Bifurcation of subharmonic in Lassa fever epidemic model

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Abstract. Standard epidemiological theory and concepts such as the basic reproductive number R_0 no longer apply or enough to determine stability, and the implications for interventions that themselves may be periodic have not been formally examined. This study considers seasonal variation of rodent on the transmission dynamics of Lassa fever. Applying perturbation method, stable subharmonic bifurcation solutions of period n are proved to coexist simultaneously and an infinite number of stable subharmonic bifurcation solutions are established. This study suggests that transmission of Lassa fever can be curbed by fighting the proliferation of rodent especially during the wet season.

Keywords: seasonal variation, subharmonic bifurcation, stability, reproduction number.

1. Introduction

Infectious diseases pose a constant threat to human well. Every individual on the earth can be affected by a disease. Proper understanding of transmission mechanism of diseases caused by existing and new pathogens may facilitate devising prevention tools. Prevention tools against transmission, including drugs and vaccines need to be developed at the similar peace to that of the microbes. Seasonal variations in rainfall, temperature and resource availability are ubiquitous and can exert strong pressures on population dynamics. Usually, vector such as rodent, mosquito, snail etc are many during the raining season depending on the lengthen of rainfall.

Lassa fever is an acute viral illness that occurs in West Africa. The reservoir, or host, of Lassa virus is a rodent known as the "multimammate rat" of

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the genus Mastomys. According to the World Health Organization, 300,000 to 500,000 cases of Lassa fever and 5000 deaths occur yearly across West Africa [2]. The symptoms of Lassa fever are: flu-like illness characterized by fever, general weakness, cough, sore throat, headache, and abdominal pain. Depending on transmission intensity, the main complication of Lassa fever after recovery is deafness. Interestingly, recovered human has permanent immunity against Lassa virus and therefore remain recovered for life [7].

Now, we briefly give account of existing literatures on subharmonic bifurcation in SIR and SEIR model. Ira and Smith formulated a mathematical model that incur permanent immunity with seasonal variations in the contact rate and analysed rigorously. They conclude that random effects in the environment could perturb the state of the dynamics from the domain of attraction from one subharmonic to another.

From the idea of Ira and Smith model, Chow and Shaw [3] considered a piecewise linear second order ordinary dfferential equation which is topo- logically equivalent to the sine-Gordon equation. Their model was subjected to a time harmonic disturbance and the behaviour of the periodic solutions was examined. The results of their analysis revealed that that subharmonic motions with period n times that of the disturbance appear via saddle-node bifurcation for n = 1,2,3,...

The work by Ira and smith was extended by Kuznetsov and Piccardi [6], they investigated the bifurcation of the periodic solutions of SEIR and SIR epidemic models with sinusoidally varying contact rate. They demonstrated by numerical simulations that the parameter portrait of the SEIR model undergoes significant structural changes when the latent period is varied.

From another perspective, Schaffer and Bronnikova discussed the bifurcation structure of SEIR model subject to seasonality. Their results suggest that combination of phenomenological equation which admit to mathematical analysis and detailed simulation will prove a recipe for progress.

In a related work, Rachel et al studied how seasonal forcing influence the system when the unforced dynamic have either monotonic, oscillatory decay or stable cycle. Their results shows that the level of oscillation in the unforced system has a larger effect on the range of behaviour when the system is seasonally forced.

Gouhei and Kazuyuki [4] examined the impact of seasonal variation pattern on epidemic dynamics. Their results suggest that accurately estimated seasonal variability is necessary for better understanding the dynamics of infectious diseases.

To the best of our knowledge no work has been done on mathematical study of Lassa fever that incorporates seasonal variation of rodent to investigate the existence of subharmonic bifurcation.

2. Model formulation

The model presented here is an extension work by [8]. The model subdivides the total human population size at time t and discrete age a_i denoted by $N_h(t,a_i)$ with $i=0,1,2,...,L$ and a_L is the maximum age of humans in the population, into susceptible humans $S_h(t,a_i)$, exposed humans $E_h(t,a_i)$, infectious humans $I_h(t,a_i)$ and recovered humans $R_h(t,a_i)$. Hence we have $N_h(t,a_i)=S_h(t,a_i)+E_h(t,a_i)+I_h(t,a_i)+R_h(t,a_i)$. The term

$$\begin{aligned}\beta_f(t) = &\frac{\sigma_2(a_i)(1+\eta(a_i)\cos(2\pi t+T))S_h(t,a_i)I_h(t,a_i)}{N_h(t,a_i)}\\ &+\frac{\sigma_1(e_j)(1+\rho(a_i)\cos(2\pi t+T))S_h(t,a_i)I_r(t,e_j)}{N_h(t,a_i)}\\ &+\frac{(1+\theta(e_j)\cos(2\pi t+T))S_h(t,a_i)}{N_h(t,a_i)}\\ &+\frac{(1+\kappa(a_i)\cos(2\pi t+T))S_h(t,a_i)}{N_h(t,a_i)}\end{aligned}$$

