A new kind of $(0, 1, 2)$-interpolation

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Abstract. In this paper, we consider the existence, explicit representation and convergence of a new kind of $(0, 1, 2)$-interpolation on non-uniformly distributed set of nodes on the unit circle.
Keywords: Legendre polynomial, explicit representation, convergence.

1. Introduction

In 1960, O. Kiš [6] extended the study on the complex plane by considering $(0, 2)$, $(0, 1, 3)$ and $(0, 1, \dots, r-2, r)$-interpolation cases in the roots of unity and showed the regularity for any $r \geq 2$. The Lagrange-Hermite interpolation process for rational functions was considered by J. F. Tarub [12]. After that J. Szabadós and A. K. Verma [10], made a study of $(0, 1, 2)$-interpolation in uniform metric on an arbitrary set of nodes and claimed that for arbitrary $f \in C[-1, 1]$, a uniquely determined polynomial of degree at most $3n-1$, can not converge uniformly. The Pál-type interpolation with different kind of interpolatory conditions was considered by M. Lénárd [7].

In 2011, H. P. Dikshit [5] examined the regularity of $(0, m)$ and $(0, 1, \dots, r-2, r)$-interpolation on the zeros of $(z^n+1)(z-\tau)$, where τ is any complex number. After that, author[1] [1] consider the convergence of $(0, 1, 2)$-interpolation on the unit circle. The Lagrange-Hermite interpolation at the Jacobi zeros with its first $(r-1)$ derivative at ± 1, was considered by G. Mastrianni, G. V. Milovanovic and I. Notarangelo [8]. Later on, the Lagrange-Hermite interpolating polynomial at some orthogonal polynomial with its first $(r-1)$ derivative at 0, was considered by G. V. Milovanovic and G. Mastrianni and P. Paster [9]. Recently, author [1] [2] considered the convergence of Lagrange-Hermite interpolation on uniformly distributed zeros of the unit circle. Also, author[1] (with Varun) [3] considered the convergence of Lagrange-Hermite interpolation on the non-uniformly distributed zeros on the unit circle obtained the explicit repre-

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sentation and convergence of the interpolatory polynomial for the same. Also, authors [4] considered the convergence of modified Hermite interpolation on the unit circle.

These have motivated us to consider different type of interpolation on the unit circle. In this paper, we consider the non – uniformly distributed zeros on the unit circle, which are obtained by projecting vertically the zeros of $(1-x^2)P_n(x)P'_n(x)$ onto the unit circle, where $P_n(x)$ stands for nth Legendre polynomial. We obtain the explicit forms of the interpolatory polynomials and establish a convergence theorem for the same. In section 2 we give some preliminaries and in section 3, we describe the problem and its existence. In section 4, we give the explicit formulae of the interpolatory polynomials. In section 5 and 6, estimation and convergence of interpolatory polynomials are given respectively.

2. Preliminaries

In this section, we shall give some well-known results, which we shall use.

The differential equation satisfied by $P_n(x)$is

$$\left(1-x^2\right)P''_n(x)-2xP'_n(x)+n(n+1)P_n(x)=0, \tag{1}$$

$$W(z)=\prod_{k=1}^{2n}(z-z_k)=K_n\ P_n\left(\frac{1+z^2}{2z}\right)z^n, \tag{2}$$

$$R_{4n-2}(z)=\prod_{k=1}^{4n-2}(z-t_k^*)=K_{1n}\ P_n\left(\frac{1+z^2}{2z}\right)P'_n\left(\frac{1+z^2}{2z}\right)z^{2n-1}, \tag{3}$$

$$R_{4n}(z)=\left(z^2-1\right)R_{4n-2}(z). \tag{4}$$

We shall require the fundamental polynomial of Lagrange interpolation based on the zeros of $R_{4n}(z)$ and $W(z)$ are respectively given as:

$$l_k(z)=\frac{R_{4n}(z)}{\left(z-t_k^*\right)R'_{4n}\left(t_k^*\right)},\qquad k=0(1)4n-1, \tag{5}$$

