi(\operatorname{grad}\ln f) - |\operatorname{grad}\ln f|^2 d\phi(\operatorname{grad}\ln\lambda) + d\phi(\nabla_{gard\ln f}\operatorname{grad}\ln f)$, which gives us

$$\begin{aligned}
&\nabla^{\phi}_{gard\ln f}d\phi(\operatorname{grad}\ln(\lambda^{2-n}f)) = (2-n)(|\operatorname{grad}\ln\lambda|^2 d\phi(\operatorname{grad}\ln f)\\
&+ d\phi(\nabla_{gard\ln f}\operatorname{grad}\ln\lambda)) + 2d\ln\lambda(\operatorname{grad}\ln f)d\phi(\operatorname{grad}\ln f)\\
&- |\operatorname{grad}\ln f|^2 d\phi(\operatorname{grad}\ln\lambda) + \frac{1}{2}d\phi(\operatorname{grad}(|\operatorname{grad}\ln f|^2))
\end{aligned} \tag{30}$$

If we replace (29) and (30) in (28), we deduce that

$$\begin{aligned}
\tau_{f,2}(\phi) &= (n-2)f^2 d\phi(\operatorname{grad}\Delta\ln\lambda) - \frac{(n-2)(n-6)}{2}f^2 d\phi(\operatorname{grad}(|\operatorname{grad}\ln\lambda|^2))\\
&- (n-2)f^2(2(\Delta\ln\lambda) + (n-2)|\operatorname{grad}\ln\lambda|^2)d\phi(\operatorname{grad}\ln\lambda)\\
&+ 2(n-2)f^2 d\phi(Ricci^M(\operatorname{grad}\ln\lambda)) + f^2(n(\Delta\ln f)\\
&+ (2n-1)|\operatorname{grad}\ln f|^2)d\phi(\operatorname{grad}\ln\lambda)\\
&- f^2 d\phi(\operatorname{grad}\Delta\ln f) - 4d\phi(\nabla_{\operatorname{grad}\ln\lambda}\operatorname{grad}\ln f)\\
&+ 4(n-2)f^2|\operatorname{grad}\ln\lambda|^2 d\phi(\operatorname{grad}\ln f)\\
&- f^2(\Delta\ln f + 2|\operatorname{grad}\ln f|^2)d\phi(\operatorname{grad}\ln f) + 4(n-2)f^2 d\phi(\nabla_{gard\ln f}\operatorname{grad}\ln\lambda)
\end{aligned}$$

$$- 6f^2 d\ln\lambda(\operatorname{grad}\ln f)d\phi(\operatorname{grad}\ln f) - \frac{3}{2}f^2 d\phi(\operatorname{grad}(|\operatorname{grad}\ln f|^2))$$
$$- 2f^2 d\phi(Ricci^M(\operatorname{grad}\ln f)).$$

Then $\tau_{f,2}(\phi) = f^2 d\phi(H(\lambda, f))$ where

$$\begin{aligned}
H(\lambda, f) &= (n-2)\operatorname{grad}\Delta\ln\lambda - \frac{(n-2)(n-6)}{2}\operatorname{grad}(|\operatorname{grad}\ln\lambda|^2)\\
&- (n-2)(2(\Delta\ln\lambda) + (n-2)|\operatorname{grad}\ln\lambda|^2)\operatorname{grad}\ln\lambda\\
&+ 2(n-2)Ricci^M(\operatorname{grad}\ln\lambda) + (n(\Delta\ln f) + (2n-1)|\operatorname{grad}\ln f|^2)\operatorname{grad}\ln\lambda\\
&- \operatorname{grad}\Delta\ln f - 4\nabla_{\operatorname{grad}\ln\lambda}\operatorname{grad}\ln f + 4(n-2)|\operatorname{grad}\ln\lambda|^2\operatorname{grad}\ln f\\
&- (\Delta\ln f + 2|\operatorname{grad}\ln f|^2)\operatorname{grad}\ln f + 4(n-2)\nabla_{gard\ln f}\operatorname{grad}\ln\lambda\\
&- 6d\ln\lambda(\operatorname{grad}\ln f)\operatorname{grad}\ln f - \frac{3}{2}\operatorname{grad}(|\operatorname{grad}\ln f|^2) - 2Ricci^M(\operatorname{grad}\ln f).
\end{aligned}$$

