Characterizations and extensions of abelian rings

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Abstract. In this paper we investigate the abelian rings on their characterizations as well as their extensions and homomorphic properties. A new viewpoint on the characterization of abelian ring is given. Then it is proved that abelian property is inherited under series of extensions such as (skew) polynomial extension, (skew) power series extension, some matrix extensions, Dorroh extension as well as Nagata extension. In the end, we discuss the condition under which abelian property can be held by a homomorphism.

Keywords: abelian rings, idempotent, extension, endomorphism.

1. Introduction

Throughout this paper, all rings are associative rings with an identity. Let $E(R)$, $C(R)$ and $UC(R)$ denote the set of all idempotents, the center and the set of all central invertible element of R respectively. We call a ring R an abelian ring, if $E(R) \subset C(R)$. Obviously, a commutative ring is an abelian ring. Many authors have studied the judgement of abelian rings. In [1], Hum shows that reduced rings(i.e. a ring with no nonzero nilpotent) and semicommutative rings (i.e. ab=0 implies arb=0 for all a,b and R in it) are abelian rings; According to [2], Kim points out that Armendariz rings are abelian rings. In [3], Liu Zhongkui proved that $\alpha - rigid$ rings(i.e. rings with the condition $a\alpha(a) = 0$ implies a=0, in which α is an endomorphism of it) is reduced so that is abelian; In [4], Kim proves that a reversible ring (i.e. ab=0 implies ba=0) is a semicommutative ring. Then every reversible ring is abelian. In [5], Wang yonghui discusses the trivial extensions and some matrix extensions of abelian rings. Motivated above,

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we will make some further discussions on the characterizations and extensions of abelian rings.

2. Characterizations of abelian rings

Definition 2.1. A ring R is called abelian, if $E(R) \subset C(R)$.

To determine the abelian property of a ring R is very general in the research of abelian rings. We induce the following definitions.

Definition 2.2. Let $A(R) = \{a \in R \mid ea = a(1-e) \text{ for some } e \in E(R)\}$, $A_r(R,\sigma) = \{a \in R \mid ea = a(1-\sigma(e))\}$, $A_l(R,\sigma) = \{a \in R \mid ea = a(1-\sigma(e))\}$, where σ is an endomorphism of R.

Many properties of $A(R)$ could be found immediately such as $A(R)$ is nonempty since $0 \in A(R)$. For any nonnegative integer n, $A^{2n+1}(R) \subseteq A(R) = 0$, and $E(R) \subseteq C(A^{2n}(R))$, etc.

And it is well known and generally applied that R is abelian if and only if $eR(1-e)$, which is an inference of the next theorem:

Theorem 2.3. *Let R be a ring with an endomorphism σ. Then the following statements are equivalent:*

(1) *R is abelian and $\sigma(e) = e$ for any $e \in E(R)$;*
(2) *$eR(1-\sigma(e)) = 0$ for each $e \in E(R)$;*
(3) *$A_r(R) = 0$;*
(4) *for any $r \in R, e \in E(R)$, $er = r\sigma(e)$.*

Proof. (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (2), (2) $\Rightarrow$ (4), (4) $\Rightarrow$ (1) Obviously.

It is suffice to proof (2) $\Rightarrow$ (3). For each $a \in A(R)$ there exists an $e \in E(R)$, such that $ea = a(1-\sigma(e)) = ea(1-\sigma(e)) = 0, (1-e)a = a\sigma(e) = (1-e)a\sigma(e) = 0$, thus $a = ea + (1-e)a = 0$ and $A_r(R) = 0$.

Corollary 2.4. *Let R be a ring with an endomorphism σ, and $A_r(R,\sigma) = 0$. Then $A(R) = 0$.*

While the converse is not true when $\sigma = 0$. This also indicates that the condition $\sigma(e) = e$ in Theorem 2.3 is not surplus.

Corollary 2.5. *Let R be a ring. Then the following statements are equivalent:*

(1) *R is abelian;*
(2) $eR(1-e) = 0$;
(3) $A(R) = 0$.

3. Extension properties of abelian rings

In this chapter, we will investigate some extension properties on abelian rings.