$$L_{1k}(z)=\frac{W(z)}{(z-z_k)W'(z_k)},\qquad k=1(1)2n. \tag{6}$$

We will also use the following results

$$\begin{cases} For\ k=1(1)2n, \\ R'_{4n}\left(t_k^*\right)=K_{1n}\frac{1}{2}\left(t_k^{*^2}-1\right)^2\left\{P'_n\left(u_k^*\right)\right\}^2 t_k^{*^{2n-3}}, \\ \text{For}\quad k=2n+1,\ldots,4n-2, \\ R'_{4n}\left(t_k^*\right)=\ K_{1n}\frac{1}{2}\left(t_k^{*^2}-1\right)^2 P_n\left(u_k^*\right)P''_n\left(u_k^*\right)\ t_k^{*^{2n-3}}. \end{cases} \tag{7}$$

We will also use the following well known inequalities:

For $-1 < x < 1$

(8) $$\left(1-x^2\right)^{1/4}\left|P_n\left(x\right)\right| \le \sqrt{\frac{2}{\pi}}n^{-1/2},$$

(9) $$\left(1-x^2\right)^{3/4}\left|P_n'\left(x\right)\right| \le \sqrt{2}\,n^{1/2}\ ,$$

(10) $$\left(1-x^2\right)^{1/2}\left|P_n\left(x\right)\right| = O\left(n^{-1}\right)\ .$$

Let $x_k = \cos\theta_k\ (k = 1, 2, \ldots, n)$ are the zeros of n^{th} Legendre polynomial $P_n\left(x\right)$, with

(11) $$1 > x_1 > x_2 >, \ldots, x_n > \ -1,$$

then

(12) $$\left(1-x_k^2\right)^{-1} \sim \left(\frac{k}{n}\right)^{-2},$$

(13) $$\left|P_n^{(s)}\left(x_k\right)\right| \sim\ k^{\ -s-1/2}\ n^{2s}, \qquad s = 0, 1, 2.$$

For more details one can see [11].

3. The problem and regularity

Let $T_n^* = \ Z_n \cup Z_n^* \ \cup \{-1, 1\}$, be obtained by projecting vertically the zeros of $\left(1-x^2\right)P_n\left(x\right)P_n'\left(x\right)$ onto the unit circle, where $P_n(x)$ stands for n^{th} Legendre polynomial,

(14) $$Z_n = \left\{\begin{array}{c} z_k = \cos\theta_k + i\sin\theta_k, \\ z_{n+k} = \overline{z_k}\ ,\quad k = 1(1)n \end{array}\right\}$$

(15) $$Z_n^* = \left\{\begin{array}{c} z_k^* = \cos\varphi_k + i\sin\varphi_k, \\ z_{(n-1)+k}^* = \overline{z_k^*}\ ,\quad k = 1\,(1)\,n-1 \end{array}\right\}$$

and

(16) $$t_0^* = 1,\ t_{4n-1}^* = -1.$$

Here we are interested to determine the following polynomial $Q_{6n+3}\left(z\right)$ of degree $\le 6n+3$ satisfying the conditions:

(17) $$\begin{cases} Q_{6n+3}\left(t_k^*\right) = \alpha_k, & k = 1(1)4n-2, \\ \left[Q_{6n+3}\left(z\right)\right]_{z=z_k}' = \beta_k, & k = 1\,(1)\,2n, \\ \left[Q_{6n+3}\left(z\right)\right]_{z=t_k^*}^{(p)} = \gamma_k, & k = 0 \text{ and } 4n-1,\ \text{p} = 0, 1, 2 \end{cases}$$

where α_k 's, β_k's and γ_k's are arbitrary complex constants.

Theorem A. *$Q_{6n+3}(z)$ is regular on T_n^*.*

Proof. It is sufficient, if we show that the unique solution of (17) is

$$Q_{6n+3}(z) \equiv 0, \tag{18}$$

when all data, $\alpha_k = \beta_k = \gamma_k = 0$.