Remark 6. Let $\phi : (M^n, g) \to (N^n, h)$, $(n \geq 3)$ to be a conformal map of dilation λ, then ϕ is bi-f-harmonic if and only if

$$\begin{aligned}
&(n-2)\operatorname{grad}\Delta\ln\lambda - \frac{(n-2)(n-6)}{2}\operatorname{grad}(|\operatorname{grad}\ln\lambda|^2)\\
&- (n-2)(2(\Delta\ln\lambda) + (n-2)|\operatorname{grad}\ln\lambda|^2)\operatorname{grad}\ln\lambda\\
&+2(n-2)Ricci^M(\operatorname{grad}\ln\lambda)+(n(\Delta\ln f)+(2n-1)|\operatorname{grad}\ln f|^2)\operatorname{grad}\ln\lambda\\
&- \operatorname{grad}\Delta\ln f - 4\nabla_{\operatorname{grad}\ln\lambda}\operatorname{grad}\ln f + 4(n-2)|\operatorname{grad}\ln\lambda|^2\operatorname{grad}\ln f\\
&- (\Delta\ln f + 2|\operatorname{grad}\ln f|^2)\operatorname{grad}\ln f + 4(n-2)\nabla_{gard\ln f}\operatorname{grad}\ln\lambda\\
&-6d\ln\lambda(\operatorname{grad}\ln f)\operatorname{grad}\ln f-\frac{3}{2}\operatorname{grad}(|\operatorname{grad}\ln f|^2)-2Ricci^M(\operatorname{grad}\ln f)=0.
\end{aligned} \tag{31}$$

In particular, if the map ϕ is biharmonic, we obtain the following result.

Corollary 5. *Let $\phi : (M^n, g) \to (N^n, h)$, $(n \geq 3)$ be a conformal biharmonic map of dilation λ. Then ϕ is bi-f-harmonic if and only if*

$$\begin{aligned}
&\operatorname{grad}\Delta\ln f + 4\nabla_{\operatorname{grad}\ln\lambda}\operatorname{grad}\ln f - 4(n-2)\nabla_{\operatorname{grad}\ln f}\operatorname{grad}\ln\lambda\\
&+ (\Delta\ln f + 2|\operatorname{grad}\ln f|^2 - 4(n-2)|\operatorname{grad}\ln\lambda|^2 + 6d\ln\lambda(\operatorname{grad}\ln f))\operatorname{grad}\ln f\\
&- (n\Delta\ln f + (2n-1)|\operatorname{grad}\ln f|^2)\operatorname{grad}\ln\lambda\\
&+ \frac{3}{2}\operatorname{grad}(|\operatorname{grad}\ln f|^2) + 2Ricci^M(\operatorname{grad}\ln f) = 0.
\end{aligned}$$

By Corollary 5, we will construct an example of bi-f-harmonic maps.

Example 5. Let $\phi : \mathbb{R}^4\backslash\{0\} \longrightarrow \mathbb{R}^4\backslash\{0\}$ be the inversion defined by $\phi(x) = \frac{x}{|x|^2}$. ϕ is a conformal biharmonic map with dilation $\lambda = \frac{1}{r^2}(r = |x|)$. We suppose that $\ln f$ is radial ($\ln f = \alpha(r)$). Then by Corollary 5, we deduce that the map $\phi : \mathbb{R}^4 \setminus \{0\} \longrightarrow \mathbb{R}^4 \setminus \{0\}$ is bi-f-harmonic if and only if the function α satisfies the following differential equation $\alpha''' + \frac{3}{r}\alpha'' - \frac{27}{r^2}\alpha' + 4\alpha'\alpha'' + \frac{5}{r}(\alpha')^2 + 2(\alpha')^3 = 0$.

Let $\beta = \alpha'$, this equation becomes $\beta'' + \frac{3}{r}\beta' - \frac{27}{r^2}\beta + 4\beta\beta' + \frac{5}{r}\beta^2 + 2\beta^3 = 0$. Looking for particular solutions of type $\beta = \frac{a}{r}$ ($a \in \mathbb{R}^*$), then $\phi : \mathbb{R}^4 \setminus \{0\} \longrightarrow \mathbb{R}^4 \setminus \{0\}$ is bi-f-harmonic if and only if $2a^2 + a - 28 = 0$. This equation has two solutions $a = -4$ and $a = \frac{7}{2}$.