denotes the rate at which the human hosts get infected by infectious human, rodent, inhalation of aerosol from urine or through reused needle when taking injection. The infectious human $I_h(t,a_i)$ gain individuals when exposed individuals becomes infectious and loses individual when they die $\mu_h(a_i)I_h(t,a_i)$, $\delta_h(a_i)I_h(t,a_i)$. After some time, exposed and infectious human recovers and moves to the recovered class $R_h(t,a_i)$. However recovered human has permanent immunity and never go back to susceptible class again. A loss of individuals is as a result of natural death $\mu_h R_h(t,a_i)$. It is assumed that $\theta(e_j)\propto\frac{C_v}{K_v}$ and $\theta(e_j)$ is generated from urine and faeces of infectious rodents, where C_v is the amount of virus in air and K_v is the saturation of virus in air. Similarly, $\kappa(a_i)$ is generated from blood of infectious individuals, $\kappa(a_i)\propto\frac{A_v}{S_v}$, where A_v is the amount of virus in needle and S_v is the saturation of virus in needle.

In similar manner, we subdivides the total rodent population size at time t and discrete age e_j denoted by $N_r(t,e_j)$ with $j=0,1,2,...,T$ and e_T is the maximum age of rodents in the population, into susceptible rodents $S_r(t,e_j)$, exposed rodents $E_r(t,e_j)$ and infectious rodents $I_r(t,e_j)$. Hence we have $N_r(t,e_j)=S_r(t,e_j)+E_r(t,e_j)+I_r(t,e_j)$. Susceptible rodent class $S_r(t,e_j)$ gain more individual into rodent population by input rate $\Lambda_r(e_j)(1+\phi_r(e_j)\cos 2\pi(t+T))$, while it loses rodents through natural death $\mu_r(e_j)S_r(t,e_j)$, hunting $\delta_r(e_j)S_r(t,e_j)$.

$$\begin{aligned}\gamma_f(t) = &\frac{\sigma_1(e_j)(1+\beta(e_j)\cos(2\pi t+T))S_r(t,e_j)I_r(t,e_j)}{N_r(t,e_j)}\\ &+\frac{(1+\theta(e_j)\cos(2\pi t+T))S_r(t,e_j)}{N_r(t,e_j)}\end{aligned}$$

denote the rate at which the rodent get infected by infectious rodent or inhalation of aerosol from urine. Transmission of Lassa virus to susceptible rodents occurs when they share unprotected storage of garbage, food stuff and water with infected rodents or from inhalation of aerosols from urine. When a susceptible rodent interact with infectious rodent, the virus enters the rodent with probability $\beta(e_j)$ and therefore the susceptible go to the exposed class $E_r(t, e_j)$. The exposed rodent then becomes infectious and enters the class $I_r(t, e_j)$ after a given time. . In this study, it is assumed that individuals who recovered from Lassa fever will never go back to susceptible class again (they remain recovered for life).

$$\begin{aligned}
&\frac{dS_h(t,a_i)}{dt}\\
&= \Lambda_h(a_i) - \sum_{i=0}^{L}\sum_{j=0}^{T} \frac{\sigma_2(a_i)(1+\eta(a_i)\cos(2\pi t+T))S_h(t,a_i)I_h(t,a_i)}{N_h(t,a_i)}\\
&+ \frac{\sigma_1(e_j)(1+\rho(a_i)\cos(2\pi t+T))S_h(t,a_i)I_r(t,e_j)}{N_h(t,a_i)}\\
&+ \frac{(1+\theta(e_j)\cos(2\pi t+T))}{N_h(t,a_i)} \times S_h(t,a_i)\\
&+ \frac{(1+\kappa(a_i)\cos(2\pi t+T))S_h(t,a_i)}{N_h(t,a_i)} - \mu_h(a_i)S_h(t,a_i),
\end{aligned} \tag{1}$$

$$\begin{aligned}
\frac{dE_h(t,a_i)}{dt} &= \sum_{i=0}^{L}\sum_{j=0}^{T} \frac{\sigma_2(a_i)(1+\eta(a_i)\cos(2\pi t+T))S_h(t,a_i)I_h(t,a_i)}{N_h(t,a_i)}\\
&+ \frac{\sigma_1(e_j)(1+\rho(a_i)\cos(2\pi t+T))S_h(t,a_i)I_r(t,e_j)}{N_h(t,a_i)}\\
&+ \frac{(1+\theta(e_j)\cos(2\pi t+T))}{N_h(t,a_i)} \times S_h(t,a_i)\\
&+ \frac{(1+\kappa(a_i)\cos(2\pi t+T))S_h(t,a_i)}{N_h(t,a_i)}\\
&- (\gamma(a_i)\alpha_1(a_i) + \epsilon_h(a_i) + \mu_h(a_i))E_h(t,a_i),
\end{aligned} \tag{2}$$