In this case, we have

$$Q_{6n+3}(z) = \left(z^2 - 1\right)^2 R_{4n-2}(z)\ q(z), \tag{19}$$

where $q(z)$ is a polynomial of degree $\leq 2n+1$ and $R_{4n-2}(z)$ is defined in (3). Obviously,

$$Q_{6n+3}(t_k^*) = 0, \quad k = 1\,(1)\,4n-2 \tag{20}$$

$$[Q_{6n+3}(z)]_{z=t_k^*}^{(p)} = 0, \quad k = 0 \text{ and } 4n-1, \quad p = 0 \text{ and } 1. \tag{21}$$

From

$$[Q_{6n+3}(z)]'_{z=z_k} = 0, \quad k = 1\,(1)\,2n \tag{22}$$

we get

$$q(z_k) = 0. \tag{23}$$

Therefore, we have

$$q(z) = (a\ z + b)\,W(z), \tag{24}$$

where a and b arbitrary constants. From

$$[Q_{6n+3}(z)]''_{z=t_k^*} = 0, \quad k = 0,\ 4n-1 \tag{25}$$

we get

$$q(\pm 1) = 0 \tag{26}$$

using (26) in (24), we get

$$a = b = 0. \tag{27}$$

Hence, the theorem follows.

4. Explicit representation of interpolatory polynomials

We shall write $Q_{6n+3}(z)$ satisfying (17) as:

$$\begin{aligned} Q_{6n+3}(z) = &\sum_{k=0}^{4n-1} \alpha_k\ D_{0,k}(z) + \sum_{k=1}^{2n} \beta_k\ D_{1,k}(z) \\ (28) \qquad &+ \beta_0 D_{1,0}(z) + \beta_{4n-1} D_{1,4n-1}(z) + \gamma_0 D_{2,0}(z) + \gamma_{4n-1} D_{2,4n-1}(z), \end{aligned}$$

where $D_{0,k}(z)(k = 0(1)4n-1)$, $D_{1,k}(z)(k = 0(1)2n, 4n-1)$ and $D_{2,k}(z)(k = 0\ \textit{and}\ 4n-1)$ are unique polynomials, each of degree at most $6n+3$ satisfying the condition:

For $k = 0\,(1)\,4n-1$

$$(29) \qquad \begin{cases} D_{0,k}(t_j^*) = \delta_{jk}, j = 0(1)4n-1 \\ [D_{0,k}(z)]'_{z=z_j} = 0, j = 1(1)2n, \\ [D_{0,k}(z)]^{(p)}_{z=t_j^*} = 0, j = 0, 4n-1, p = 1, 2. \end{cases}$$

For $k = 1(1)2n$ and $k = 0, 4n-1$

$$(30) \qquad \begin{cases} \begin{cases} \text{For } k = 1\,(1)\,2n \text{ and } k = 0, \quad 4n-1 \\ D_{1,k}\left(t_j^*\right) = 0, \qquad j = 1\,(1)\,4n-2 \\ [D_{1,k}\,(z)]'_{z=z_j} = \delta_{jk}, \qquad j = 1\,(1)\,2n, \end{cases} \\ [D_{1,k}\,(z)]'_{z=t_j^*} = \delta_{jk}, \qquad j,k = 0 \text{ and } 4n-1 \\ [D_{1,k}\,(z)]^{(p)}_{z=t_j^*} = 0, \quad k = 1(1)2n, \qquad j,k = 0 \text{ and } 4n-1,\ p = 0, 2. \end{cases}$$

For $k = 0$ and $4n-1$

$$(31) \qquad \begin{cases} D_{2,k}(t_j^*) = 0, j = 1(1)4n-2 \\ [D_{2,k}(z)]'_{z=z_j} = 0, j = 1(1)2n, \\ [D_{2,k}(z)]^{(p)}_{z=t_j^*} = 0, j = 0,\ 4n-1, \quad p = 0, 1 \\ [D_{2,k}(z)]''{}_{z=t_j^*} = \delta_{jk}, j = 0, 4n-1, \end{cases}$$

Theorem 4.1. *For $k = 0$ and $4n-1$, we have*

$$(32) \qquad D_{2,k}\,(z) = \left(z^2-1\right)^2 R_{4n-2}\,(z)\ \ W\,(z) \left\{c_{1k} + c_{2k}(z - t_k^*)\right\},$$

where

$$(33) \qquad c_{1k} = \frac{1}{8\ R_{4n-2}\left(t_k^*\right) W\left(t_k^*\right)}\ , \quad k = 0 \ \textit{and}\ 4n-1$$

$$(34) \qquad c_{2k} = (-1)^k \frac{1}{2} c_{1k}\ , \quad k = 0\ \textit{and}\ 4n-1.$$