1. For $a = -4$, we obtain $f(r) = Cr^{-4}$ and in this case the inversion $\phi : \mathbb{R}^4 \setminus \{0\} \longrightarrow \mathbb{R}^4 \setminus \{0\}$ is f-harmonic so bi-f-harmonic.
2. For $a = \frac{7}{2}$, we obtain $f(r) = Cr^{\frac{7}{2}}$; it follows that the inversion $\phi : \mathbb{R}^4 \setminus \{0\} \longrightarrow \mathbb{R}^4 \setminus \{0\}$ is bi-f-harmonic (not f-harmonic).

Now, we calculate the stress bi-f-energy tensor for a conformal map ϕ when we prove that $S_{f,2}(\phi)$ depends only the dilation and the function f.

Theorem 5. *Let $\phi : (M^n, g) \to (N^n, h)$ be a conformal map with dilation λ, then we have*

$$
\begin{aligned}
S_{f,2}(\phi) &= f^2\lambda^2(-\frac{(n-2)^2}{2}|\operatorname{grad}\ln\lambda|^2 + \frac{3}{2}|\operatorname{grad}\ln f|^2)g \\
&+ f^2\lambda^2((2-n)d\ln\lambda(\operatorname{grad}\ln f) + (2-n)\Delta\ln\lambda + \Delta\ln f)g \\
&+ f^2\lambda^2(2(n-2)sym(d\ln\lambda \odot d\ln f) - 2(d\ln f)^2) \\
&- 2f^2\lambda^2((2-n)\nabla d\ln\lambda + \nabla d\ln f)
\end{aligned}
\tag{32}
$$

and the trace of $S_{f,2}$ is given by

$$
\begin{aligned}
Tr_g S_{f,2}(\phi) &= -(n-2)^2\Delta\ln\lambda - \frac{n(n-2)^2}{2}|\operatorname{grad}\ \ln\lambda|^2 + (n-2)\Delta\ln f \\
&+ \frac{3n-4}{2}|\operatorname{grad}\ \ln f|^2 - (n-2)^2 d\ln\lambda(\operatorname{grad}\ \ln f).
\end{aligned}
\tag{33}
$$

Proof of Theorem 5. By definition, the stress bi-f-energy tensor is given by: $S_{f,2}(\phi) = (\frac{1}{2}|\tau_f(\phi)|^2 + fTr_g h(\nabla\tau_f(\phi), d\phi))g - 2fsymh(\nabla\tau_f(\phi), d\phi)$. Since ϕ is a conformal map with dilation λ, we have $\tau_f(\phi) = (2-n)fd\phi(\operatorname{grad}\ln\lambda) + fd\phi(\operatorname{grad}\ln f)$ and

$$
\begin{aligned}
|\tau_f(\phi)|^2 &= (n-2)^2 f^2\lambda^2|\operatorname{grad}\ln\lambda|^2 + f^2\lambda^2|\operatorname{grad}\ln f|^2 \\
&+ 2(2-n)f^2\lambda^2 d\ln\lambda(\operatorname{grad}\ln f).
\end{aligned}
\tag{34}
$$

Calculate the term $Tr_g h(\nabla\tau_f(\phi), d\phi)$, let $(e_i)_{1\leq i\leq n}$ be an orthonormal frame on M, we have

$$
\begin{aligned}
&Tr_g h(\nabla\tau_f(\phi), d\phi) = (2-n)h(\nabla^\phi_{e_i} fd\phi(\operatorname{grad}\ln\lambda), d\phi(e_i)) \\
&+ h(\nabla^\phi_{e_i} fd\phi(\operatorname{grad}\ln f), d\phi(e_i)) \\
&= (2-n)fh(\nabla^\phi_{e_i} d\phi(\operatorname{grad}\ln\lambda), d\phi(e_i)) + (2-n)h(e_i(f)d\phi(\operatorname{grad}\ln\lambda), d\phi(e_i)) \\
&+ fh(\nabla^\phi_{e_i} d\phi(\operatorname{grad}\ln f), d\phi(e_i)) + h(e_i(f)d\phi(\operatorname{grad}\ln f), d\phi(e_i)) \\
&= (2-n)fh(\nabla^\phi_{e_i} d\phi(\operatorname{grad}\ln\lambda), d\phi(e_i)) + (2-n)fh(d\phi(\operatorname{grad}\ln\lambda), d\phi(\operatorname{grad}\ln f))
\end{aligned}
$$