$$\frac{dI_h(t,a_i)}{dt} = \sum_{i=0}^{L} \epsilon_h(a_i)E_h(t,a_i) - (\psi(a_i)\alpha_2(a_i) + \mu_h(a_i) + \delta_h(a_i))I_h(t,a_i), \tag{3}$$

$$\frac{dR_h(t,a_i)}{dt} = \sum_{i=0}^{L} \gamma(a_i)\alpha_1(a_i)E_h(t,a_i) + \psi(a_i)\alpha_2(a_i)I_h(t,a_i) - \mu_h(a_i)R_h(t,a_i), \tag{4}$$

$$\begin{aligned}
\frac{dS_r(t,e_j)}{dt} &= \Lambda_r(e_j)(1+\phi_r(e_j)\cos(2\pi t+T))\\
&+ \sum_{j=0}^{T} \frac{\sigma_1(e_j)(1+\beta(e_j)\cos(2\pi t+T))}{N_r(t,e_j)} \times S_r(t,e_j)I_r(t,e_j)\\
&+ \frac{(1+\theta(e_j)\cos(2\pi t+T))S_r(t,e_j)}{N_r(t,e_j)} - (\mu_r(e_j) + \delta_r(e_j))S_r(t,e_j),
\end{aligned} \tag{5}$$

$$\frac{dE_r(t,e_j)}{dt} = \sum_{j=0}^{T} \frac{\sigma_1(e_j)(1+\beta(e_j)\cos(2\pi t+T))S_r(t,e_j)I_r(t,e_j)}{N_r(t,e_j)} + \frac{(1+\theta(e_j)\cos(2\pi t+T))S_r(t,e_j)}{N_r(t,e_j)} - (\epsilon_r(e_j)+\mu_r(e_j)+\delta_r(e_j))E_r(t,e_j), \tag{6}$$

$$\frac{dI_r(t,e_j)}{dt} = \sum_{j=0}^{T} \epsilon_r(e_j)E_r(t,e_j) - (\mu_r(e_j)+\delta_r(e_j))I_r(t,e_j) \tag{7}$$

together with the initial conditions

$$\begin{aligned} &S_h(0,a_i) = S_{0h}(a_i), E_h(0,a_i) = E_{0h}(a_i),\\ &I_h(0,a_i) = I_{0h}(a_i), R_h(0,a_i) = R_{0h}(a_i),\\ &S_r(0,e_j) = S_{0r}(e_j), E_r(0,e_j) = E_{0r}(e_j), I_r(0,e_j) = I_{0r}(e_j). \end{aligned} \tag{8}$$

3. Model analysis

Here we present theoretical results and numerical simulation of the model presented above.

3.1 Existence and positivity of solutions

Lassa fever model governed by system (1)-(7) is mathematical well-posed in a feasible region $\mathcal{D}$ defined by $\mathcal{D} = \mathcal{D}_h \times \mathcal{D}_r \subset \mathcal{R}^4_+ \times \mathcal{R}^3_+$ where

$$\mathcal{D}_h = \left\{ S_h(t,a_i), E_h(t,a_i), I_h(t,a_i), R_h(t,a_i) \in \mathcal{R}^4_+ : N_h \leq \sum_{i=0}^{L} \frac{\Lambda_h(a_i)}{\mu_h(a_i)} \right\}$$

and

$$\mathcal{D}_r = \left\{ S_r(t,e_j), E_r(t,e_j), I_r(t,e_j), \in \mathcal{R}^3_+ : N_r \leq \sum_{j=0}^{T} \frac{\Lambda_r(e_j)(1+\phi_r(|\cos T|+|\sin T|))}{\mu_r(e_j)+\delta_r(e_j)} \right\}.$$

3.2 Disease-free equilibrium π_0

Disease-free equilibrium points are steady-state solutions where there is no Lassa fever infection. Thus, the disease-free equilibrium point, π_{01} for the Lassa fever model (1)-(7) implies that $S^*_h(a_i) \neq 0, E^*_h(a_i) = I^*_h(a_i) = 0, S^*_r \neq 0, E^*_r = I^*_r = 0$ and putting these into (1), (4) and (5) yields $R^*_h(a_i) = 0, S^*_h(a_i) = \frac{\Lambda_h(a_i)}{\mu_h(a_i)}$, and $S^*_r(e_j) = \frac{\Lambda_r(e_j)(1+\phi_r(e_j)\cos(2\pi t+T))}{\mu_r(e_j)+\delta_r(e_j)}$ respectively. Consequently, we obtain π_0 as

$$\pi_0 = \left(\frac{\Lambda_h(a_i)}{\mu_h(a_i)}, 0, 0, 0, \frac{\Lambda_r(e_j)(1+\phi_r(e_j)\cos(2\pi t+T)}{\mu_r(e_j)+\delta_r(e_j)}, 0, 0 \right). \tag{9}$$