Proof. From equation (32), it is obvious

(35) $$[D_{2,k}(z)]^{(p)}_{z=t_j^*} = 0, \quad j = 0, 4n-1 \quad \text{and} \quad p = 0, 1.$$

(36) $$D_{2,k}\left(t_j^*\right) = 0, \quad j = 1\,(1)\,4n-2.$$

(37) $$[D_{2,k}(z)]'_{z=z_j} = 0, \quad j = 1\,(1)\,2n,$$

From

(38) $$[D_{2,k}(z)]''_{z=t_j^*} = \delta_{jk}, \quad j = 0,\ 4n-1$$

we get (33)-(34).

Theorem 4.2. *For* $k = 1\,(1)\,2n$ *we have*

(39) $$D_{1,k}(z) = b_k\left(z^2-1\right)^3 R_{4n-2}(z)\, L_{1k}(z),$$

where

(40) $$b_k = \frac{1}{\left(z_k^2-1\right)^3 R'_{4n-2}(z_k)}.$$

For $k = 0, 4n-1$

(41) $$D_{1,k}(z) = \frac{(z+t_k^*)^2 R_{4n}(z)\ \ W(z)}{4R'_{4n}\left(t_k^*\right)\quad W\left(t_k^*\right)} + \frac{\left(z^2-1\right) R_{4n}(z)\ \ W(z)}{4R'_{4n}\left(t_k^*\right)\quad W\left(t_k^*\right)} S_k(z),$$

where

(42) $$S_k(z) = -\left[\frac{R''_{4n}\left(t_k^*\right)}{2R'_{4n}\left(t_k^*\right)} + \frac{W'\left(t_k^*\right)}{W\left(t_k^*\right)} + t_k^*\right](z+t_k^*).$$

Proof. For $k = 1\,(1)\,2n$, from equation (39), it is obvious,

(43) $$[D_{1,k}(z)]^{(p)}_{z=t_j^*} = 0, \quad j = 0, 4n-1 \quad \text{and} \quad p = 0, 1, 2.$$

(44) $$D_{1,k}\left(t_j^*\right) = 0, \quad j = 1\,(1)\,4n-2.$$

For $j = 1\,(1)\,2n$,

(45) $$D'_{1k}(z_j) = 0, \qquad \text{for} \qquad j \neq k$$

and for $j = k$, we get (40).

For $k = 0,\ 4n-1$, from equation (41), it is obvious

(46) $$D_{1,k}\left(t_j^*\right) = 0, \quad j = 0\,(1)\,4n-1,$$

(47) $$D'_{1k}(z_j) = 0, \quad j = 1\,(1)\,2n,$$

(48) $$D'_{1k}\left(t_j^*\right) = \delta_{jk}, \quad j = 0, 4n-1.$$

From

$$D''_{1k}\left(t_j^*\right)=0, \qquad j=0 \text{ and } 4n-1 \tag{49}$$

we get (42).

Theorem 4.3. *For $k=1\,(1)\,2n$ we have*

$$\begin{aligned} D_{0,k}(z) &= \frac{\left(z^2-1\right)^2}{\left(t_k^{*2}-1\right)^2} L_{1k}(z)\, l_k(z) \\ &\cdot \left[1-\left(z-t_k^*\right)\left\{L'_{1k}(z_k)+l'_k(z_k)+\frac{4z_k}{\left(z_k^2-1\right)}\right\}\right]. \end{aligned} \tag{50}$$

For $k=2n+1,\ldots,4n-2$

$$D_{0,k}(z)=\frac{\left(z^2-1\right)^2}{\left(t_k^{*2}-1\right)^2}\frac{W(z)}{W\left(t_k^*\right)} l_k(z). \tag{51}$$