$$
\begin{aligned}
&+ fh(\nabla^{\phi}_{e_i} d\phi(\operatorname{grad}\ln f), d\phi(e_i)) + fh(d\phi(\operatorname{grad}\ln f), d\phi(\operatorname{grad}\ln f))\\
&= (2-n)fh(\nabla^{\phi}_{e_i} d\phi(\operatorname{grad}\ln\lambda), d\phi(e_i)) + (2-n)f\lambda^2 d\ln\lambda(\operatorname{grad}\ln f)\\
&+ fh(\nabla^{\phi}_{e_i} d\phi(\operatorname{grad}\ln f), d\phi(e_i)) + f\lambda^2|\operatorname{grad}\ln f|^2.
\end{aligned}
$$

It is known that (see [2]) $h(\nabla^{\phi}_{e_i} d\phi(\operatorname{grad}\ln\lambda), d\phi(e_i)) = h(\nabla d\phi(e_i, \operatorname{grad}\ln\lambda), d\phi(e_i)) + h(d\phi(\nabla_{e_i}\operatorname{grad}\ln\lambda), d\phi(e_i)) = h(|\operatorname{grad}\ln\lambda|^2 d\phi(e_i), d\phi(e_i)) + \lambda^2 g(\nabla_{e_i}\operatorname{grad}\ln\lambda, e_i) = n\lambda^2|\operatorname{grad}\ln\lambda|^2 + \lambda^2\Delta\ln\lambda$ and

$$
\begin{aligned}
&h(\nabla^{\phi}_{e_i} d\phi(\operatorname{grad}\ln f), d\phi(e_i)) = h(\nabla d\phi(e_i, \operatorname{grad}\ln f), d\phi(e_i))\\
&+ h(d\phi(\nabla_{e_i}\operatorname{grad}\ln f), d\phi(e_i))\\
&= h(e_i(\ln\lambda) d\phi(\operatorname{grad}\ln f), d\phi(e_i)) + h(d\ln\lambda(\operatorname{grad}\ln f) d\phi(e_i), d\phi(e_i))\\
&- h(e_i(\ln f) d\phi(\operatorname{grad}\ln\lambda), d\phi(e_i)) + \lambda^2 g(\nabla_{e_i}\operatorname{grad}\ln f, e_i)\\
&= h(d\phi(\operatorname{grad}\ln f), d\phi(\operatorname{grad}\ln\lambda)) + d\ln\lambda(\operatorname{grad}\ln f) h(d\phi(e_i), d\phi(e_i))\\
&- h(d\phi(\operatorname{grad}\ln\lambda), d\phi(\operatorname{grad}\ln f)) + \lambda^2 g(\nabla_{e_i}\operatorname{grad}\ln f, e_i)\\
&= n\lambda^2 d\ln\lambda(\operatorname{grad}\ln f) + \lambda^2\Delta\ln f.
\end{aligned}
$$

Which gives us

$$
\begin{aligned}
Tr_g h(\nabla\tau_f(\phi), d\phi) &= (2-n)nf\lambda^2|\operatorname{grad}\ln\lambda|^2 + (2-n)f\lambda^2\Delta\ln\lambda\\
&+ 2f\lambda^2 d\ln\lambda(\operatorname{grad}\ln f) + f\lambda^2\Delta\ln f + f\lambda^2|\operatorname{grad}\ln f|^2.
\end{aligned} \tag{35}
$$

Now, let us look at the last term $symh(\nabla\tau_f(\phi), d\phi)$; we have by definition for any $X, Y \in \Gamma(TM)$

$$
symh(\nabla\tau_f(\phi), d\phi)(X, Y) = \frac{1}{2}(h(\nabla_X\tau_f(\phi), d\phi(Y)) + h(\nabla\tau_f(\phi), d\phi(X))).
$$