Using the next generation matrix technique as described in [7], we obtain the threshold parameter known as basic reproduction number, $\mathcal{R}_0$, as

$$\mathcal{R}_0(a,t)=\sum_{i=0}^{L}\frac{\sigma_2(a_i)(1+\eta(a_i)\cos(2\pi t+T))\epsilon_h(a_i)}{(\gamma(a_i)\alpha_1(a_i)+\epsilon_h(a_i)+\mu_h(a_i))(\psi(a_i)\alpha_2(a_i)+\mu_h(a_i)+\delta_h(a_i))}. \tag{10}$$

3.3 Endemic equilibrium point E_{e2}

Here, we shall show that the formulated model (1)-(7) has an endemic equilibrium point, E_e. The endemic equilibrium point is a positive steady state solution where the disease persists in the population.

Theorem 1. *Lassa fever model (1) - (7) has no endemic equilibrium when $\mathcal{R}_0(a,t)<1$ and a unique endemic equilibrium exist when $\mathcal{R}_0(a,t)>1$.*

Proof. Let $E_e=(S_h^{**},E_h^{**},I_h^{**},R_h^{**},S_r^{**},E_r^{**},I_r^{**})$ be a non-trivial equilibrium of system (1) -(2.7). That is, all component of E_e are positive. Then the Lassa fever model (1)-(7) at steady state becomes

$$\begin{aligned}I_r^{**}(e_j)&=\Lambda_r(e_j)(1+\phi_r\cos(2\pi t+T))\epsilon_r(e_j)\sigma_1(e_j)(1+\beta(e_j)\cos(2\pi t+T)\\&\quad-((1+\theta\cos(2\pi t+T))\Lambda_r(e_j)(1+\phi_r\cos(2\pi t+T))\\&\quad\times(\mu_r(e_j)+\delta_r(e_j))(\epsilon_r(e_j)+\mu_r(e_j)+\delta_r(e_j))+P,\end{aligned} \tag{11}$$

$$\begin{aligned}&E_r^{**}(e_j)\\&=\sum_{j=0}^{T}\frac{\sigma_1(e_j)(1+\beta(e_j)\cos(2\pi t+T)I_r^{**}(e_j)+(1+\theta(e_j)\cos(2\pi t+T))\Lambda_{0r}}{(\epsilon_r(e_j)+\mu_r(e_j)+\delta_r(e_j))[\sigma_1(e_j)(1+\beta(e_j)\cos(2\pi t+T))I_r^{**}(e_j)+E_{or}]}\end{aligned} \tag{12}$$

and

$$\begin{aligned}&S_r^{**}(e_j)\\&=\frac{\Lambda_r^2(e_j)(1+\phi_r(e_j)\cos(2\pi t+T))}{\sum_{j=0}^{T}[\sigma_1(e_j)(1+\beta(e_j)\cos(2\pi t+T))I_r^{**}(e_j)+(1+\theta(e_j)\cos(2\pi t+T))+\Lambda_{0r}]r_d},\end{aligned} \tag{13}$$

$$\begin{aligned}R_h^{**}(a_i)&=\frac{\sum_{i=0}^{L}(\psi(a_i)\alpha_2(a_i)+\mu_h(a_i)+\delta_h(a_i))\gamma(a_i)\alpha_1(a_i)I_h^{*}(a_i)}{\mu_h(a_i)\sum_{i=0}^{L}\epsilon_h(a_i)}\\&\quad+\frac{\psi(a_i)\alpha_2(a_i)I_h^{**}(a_i)}{\mu_h(a_i)},\end{aligned} \tag{14}$$

$$E_h^{**}(a_i) = \frac{I_h^{**}(a_i)(\psi(a_i)\alpha_2(a_i) + \mu_h(a_i) + \delta_h(a_i))}{\sum_{i=0}^{L} \epsilon_h(a_i)} \tag{15}$$

and

$$S_h^{**}(a_i) = \frac{I_h^{**}(a_i)\Lambda_h(a_i)(\psi(a_i)\alpha_2(a_i)+\mu_h(a_i)+\delta_h(a_i))(\gamma(a_i)\alpha_1(a_i)+\mu_h(a_i)+\epsilon_h(a_i))}{\sum_{i=0}^{L}\sum_{j=0}^{T}\mu_h(a_i)\epsilon_h(a_i)U}, \tag{16}$$