Let σ be an endomorphism of a ring R. Let $R[x,\sigma], R[[x,\sigma]]$ denote the skew polynomial ring and the skew power series ring, where $xa = \sigma(a)x$ for all $a \in R$, while the polynomial ring and power series ring over R is denoted by $R[x]$ and $R[[x]]$ respectively. Let σ^* denotes the endomorphism on $R[[x]]$ such that $\sigma^*(\sum_{j=0}^{\infty} a_j x^j) = \sum_{j=0}^{\infty} \sigma(a_j)x^j$. And in the next paragraph, $A_r(R[x,\sigma])(A_r(R[[x,\sigma]]))$ denotes $A_r(R[x,\sigma],\sigma^*)(A_r(R[[x,\sigma]],\sigma^*))$.

Lemma 3.1. *Let R be a ring , if $e(x) = \sum_{j=0}^{\infty} a_j x^j \in E(R[[x]])$, then $a_0 \in E(R)$ and $a_1 \in A(R)$.*

Proof. $e^2(x) = e(x)$, then $e_0^2 = e_0$ and $e_0e_1 + e_1e_0 = e_1$, thus the conclusion follows. More generally, if $e(x) = a_0 + \sum_{j=k}^{\infty} a_j x^j \in E(R[[x]])$, then $a_0 \in E(R)$ and $a_k \in A(R)$.

Theorem 3.2. *Let R be a ring with an endomorphism σ, then the following statements are equivalent:*

(1) *R is abelian and $\sigma(e) = e$ for any $e \in E(R)$;*
(2) *$E(R) = E(R[x,\sigma]) = E(R[[x,\sigma]])$;*
(3) *$R[x,\sigma]$ is abelian;*
(4) *$R[[x,\sigma]]$ is abelian;*
(5) *$A_r(R) = A_r(R[x,\sigma]) = A_r(R[[x,\sigma]])$.*

Proof. (1) $\Rightarrow$ (2) Clearly $E(R) \subseteq E(R[x,\sigma]) \subseteq E(R[[x,\sigma]])$, it is enough to show that $E(R[[x,\sigma]]) \subseteq E[R]$. for each $e(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x,\sigma]]$, by Lemma 3.1, $e_0 \in E(R)$, $e_1 \in A_r(R) = 0 = A(R)$, since R is abelian. By the same proceeding we have $a_2 = a_3 = ... = a_n = 0$.

(2) $\Rightarrow$ (4) For each $ef(x)(1-\sigma(e))$, where $f(x) \in R[[x,\sigma]]$, let $g(x) = e + ef(x)(1-\sigma(e))x$, it is easy to find that $g^2(x) = g(x) \in E(R) = E(R[[x,\sigma]])$, thus $ef(x)(1-\sigma(e)) = 0$ and $eR[x,\sigma](1-e) = 0$, therefore $R[[x,\sigma]]$ is abelian.

It is obvious (4) $\Rightarrow$ (3) $\Rightarrow$ (1), since it is well known that abelian property is closed by subset, and since $R[x,\sigma]$ is abelian, then for any $e \in E(R)$, $ex = xe = \sigma(e)x$, thus $\sigma(e) = e$.

(4) $\Rightarrow$ (5) $R[[x,\sigma]]$ is abelian, then $0 \subseteq A_r(R) \subseteq A_r(R[x,\sigma]) \subseteq A_r(R[[x,\sigma]]) = 0$, thus the conclusion follows.

(5) $\Rightarrow$ (1) For each $a \in A_r(R)$ such that $ea = a(1-\sigma(e))$, $ax \in A(R[[x,\sigma]]) \subseteq A(R[[x,\sigma]]) = A_r(R)$, thus $a = 0$, and $A_r(R) = 0$. Therefore R is abelian and $\sigma(e) = e$ for any $e \in E(R)$.

The next example will show that the condition $\sigma(e) = e$ is not surplus

Example 3.3. Let $R = \bigotimes_{i=0}^{\infty} Z_4$, where Z_4 denotes the ring of integers modulo 4, and let σ be an endomorphism of R such that $\sigma((a_0, a_1, a_2, ...)) = (0, a_0, a_1, a_2, ...)$, $a_i \in Z_4$. Let $i \geq 1$ be an integer and e_i denote the element $(a_0, a_1, a_2, ...)$ such that $a_{i-1} = 1$, while the others is 0. Obviously, $e_i \in E(R)$ and $\sigma(e_i) = e_{i+1}$ for each i.