For the term $h(\nabla_X\tau_f(\phi), d\phi(Y))$, we have

$$
\begin{aligned}
&h(\nabla_X\tau_f(\phi), d\phi(Y)) = (2-n)h(\nabla_X f d\phi(\operatorname{grad}\ln\lambda), d\phi(Y))\\
&+ h(\nabla_X f d\phi(\operatorname{grad}\ln f), d\phi(Y))\\
&= (2-n)fh(\nabla_X d\phi(\operatorname{grad}\ln\lambda), d\phi(Y)) + (2-n)X(f)h(d\phi(\operatorname{grad}\ln\lambda), d\phi(Y))\\
&+ fh(\nabla_X d\phi(\operatorname{grad}\ln f), d\phi(Y)) + X(f)h(d\phi(\operatorname{grad}\ln f), d\phi(Y))\\
&= (2-n)fh(\nabla_X d\phi(\operatorname{grad}\ln\lambda), d\phi(Y)) + (2-n)f\lambda^2 X(\ln f)Y(\ln\lambda)\\
&+ fh(\nabla_X d\phi(\operatorname{grad}\ln f), d\phi(Y)) + f\lambda^2 X(\ln f)Y(\ln f).
\end{aligned}
$$

Since ϕ is a conformal map, a simple calculate gives

$$
h(\nabla_X d\phi(\operatorname{grad}\ln\lambda), d\phi(Y)) = \lambda^2|\operatorname{grad}\ln\lambda|^2 g(X, Y) + \lambda^2\nabla d\ln\lambda(X, Y)
$$

and

$$
\begin{aligned}
h(\nabla_X d\phi(\operatorname{grad}\ln f), d\phi(Y)) &= \lambda^2 X(\ln\lambda)Y(\ln f) + \lambda^2 d\ln\lambda(\operatorname{grad}\ln f)g(X, Y)\\
&- \lambda^2 X(\ln f)Y(\ln\lambda) + \lambda^2\nabla d\ln f(X, Y),
\end{aligned}
$$

then

$$h(\nabla_X \tau_f(\phi), d\phi(Y)) = (2-n) f\lambda^2 |\operatorname{grad}\ln\lambda|^2 g(X,Y) + f\lambda^2 d\ln\lambda(\operatorname{grad}\ln f) g(X,Y)$$
$$+ (2-n) f\lambda^2 \nabla d\ln\lambda(X,Y) + f\lambda^2 X(\ln\lambda) Y(\ln f) + f\lambda^2 \nabla d\ln f(X,Y)$$
$$- (n-1) f\lambda^2 X(\ln f) Y(\ln\lambda) + f\lambda^2 X(\ln f) Y(\ln f).$$

A similar calculation gives us

$$h(\nabla_Y \tau_f(\phi), d\phi(X)) = (2-n) f\lambda^2 |\operatorname{grad}\ln\lambda|^2 g(X,Y) + f\lambda^2 d\ln\lambda(\operatorname{grad}\ln f) g(X,Y)$$
$$+ (2-n) f\lambda^2 \nabla d\ln\lambda(X,Y) + f\lambda^2 X(\ln f) Y(\ln\lambda) + f\lambda^2 \nabla d\ln f(X,Y)$$
$$- (n-1) f\lambda^2 X(\ln\lambda) Y(\ln f) + f\lambda^2 X(\ln f) Y(\ln f).$$

It follows that

$$\begin{aligned} sym h(\nabla \tau_f(\phi), d\phi)(X,Y) &= (2-n) f\lambda^2 |\operatorname{grad}\ln\lambda|^2 g(X,Y) \\ &+ f\lambda^2 d\ln\lambda(\operatorname{grad}\ln f) g(X,Y) \\ &+ (2-n) f\lambda^2 \nabla d\ln\lambda(X,Y) + f\lambda^2 \nabla d\ln f(X,Y) + f\lambda^2 (d\ln f)^2(X,Y) \\ &- (n-2) f\lambda^2 sym(d\ln\lambda \odot d\ln f)(X,Y), \end{aligned} \tag{36}$$

where $(d\ln f)^2(X,Y) = d\ln f(X) d\ln f(Y) = X(\ln f) Y(\ln f)$ and

$$\begin{aligned} sym(d\ln\lambda \odot d\ln f)(X,Y) &= \frac{1}{2}((d\ln\lambda \otimes d\ln f)(X,Y) + (d\ln\lambda \otimes d\ln f)(Y,X)) \\ &= \frac{1}{2}(X(\ln\lambda) Y(\ln f) + X(\ln f) Y(\ln\lambda)). \end{aligned}$$