where

$$\begin{aligned} U = {} & [\sigma_1(e_j)(1+\rho(a_i)\cos(2\pi t+T))\Lambda_{0r}\epsilon_r(e_j)\sigma_1(1+\beta(e_j)\cos(2\pi t+T)) \\ & - (1+\theta(e_j)\cos(2\pi t+T)) + \Lambda_{0r}\sigma_1(e_j)(1+\rho(a_i)\cos(2\pi t+T))(\mu_r(e_j) \\ & + \delta_r(e_j))(\epsilon_r(e_j)+\mu_r(e_j)+\delta_r(e_j)) + \sigma_1(e_j)(1+\rho(a_i)\cos(2\pi t+T))P \\ & + \sigma_2(a_i)(1+\eta(a_i)\cos(2\pi t+T))I_h^{**}(a_i) \\ & + (1+\theta(e_j)\cos(2\pi t+T)) + (1+\kappa(a_i)\cos(2\pi t+T))], \end{aligned}$$

$$P = \frac{\sqrt{[(\theta_0+\Lambda_{0r})(\mu_r(e_j)+\delta_r(e_j))(\epsilon_r(e_j)+\mu_r(e_j)+\delta_r(e_j)) - \Lambda_{0r}\epsilon_r(e_j)\beta_0]^2 + M}}{2\beta_0(e_j)(\mu_r(e_j)+\delta_r(e_j))(\epsilon_r(e_j)+\mu_r(e_j)+\delta_r(e_j))},$$

$$\begin{aligned} & \Lambda_{0r} = \Lambda_r(e_j)(1+\phi_r(e_j)\cos(2\pi t+T)), \\ & \beta_0 = \sigma_1(e_j)(1+\beta(e_j)\cos(2\pi t+T)), \theta_0 = (1+\theta(e_j)\cos(2\pi t+T)), \\ & r_d = (\mu_r(e_j)+\delta_r(e_j)), E_{0r} = (1+\theta(e_j)\cos(2\pi t+T)) + \Lambda_{0r}, \end{aligned}$$

with

$$\begin{aligned} M = {} & 4\sigma_1(e_j)(1+\beta(e_j)\cos(2\pi t+T))\Lambda_{0r}\epsilon_r(e_j) \\ & \times (1+\theta(e_j)\cos(2\pi t+T))(\mu_r(e_j)+\delta_r(e_j))(\epsilon_r(e_j)+\mu_r(e_j)+\delta_r(e_j)), \end{aligned}$$

$$K_1(I_h^{**})^2 + K_2 I_h^{**} + K_3 = 0, \tag{17}$$

where

$$\begin{aligned} K_1 = {} & \sigma_2(a_i)(1+\eta(a_i)\cos(2\pi t+T))\mu_h(a_i), \\ K_2 = {} & \sigma_1(e_j)(1+\rho(a_i)\cos(2\pi t+T)(1+\beta(e_j)\cos(2\pi t+T))\epsilon_r(e_j)\mu_h(a_i)\Lambda_{0r} \\ & + \mu_h(a_i)\sigma_1(e_j)(1+\beta(e_j)\cos(2\pi t+T))((1+\theta(e_j)\cos(2\pi t+T)) \\ & + \Lambda_{0r})(\mu_r(e_j)+\delta_r(e_j)) \times (\epsilon_r(e_j)+\mu_r(e_j)+\delta_r(e_j)) \\ & + \mu_h(a_i)\sigma_1(e_j)(1+\rho(a_i)\cos(2\pi t+T))P \\ & + \mu_h(a_i)((1+\theta(e_j)\cos(2\pi t+T)) + (1+\kappa(a_i)\cos(2\pi t+T)) \\ & + \Lambda_h(a_i)\mu_h(a_i)(1-\mathcal{R}_0(a,t)), \\ K_3 = {} & d_1 - (d_2+d_3), \end{aligned}$$

with

$$d_1 = \frac{\begin{array}{l}\mu_h(a_i)\epsilon_h(a_i)\Lambda_h(a_i)\sigma_1(e_j)(1 \quad +\rho(a_i)\cos(2\pi t+T))(\theta_0+\Lambda_{0r})(\mu_r(e_j) \\ \qquad\qquad\qquad\qquad\qquad +\delta_r(e_j))(\epsilon_r(e_j)+\mu_r(e_j)+\delta_r(e_j))\end{array}}{(\psi(a_i)\alpha_2(a_i)+\mu_h(a_i)+\delta_h(a_i))(\gamma(a_i)\alpha_1(a_i)+\mu_h(a_i)+\epsilon_h(a_i))},$$