Consider $f_k(x) = \sum_{i=0}^{k} e_i x^j$.

$$f_k^2(x) = \sum_{s=0}^{2k} \sum_{i+j=s} e_i \sigma^i(e_j) x^s = \sum_{s=0}^{2k} \sum_{i+j=s} e_i e_s x^s$$
$$= \sum_{s=0}^{k} e_s x^s = f_k(x) \in E([R,\sigma]) - E(R).$$

Similarly Let $f(x) = \sum_{i=0}^{\infty} e_i x^j$, we can also get $f^2 = f$ by the same proceeding. Thus $f(x) \in E(R[[x,\sigma]]) - E(R[x,\sigma])$. Choose $g(x) = (2e_0 + 3e_1)x$, $e_0 g(x) = 2e_0 x$ while $g(x)e_0 = 3e_1 x \neq e_0 g(x)$. So $R[x,\sigma]$, $R[[x,\sigma]]$ is not abelian.

When σ is an identical endomorphism, we can immediately get:

Corollary 3.4. *Let R be a ring. Then the following statements are equivalent:*
(1) *R is abelian;*
(2) $E(R) = E(R[x]) = E(R[[x]])$;
(3) *$R[x]$ is abelian;*
(4) *$R[[x]]$ is abelian;*
(5) $A(R) = A(R[x]) = A(R[[x]])$.

Given a ring R and a $R-R-$bimodule M, the trivial extension of R by M is the ring $T(R,M) = \{(a,b) \mid a \in R, b \in R\}$ with the usual addition and the multiplication $(a_1,b_1)(a_2,b_2) = (a_1a_2, a_1b_2 + a_2b_1)$ where $a_i, b_i \in R$. Let $n(\geq 2)$ be an integer, $T_n(R)$ denotes $T(R, T_{n-1}(R))$. It is well known that $T_n(R)$ is isomorphic to the ring $R[x]/(x^n)$ which is also isomorphic to of all matrices

$$P_n(R) = \{ \begin{bmatrix} a & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a & a_{12} & \dots & a_{1n-1} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & a & a_{12} \\ 0 & 0 & 0 & \dots & a \end{bmatrix} \mid a, a_{ij} \in R\}.$$

By the same method we have:

Corollary 3.5. *Let R be a ring. Then the following statements are equivalent:*
(1) *R is abelian;*
(2) $E(R) = E(R[x]/(x^n)) = E(T_n(R)) = E(P_n(R))$;
(3) *$R[x]/(x^n)$ is abelian;*
(4) *$T_n(R)$ is abelian;*
(5) *$P_n(R)$ is abelian;*
(6) $A(R) = A(R[x]/(x^n)) = A(T_n(R)) = A(P_n(R))$.

More generally, let

$$R_n = \{ \begin{bmatrix} a & a_{12} & \dots & a_{1n} \\ & a & \dots & a_{2n} \\ & & \ddots & \vdots \\ & & & a \end{bmatrix} \mid a, a_{ij} \in R\},$$

and the abelian property of R can also be held by R_n.

Theorem 3.6. *Let R be a ring. Then the following statements are equivalent:*

(1) *R is abelian;*

(2) *Let $\sigma : R \to R_n$ be a homomorphism such that $\sigma(a) = aI$ where $I \in R_n$ is unit matrix. Then $\sigma(E(R)) = E(R_n)$;*

(3) *R_n is abelian;*

(4) *$\sigma(A(R)) = A(R_n)$.*

Proof. (1) $\Rightarrow$ (2) It is enough to show $E(R_n) \subseteq E(R)$. Since R is abelian, then $A(R) = 0$. Let $E = \begin{bmatrix} a & a_{12} & ... & a_{1n} \\ & a & ... & a_{2n} \\ & & \ddots & \vdots \\ & & & a \end{bmatrix} \in E(R_n)$. We have $a \in E(R)$, and $a_{12} = aa_{12} + a_{12}a$, then $a_{12} \in A(R) = 0$, then we have $a_{13} = aa_{13} + a_{13}a$, by the same proceeding we can get $a_{13} = 0$, repeating this process we will finally get $a_{12} = a_{13} = a_{14} = ... = a_{23} = ...a_{2n} = ... = a_{nn-1} = 0$, so the conclusion follows.

(2) $\Rightarrow$ (3) For each $r \in R, e \in E(R)$, Let $A = \begin{bmatrix} 0 & 0 & ... & r \\ & 0 & ... & 0 \\ & & \ddots & \vdots \\ & & & 0 \end{bmatrix}$. Let $B = eI + eA(1 - e)$, then $B^2 = B \in E(R_n) = E(\sigma(R))$. Thus, $er(1 - e) = 0$, this indicate that for each $C \in R$, $eC(1-e) = 0$, therefore the conclusion follows.