By (34), (35) and (36), we conclude that

$$\begin{aligned} S_{f,2}(\phi)(X,Y) &= f^2\lambda^2\Big(-\frac{(n-2)^2}{2}|\operatorname{grad}\ln\lambda|^2 + \frac{3}{2}|\operatorname{grad}\ln f|^2\Big) g(X,Y) \\ &+ f^2\lambda^2((2-n) d\ln\lambda(\operatorname{grad}\ln f) + (2-n)\Delta\ln\lambda + \Delta\ln f) g(X,Y) \\ &+ f^2\lambda^2(2(n-2) sym(d\ln\lambda \odot d\ln f) - 2(d\ln f)^2)(X,Y) \\ &- 2f^2\lambda^2((2-n)\nabla d\ln\lambda + \nabla d\ln f)(X,Y). \end{aligned}$$

It follows that

$$\begin{aligned} S_{f,2}(\phi) &= f^2\lambda^2\Big(-\frac{(n-2)^2}{2}|\operatorname{grad}\ln\lambda|^2 + \frac{3}{2}|\operatorname{grad}\ln f|^2\Big) g \\ &+ f^2\lambda^2((2-n) d\ln\lambda(\operatorname{grad}\ln f) + (2-n)\Delta\ln\lambda + \Delta\ln f) g \\ &+ f^2\lambda^2(2(n-2) sym(d\ln\lambda \odot d\ln f) - 2(d\ln f)^2) \\ &- 2f^2\lambda^2((2-n)\nabla d\ln\lambda + \nabla d\ln f). \end{aligned}$$

Calculate now the trace of stress bi-f-energy tensor. Let $(e_i)_{1\leq i\leq n}$ be an orthonormal frame on M, we have

$$\begin{aligned}
Tr_g S_{f,2}(\phi) &= S_{f,2}(\phi)(e_i,e_i)\\
&= f^2\lambda^2(-\frac{(n-2)^2}{2}|\operatorname{grad}\ln\lambda|^2+\frac{3}{2}|\operatorname{grad}\ln f|^2)g(e_i,e_i)\\
&+ f^2\lambda^2((2-n)d\ln\lambda(\operatorname{grad}\ln f)+(2-n)\Delta\ln\lambda+\Delta\ln f)g(e_i,e_i)\\
&+ f^2\lambda^2(2(n-2)sym(d\ln\lambda\odot d\ln f)-2(d\ln f)^2)(e_i,e_i)\\
&- 2f^2\lambda^2((2-n)\nabla d\ln\lambda+\nabla d\ln f)(e_i,e_i)\\
&= f^2\lambda^2(-\frac{n(n-2)^2}{2}|\operatorname{grad}\ln\lambda|^2+\frac{3n}{2}|\operatorname{grad}\ln f|^2)\\
&+ f^2\lambda^2((2-n)nd\ln\lambda(\operatorname{grad}\ln f)+(2-n)n\Delta\ln\lambda+n\Delta\ln f)\\
&+ f^2\lambda^2(2(n-2)d\ln\lambda(\operatorname{grad}\ln f)-2|\operatorname{grad}\ln f|^2)\\
&- 2f^2\lambda^2((2-n)\Delta\ln\lambda+\Delta\ln f),
\end{aligned}$$

then

$$Tr_g S_{f,2}(\phi) = -(n-2)^2\Delta\ln\lambda-\frac{n(n-2)^2}{2}|\operatorname{grad}\ln\lambda|^2+(n-2)\Delta\ln f$$

$$+\frac{3n-4}{2}|\operatorname{grad}\ln f|^2-(n-2)^2 d\ln\lambda(\operatorname{grad}\ln f).$$

Remark 7. Let $\phi:(M^n,g)\to(N^n,h)$ to be a conformal map of dilation λ, we have $\operatorname{div}S_{f,2}(\phi)=h(\tau_{f,2}(\phi),d\phi)+(Tr_g h(\nabla\tau_f(\phi),d\phi))df$. A simple calculation gives us $Tr_g h(\nabla\tau_f(\phi),d\phi)=f\lambda^2((2-n)\Delta\ln\lambda+(2-n)n|\operatorname{grad}\ln\lambda|^2+\Delta\ln f+3d\ln\lambda(\operatorname{grad}\ln f))$. If we suppose that the map ϕ is bi-f-harmonic, then $\operatorname{div}S_{f,2}(\phi)$ is zero if and only if the functions λ and f satisfy the following equation $(2-n)\Delta\ln\lambda+\Delta\ln f+(2-n)n|\operatorname{grad}\ln\lambda|^2+3d\ln\lambda(\operatorname{grad}\ln f)=0$.

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