$$d_2 = \frac{\begin{array}{l}\Lambda_h(a_i)H_0\Lambda_{0r}\epsilon_r(e_j)(1 \quad +\beta(e_j)\cos(2\pi t+T))\mu_h(a_i)\epsilon_h(a_i) \\ \qquad\qquad\qquad\qquad +H_0\mu_h(a_i)\epsilon_h(a_i)\Lambda_h(a_i)P\end{array}}{(\psi(a_i)\alpha_2(a_i)+\mu_h(a_i)+\delta_h(a_i))(\gamma(a_i)\alpha_1(a_i)+\mu_h(a_i)+\epsilon_h(a_i))},$$

$$d_3 = \frac{\Lambda_h(a_i)\epsilon_h(a_i)\mu_h(a_i)((1+\theta(e_j)\cos(2\pi t+T))+(1+\kappa(a_i)\cos(2\pi t+T))}{(\psi(a_i)\alpha_2(a_i)+\mu_h(a_i)+\delta_h(a_i))(\gamma(a_i)\alpha_1(a_i)+\mu_h(a_i)+\epsilon_h(a_i))}$$

$H_0 = \sigma_2(a_i)(1+\rho(a_i)\cos(2\pi t+T))$. It is clearly seen that $K_1 > 0$. When $\mathcal{R}_0(a,t) < 1$, one sees that $K_2 > 0$. However, when $R_0(a,t) > 1$, then $K_2 < 0$ and endemic equilibrium exists. Finally, $K_3 < 0$ if $d_1 < (d_2 + d_3)$. □

3.4 Subharmonic bifurcation analysis

Here, we are interested in the case that the rodent birth and contact rate are periodic of period one year. For this reason it is convenient to take a year as our unit of time. With this convention, the human birth and death rate, $\Lambda_h(a_i)$, $\mu_h(a_i)$ and rodent birth and death rate $\Lambda_r(e_j)(1+\phi_r\cos(2\pi t+T))$ and μ_r will be such that $\frac{1}{\Lambda_h(a_i)}$, $\frac{1}{\mu_h(a_i)}$, $\frac{1}{\Lambda_r(e_j)(1+\phi_r\cos(2\pi t+T))}$ and $\frac{1}{\mu_r(e_j)}$ are 50 years, the latent period prior to becoming infectious $\frac{1}{\mu_h(a_i)+\epsilon_h(a_i)+\gamma(a_i)\alpha_1(a_i)}$, $\frac{1}{\mu_r(e_j)+\delta_r(e_j)+\epsilon_r(e_j)}$ for human and mosquito are typically a few days to a week, the same is true for average infectious period $\frac{1}{\mu_h(a_i)+\delta(a_i)+\psi(a_i)\alpha_2(a_i)}$ and $\frac{1}{\delta_r(e_j)+\mu_r(e_j)}$. We exploit the fact that $\mu_h(a_i)$, $\Lambda_h(a_i)$, $\mu_r(e_j)$, $\Lambda_r(e_j)(1+\phi_r(e_j)\cos(2\pi t+T))$, $\frac{1}{\mu_h(a_i)+\epsilon_h(a_i)+\gamma(a_i)\alpha_1(a_i)}$, $\frac{1}{\mu_h(a_i)+\delta(a_i)+\psi(a_i)\alpha_2(a_i)}$, $\frac{1}{\mu_r(e_j)+\epsilon_r(e_j)}$ and $\frac{1}{\delta_r(e_j)+\mu_r(e_j)}$ are $0(10^{-2})$ by introducing a small parameters ε as follows:

$$\begin{aligned}
& I_h^{**}(a_i) = \Lambda_h(a_i) \\
& \quad = \frac{I_h^{**}(a_i)\Lambda_h(a_i)(\psi(a_i)\alpha_2(a_i)+\mu_h(a_i)+\delta_h(a_i))\gamma^0\mathcal{R}_{02}^2(a,t)}{\mu_h(a_i)\epsilon_h(a_i)U(\mathcal{R}_{02}^2(a,t)-1)} = \varepsilon, \\
& \mu_h(a_i) = \frac{I_h^{**}(a_i)}{\mathcal{R}_{02}^2(a,t)-1} = \varepsilon, \\
& \epsilon_h(a_i)+\mu_h(a_i)+\gamma(a_i)\alpha_1(a_i) \\
& \quad = \frac{I_h^{**}(a_i)\Lambda_h(a_i)\epsilon_h(a_i)}{(\psi(a_i)\alpha_2(a_i)+\mu_h(a_i)+\delta_h(a_i))\mu_h(a_i)\mu_r(e_j)\mathcal{R}_{01}^2(a,t)} = \frac{\Delta_2}{\varepsilon}, \\
& \psi(a_i)\alpha_2(a_i)+\delta_h(a_i)+\mu_h(a_i) = \frac{\Delta_3}{\varepsilon}, \qquad (18)
\end{aligned}$$