For $k=0$ and $4n-1$,

$$D_{0,k}(z)=\frac{\left(z+t_k^*\right)^3 R_{4n-2}(z)\, W(z)}{8\; R_{4n-2}\left(t_k^*\right) W\left(t_k^*\right)}+a_{1k}D_{1,k}(z)+a_{2k}D_{2,k}(z), \tag{52}$$

where

$$a_{1k}=-\left[\frac{3}{2}+\frac{W'\left(t_k^*\right)}{W\left(t_k^*\right)}+\frac{R'_{4n-2}\left(t_k^*\right)}{R_{4n-2}\left(t_k^*\right)}\right] \tag{53}$$

$$\begin{aligned} a_{2k} = &-\left[\frac{3}{2}t_k^*+\frac{R''_{4n-2}\left(t_k^*\right)}{R_{4n-2}\left(t_k^*\right)}+2\frac{R'_{4n-2}\left(t_k^*\right)}{R_{4n-2}\left(t_k^*\right)}\frac{W'\left(t_k^*\right)}{W\left(t_k^*\right)}+\frac{W''\left(t_k^*\right)}{W\left(t_k^*\right)}\right] \\ &-3\left\{\frac{W'\left(t_k^*\right)}{W\left(t_k^*\right)}+\frac{R'_{4n-2}\left(t_k^*\right)}{R_{4n-2}\left(t_k^*\right)}\right\}. \end{aligned} \tag{54}$$

Proof. For $k=1\,(1)\,2n$, let

$$D_{0,k}(z)=\frac{\left(z^2-1\right)^2}{\left(t_k^{*2}-1\right)^2} L_{1k}(z)\, l_k(z)\, h_k(z), \tag{55}$$

where $h_k(z)$ is a linear polynomial such that

$$h_k(z)=a_k+b_k\left(z-t_k^*\right). \tag{56}$$

From (55), obviously

$$D_{0,k}\left(t_j^*\right)=0, \quad j=0 \text{ and } 4n-1, \text{ for } j\neq k \tag{57}$$

and for $j = k$, we must have

$$h_k\left(t_k^*\right) = 1. \tag{58}$$

Again, from (55),

$$D'_{0k}\left(z_j\right) = 0,\ j = 1(1)2n,\ \text{for } j \neq k \tag{59}$$

and for $j = k$, we get

$$h'_k\left(z_k\right) = -\left\{L'_{1k}\left(z_k\right) + l'_k\left(z_k\right) + \frac{4z_k}{\left(z_k^2 - 1\right)}\right\}, \tag{60}$$

using (56), (58) and (60) in (55), we get (50).

From (51), for $k = 2n + 1, \ldots, 4n - 2$, obviously

$$D_{0,k}\left(t_j^*\right) = \delta_{jk},\quad j = 1(1)4n - 2, \tag{61}$$

$$D'_{0,k}\left(z_j\right) = 0,\ j = 1(1)2n. \tag{62}$$

From (52), for $k = 0,\ 4n - 1$

$$D_{0,k}\left(t_j^*\right) = 0,\quad j = 1\,(1)\,4n - 2, \tag{63}$$

$$D_{0,k}\left(t_j^*\right) = \delta_{jk},\quad j = 0, 4n - 1. \tag{64}$$

From

$$D'_{0k}\left(t_j^*\right) = 0,\quad j = 0 \text{ and } 4n - 1, \tag{65}$$

we get (53).

From

$$D''_{0k}\left(t_j^*\right) = 0,\quad j = 0 \text{ and } 4n - 1, \tag{66}$$

we get (54).

5. Estimation of fundamental polynomials

Lemma. *Let $D_{2,k}(z), D_{1,k}(z)$ and $D_{0,k}(z)$ be defined in section 4. Then for $|z| \leq 1$, we get*

$$|D_{2,k}(z)| \leq c\, n^{-2},\ k = 0 \text{ and } 4n - 1 \tag{67}$$

$$|D_{1,k}(z)| \leq c\, n^{-1},\quad k = 0 \text{ and } 4n - 1, \tag{68}$$

$$\sum_{k=1}^{2n} |D_{1,k}(z)| \leq c\, n^{-1} \log n, \tag{69}$$

$$|D_{0,k}(z)| \leq c,\quad k = 0 \text{ and } 4n - 1, \tag{70}$$

$$\sum_{k=1}^{4n-2} |D_{0,k}(z)| \leq c \log n, \tag{71}$$

where c is a constant independent of n and z.