(3) $\Rightarrow$ (1), (3) $\Rightarrow$ (4) Obviously.

(4) $\Rightarrow$ (1) For each $a \in A(R)$, let $A = \begin{bmatrix} 0 & 0 & ... & a \\ & 0 & ... & 0 \\ & & \ddots & \vdots \\ & & & 0 \end{bmatrix} \in E(R_n)$, clearly $A \in A(R_n) = \sigma(A(R))$, thus $a = 0$ and $A(R) = 0$. Therefore R is abelian.

The next example will show that R_n in Theorem 3.6 could not be replaced by $U_n(R) = \{ \begin{bmatrix} a_1 & * & \dots & * \\ & a_2 & \dots & * \\ & & \ddots & \vdots \\ & & & a_n \end{bmatrix} \mid a_i \in R\}$ when R is abelian.

Example 3.7. Let $R = Z$, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in U_2(Z)$, it is clear that R is abelian and $A \in U_2(Z)$, while A could not commute with B, thus $U_n(R)$ is not abelian.

Proposition 3.8. Let R be a ring and $S = \{\begin{bmatrix} a & 0 & 0 \\ b & a & c \\ 0 & 0 & a \end{bmatrix} \mid a, b, c \in R\}$, then the following statements are equivalent:

(1) R is abelian;

(2) Let $\sigma : R \to S$ be a homomorphism such that $\sigma(a) = aI$ where $I \in R_n$ is unit matrix. $\sigma(E(R)) = E(S)$;

(3) S is abelian;

(4) $\sigma(A(R)) = A(S)$.

Proof. (1) $\Rightarrow$ (2) Let $E^2 = E = \begin{bmatrix} a & 0 & 0 \\ b & a & c \\ 0 & 0 & a \end{bmatrix} \in R$. Then

$$\begin{bmatrix} a^2 & 0 & 0 \\ ba + ab & a^2 & ac + ca \\ 0 & 0 & a^2 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ b & a & c \\ 0 & 0 & a \end{bmatrix},$$

thus $a \in E(R), b, c \in A(R) = 0$ Since R is abelian. Thus $E = aI \in \sigma(E(R))$.

(2) $\Rightarrow$ (3) For each $r \in R, e \in E(R)$, and let $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & r \\ 0 & 0 & 0 \end{bmatrix}$, let $B = eI + eA(1 - e)$, then $B^2 = B \in E(S) = E(\sigma(R))$. Thus $r = 0$, this indicates that for each $C \in S$, $eC(1 - e) = 0$ and therefore S is abelian.

(3) $\Rightarrow$ (1), (3) $\Rightarrow$ (4) It is obvious.

(4) $\Rightarrow$ (1) For each $a \in A(R)$, Let $A = \begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then $A \in A(S) = A(\sigma(R))$, thus $a = 0$ and $A(R) = 0$. Therefore R is abelian.

Let R be a ring, and σ be a monomorphism of R. Recall that the Nagata extension of R by R and σ is the ring $N(R, R, \sigma) = \{(a, b) \mid a \in R, b \in R\}$ with the usual addition and multiplication $(a_1, b_1)(a_2, b_2) = (a_1a_2, \sigma(a_1)b_2 + a_2b_1)$ where $a_i, b_i \in R$.

Proposition 3.9. Let R be a ring, and let σ be a monomorphism of R. Then the following statements are equivalent:

(1) R is abelian, $\sigma(e) = e$ for any $e \in E(R)$;

(2) Let $\phi : R \to S$ be a homomorphism such that $\phi(a) = (a, 0)$. $\phi(E(R)) = E(N(R, R, \sigma))$;

(3) $N(R, R, \sigma)$ is abelian;

(4) $\phi(A_l(R)) = A_l(N(R, R, \sigma))$.

Proof. (1) $\Rightarrow$ (2) Since R is abelian, then $A(R) = 0$. For each $(a, b) \in E(R)$, we have $(a^2, \sigma(a)b + ab) = (a^2, 2ab) = (a, b)$, then $a \in E(R)$ and $b \in A(R) = 0$, thus $(a, b) = (a, 0) \in \phi(E(R))$. The conclusion follows.