$$I_r^{**}(e_j)=\mu_r(e_j)=[\mu_r(e_j)+\sigma_1(e_j)(1+\beta(e_j)\cos(2\pi t+T)))I_r^{**}(e_)+\theta^0]\mu^0=\varepsilon,$$
$$\epsilon_r(e_j)+\mu_r(e_j)+\delta_r(e_j)=\frac{\epsilon_r(e_j)+\mu_r(e_j)+\delta_r(e_j)}{(1+\beta\cos(2\pi t+T))+E_{0r}}=\frac{\Delta_4}{\varepsilon},$$
$$\mu_r(e_j)+\delta_r(e_j)=\Lambda_r(e_j)(1+\phi_r(e_j)\cos(2\pi t+T))\epsilon(e_j)\sigma^0=\frac{\Delta_5}{\varepsilon},$$

where $\gamma^0=(\gamma(a_i)\alpha_1(a_i)+\mu_h(a_i)+\epsilon_h(a_i))$, $\theta^0=(1+\theta(e_j)\cos(2\pi t+T))+\Lambda_{0r}$, $\mu^0=\mu_r(e_j)+\delta_r(e_j)$, $\sigma^0=\sigma_1(e_j)(1+\beta(e_j)\cos(2\pi t+T))-(1+\theta(e_j)\cos(2\pi t+T)(\mu_r(e_j)+\delta_r(e_j))(\epsilon_r(e_j)+\mu_r(e_j)+\delta_r(e_j))+P$. New state variables (x,y,z,p,q,r) are introduced by setting

$$\begin{aligned}
&S_h(t,a_i)=S_h^{**}(a_i)(1+x^0), E_h(t,a_i)=E_h^{**}(a_i)(1+y^0),\\
&I_h(t,a_i)=I_h^{**}(a_i)(1+z^0),\\
&S_r(t,E_j)=S_r^{**}(e_j)(1+p^0), E_r(t,e_j)=E_r^{**}(e_j)(1+q^0),\\
&I_r(t,e_j))=I_r^{**}(e_j)(1+r^0).
\end{aligned}\tag{19}$$

Incorporating (18) and (19) in model (1)-(7) yield the following system of equations which has the property that when $\rho(a_i)=\eta(a_i)=\kappa(a_i)=\beta(e_j)=\theta(e_j)=0$ and $\phi_r=0$, the endemic equilibrium becomes $(x^0,y^0,z^0,p^0,q^0,r^0)=0$

$$\begin{aligned}
\dot{x}^0&=-\varepsilon[(\Theta+(\rho(a_i)+\eta(a_i)+\kappa(a_i)\theta(e_j))\cos(2\pi t+T))x^0\\
&+(1+(\rho(a_i)+\eta(a_i)+\kappa(a_i)\theta(e_j))\cos(2\pi t+T))z^0\\
&+(\rho(a_i)+\eta(a_i)+\kappa(a_i)\theta(e_j))\cos(2\pi t+T)\\
&+(1+(\rho(a_i)+\eta(a_i)+\kappa(a_i)\theta(e_j))\cos(2\pi t+T)x^0z^0],\\
\dot{y}^0&=\frac{\Delta_{02}}{\varepsilon}[(\rho(a_i)+\eta(a_i)+\kappa(a_i)\theta(e_j))\cos(2\pi t+T)\\
&+z^0(1+(\rho(a_i)+\eta(a_i)+\kappa(a_i)\theta(e_j))\cos(2\pi t+T))\\
&+x^0(1+(\rho(a_i)+\eta(a_i)+\kappa(a_i)\theta(e_j))\cos(2\pi t+T))\\
&+r^0x^0(1+(\rho(a_i)+\eta(a_i)+\kappa(a_i)\theta(e_j))\cos(2\pi t+T))-y^0],\\
\dot{z}^0&=\frac{\Delta_{03}}{\varepsilon}[y^0-z^0],\\
\dot{p}^0&=-\varepsilon[p^0+1+(\beta(e_j)+\theta(e_j))\cos(2\pi t+T)\\
&+p^0(1+(\beta(e_j)+\theta(e_j))\cos(2\pi t+T))\\
&+r^0(1+(\beta(e_j)+\theta(e_j))\cos(2\pi t+T))\\
&+p^0z^0(1+(\beta(e_j)+\theta(e_j))\cos(2\pi t+T))],\\
\dot{q}^0&=\frac{\Delta_0}{\varepsilon}[(\beta(e_j)+\theta(e_j))\cos(2\pi t+T)\\
&+p^0(1+(\beta(e_j)+\theta(e_j))\cos(2\pi t+T))
\end{aligned}\tag{20}$$