6. Convergence

Theorem B. *Let $f(z)$ be continuous for $|z| \leq 1$ and analytic for $|z| < 1$. Let the arbitrary numbers $\beta_k' s$ and $\gamma_k' s$ be such that:*

$$|\beta_k| = O\left(n\, \omega\left(f, n^{-1}\right)\right), \qquad k = 1\,(1)\,2n, \quad k = 0 \text{ and } 4n-1, \tag{72}$$

$$|\gamma_k| = O\left(n^2 \omega\left(f, n^{-1}\right)\right), \qquad k = 0 \text{ and } 4n-1. \tag{73}$$

Then $\{Q_{6n+3}(z)\}$ be defined by

$$\begin{aligned} Q_{6n+3}(z) = &\sum_{k=0}^{4n-1} f(t_k^*)\, D_{0,k}(z) + \sum_{k=1}^{2n} \beta_k D_{1,k}(z) + \beta_0 D_{1,0}(z) \\ &+ \beta_{4n-1} D_{1,4n-1}(z) + \gamma_0 D_{2,0}(z) + \gamma_{4n-1} D_{2,4n-1}(z) \end{aligned} \tag{74}$$

satisfies the relation

$$|Q_{6n+3}(z) - f(z)| = O\left(\omega\left(f, n^{-1}\right) \log n\right), \tag{75}$$

where $\omega\left(f, n^{-1}\right)$ be the modulus of continuity of $f(z)$.

To prove the Theorem B, we shall need the followings:

Remark. Let $f(z)$ be continuous for $|z| \leq 1$ and analytic for $|z| < 1$. Then there exist a polynomial $F_n(z)$ of degree $\leq 6n+3$ satisfying Jackson's inequality

$$|f(z) - F_n(z)| \leq c\, \omega\left(f, n^{-1}\right), \quad z = e^{i\theta}\,(0 \leq \theta < 2\pi). \tag{76}$$

Also an inequality due to O. Kiš ([6]),

$$\left|F_n^{(m)}(z)\right| \leq c\, n^m\, \omega\left(f, n^{-1}\right), \quad m \in I^+. \tag{77}$$

Proof. Since $Q_{6n+3}(z)$ be a uniquely determined polynomial of degree $\leq 6n+3$ and the polynomial $F_n(z)$ of degree $\leqslant 6n+3$ satisfying (76) and (77) can be expressed as

$$\begin{aligned} F_n(z) = &\sum_{k=0}^{4n-1} F_n(t_k^*)\, D_{0,k}(z) + \sum_{k=1}^{2n} F_n'(z_k)\, D_{1,k}(z) \\ &+ F_n'(t_0^*)\, D_{1,0}(z) + F_n'\left(t_{4n-1}^*\right) D_{1,4n-1}(z) \\ &+ F_n''(t_0^*) D_{2,0}(z) + F_n''\left(t_{4n-1}^*\right) D_{2,4n-1}(z). \end{aligned}$$

Then

$$\begin{aligned} &|Q_{6n+3}(z) - f(z)| \leq |Q_{6n+3}(z) - F_n(z)| + |F_n(z) - f(z)| \\ &\leq \sum_{k=0}^{4n-1} |f(t_k^*) - F_n(t_k^*)|\, |D_{0,k}(z)| + \sum_{k=1}^{2n} \left\{|\beta_k| + \left|F_n'(z_k)\right|\right\} |D_{1,k}(z)| \\ &+ \left\{|\beta_0| + \left|F_n'(t_0^*)\right|\right\} |D_{1,0}(z)| \left\{|\beta_{4n-1}| + \left|F_n'\left(t_{4n-1}^*\right)\right|\right\} |D_{1,4n-1}(z)| \\ &+ \left\{|\gamma_0| + \left|F_n''(t_0^*)\right|\right\} |D_{2,0}(z)| + \left\{|\gamma_{4n-1}| + \left|F_n''\left(t_{4n-1}^*\right)\right|\right\} |D_{2,4n-1}(z)| \\ &+ |F_n(z) - f(z)|, \end{aligned} \tag{78}$$

using (72)-(73), (76)-(77) and Lemma in section 5, we get (75).

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