(2) $\Rightarrow$ (3) For each $r \in R, e \in E(R)$ consider $(e, \sigma(e)r(1-e))$, we have $(e, \sigma(e)r(1-e))^2 = (e, \sigma(e)r(1-e)) \in E(N(R,R,\sigma)) = E(\phi(R))$, thus $\sigma(e)r(1-e) = 0$, therefore $N(R,R,\sigma)$ is abelian.

(3) $\Rightarrow$ (1), (3) $\Rightarrow$ (4) Obviously.

(4) $\Rightarrow$ (1) For each $a \in A(R)$, $(0,a) \in A(N(R,R,\sigma)) \subseteq A_l(N(R,R,\sigma)) = \phi(A_l(R))$, thus $a = 0$ and $A_l(R) = 0$. Therefore R is ablian and $\sigma(e) = e$ for any $e \in E(R)$.

Let R be an algebra over a commutative ring S. The Dorroh extension of R by S is the ring $D(R,S) = \{(r,s) \mid r \in R, s \in S\}$ with usual addition and multiplication $(a,s)(b,t) = (ab + ta + sb, st)$ where $a, b \in$R and $s, t \in S$

Let R be a ring, and Consider $E_c(R)$ =$\{a \in R \mid a^2 = ua$, for some $u \in UC(R)\}$, where $UC(R)$ denotes the subset of central invertible element of R. Clearly $E(R) \subseteq E_c(R)$.

Lemma 3.10. *Let R be a ring. Then the following statements are equivalent:*

(1) *R is abelian;*

(2) $E_c(R) \subseteq C(R)$.

Proof. (2) $\Rightarrow$ (1) Obviously.

(1) $\Rightarrow$ (2) Since $(ua)^2 = u^2a^2 = ua$ and R is abelian, then $ua \in C(R)$, because $u \in UC(R)$, then $a \in C(R)$.

Proposition 3.11. Let R be a ring, which is an algebra over a commutative ring S. Then the following statements are equivalent:

(1) R is abelian;

(2) $D(R,S)$ is abelian.

Proof. Let (e,s) be an idempotent of $D(R,S)$, $(e,s)^2 = (e,s)$ and we have $(e^2 + 2se, s^2) = (e,s)$, then $e^2 = e(1-2s), s^2 = s$, by Lemma 3.10, $e \in C(R)$, thus $D(R,S)$ is abelian.

4. Abelian rings with homomorphisms

Let R and S be rings, $\phi : R \longrightarrow S$ and $\varphi : S \longrightarrow R$ are ring homomorphisms. At the end, We give the next Theorem on abelian properties between R and S.

Theorem 4.1. *Let R and S be rings, $\phi : R \longrightarrow S$ and $\varphi : S \longrightarrow R$ are ring homomorphisms such that $\phi\varphi = 1_s$, Then:*

(1) *If R is ablian, then so is S;*

(2) *If $\varphi\phi(E(S)) = E(R)$, and S is abelian, then so is R.*

Proof. (1) Clearly ϕ is an epimorphism. For each $r \in R$, $er(1-e) = \phi\varphi(er(1-e)) = 0$, since R is abelian. Thus S is abelian.

(2) Firstly we show $R = \varphi\phi(R) \oplus Ker\phi$. For each $a \in R$, $a = a - \varphi\phi(a) + \varphi\phi(a)$. Clearly $\varphi\phi(a) \in \varphi\phi(R)$ and $\phi(a - \varphi\phi(a)) = 0$ so that $a - \varphi\phi(a) \in$

$Ker\phi$. For every $b \in \varphi\phi(R) \cap Ker\phi$, there exists an element $c \in R$ such that $b = \varphi\phi(c)$ and $\phi(b) = \phi\varphi\phi(c) = \phi(c) = 0$, then $b = \varphi\phi(c) = \varphi(0) = 0$, thus $R = \varphi\phi(R) \oplus Ker\phi$.

Let $a = r - \varphi\phi(r) \in Ker\phi, b = \varphi\phi(r) \in \varphi\phi(R)$, clearly $r = a + b$, consider $c = e + er(1-e) = e + ea(1-e) + eb(1-e) = e + ea(1-e), c^2 = c$, since $\varphi\phi(R) = \varphi(S) \cong S$ is abelian. Thus $c = e+ea(1-e) \in E(R) = \varphi(E(S)) \in \varphi(S)$, and $ea(1-e) = c - e \in \varphi(S)$. Note that $ea(1-e) \in Ker\phi$, since $Ker\phi$ is an ideal of R. Therefore $ea(1-a) = 0$, $er(1-e) = 0$, then R is abelian.

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