$$+ r^0(1 + (\beta(e_j) + \theta(e_j))\cos(2\pi t + T))$$
$$+ p^0 z^0(1 + (\beta(e_j) + \theta(e_j))\cos(2\pi t + T)) - q^0],$$
$$\dot{r}^0 = \frac{\Delta_{05}}{\varepsilon}[q^0 - r^0],$$

where $\Theta = \frac{\mathcal{R}_0(a,t)}{\mathcal{R}_0(a,t)-1}$. Before proceeding further in analyzing the periodic system (20) we need information concerning the eigenvalues of the linearized system about the endemic steady state when $\rho(a_i) = \eta(a_i) = \kappa(a_i) = \beta(e_j) = \theta(e_j) = 0$. For $\rho(a_i) = \eta(a_i) = \kappa(a_i) = \beta(e_j) = \theta(e_j) = 0$ the endemic steady state is the origin and we have the following lemma.

Lemma 1. *The eigenvalues corresponding to the linearized system*

$$\begin{pmatrix} x^0 \\ y^0 \\ z^0 \end{pmatrix}' = \begin{pmatrix} -\varepsilon\Theta & 0 & -\varepsilon \\ \frac{\Delta_{02}}{\varepsilon} & -\frac{\Delta_{02}}{\varepsilon} & \frac{\Delta_{02}}{\varepsilon} \\ 0 & \frac{\Delta_{03}}{\varepsilon} & -\frac{\Delta_{03}}{\varepsilon} \end{pmatrix} \begin{pmatrix} x^0 \\ y^0 \\ z^0 \end{pmatrix}$$

are given by $\lambda_+, \lambda_-, \lambda_3$ *below:*$\lambda_\pm(\varepsilon) = \varepsilon r_1 \pm i\nu + 0(\varepsilon^2)$, *where*

$$\nu = \sqrt{\frac{\Delta_{02}\Delta_{03}}{\Delta_2 + \Delta_3}},$$
$$r_1 = \frac{\Delta_{02}\Delta_{03} - (\Delta_{02} + \Delta_0 3)^2\eta}{2(\Delta_{02} + \Delta_{03})^2} < 0, \tag{21}$$
$$\lambda_3(\varepsilon) = -\frac{(\Delta_{02} + \Delta_{03})}{\varepsilon} + 0(\varepsilon).$$

We observed that the endemic steady state is asymptotically stable but the attraction is weak. There is a rapid relaxation onto a center manifold on which orbits slowly spiral into the origin. The underlying mechanism we will exploit is the weakly damped oscillation on the center manifold. to leading order, the frequency of this oscillation, ν depends on the reproduction rate $\mathcal{R}_0(a,t)$, the birth rate $\Lambda(a_i)$ and the sum of the exposed and infectious period.

Our goal at this point is to make change of variable in (20) when $\rho(a_i), \eta(a_i), \kappa(a_i), \beta(e_j), \theta(e_j)$ are equal to zero. In addition, further analysis will require the assumption of a small perturbation in (2.20). Inspection of (2.20) shows that this will be the case only if $\frac{\rho(a_i)}{\varepsilon}, \frac{\eta(a_i)}{\varepsilon}, \frac{\kappa(a_i)}{\varepsilon}, \frac{\theta(e_j)}{\varepsilon}$ are small. With this remarks as motivation we proceed as follows. Let

$$\bar{\rho}(a_i), \bar{\eta}(a_i), \bar{\kappa}(a_i), \bar{\theta}(e_j) = \frac{\rho(a_i)}{\varepsilon}, \frac{\eta(a_i)}{\varepsilon}, \frac{\kappa(a_i)}{\varepsilon}, \frac{\theta(e_j)}{\varepsilon},$$
$$\bar{x}^0 = v\left[\frac{x^0}{\varepsilon} - \frac{\varepsilon\Delta_{03}(z^0 - y^0)}{(\Delta_{02} + \Delta_{03})^2}\right], \tag{22}$$
$$\bar{y}^0 = \frac{\Delta_{03}y^0 + \Delta_{02}z^0}{\Delta_{02} + \Delta_{03}},$$
$$\bar{z}^0 = z^0 - y^0,$$

where ν is given in Lemma 1. While this transformation appears to mix up the various epidemiological classes, a glance at (20) indicates that one should expect $z^0 - y^0$ to be small, say order ε, hence (22) should take the particularly simple form

$$\begin{aligned} x^0 &= \frac{\varepsilon \bar{x}^0}{\nu} + 0(\varepsilon^3), \\ y^0 &= \bar{y}^0 + 0(\varepsilon), \\ z^0 &= \bar{y}^0 - 0(\varepsilon). \end{aligned} \tag{23}$$

Epidemiologically, equation (23) has the interpretation that the ratio of infectives to exposed individuals is $\frac{\psi(a_i)lpha_2(a_i)}{\epsilon_h(a_i)}$ to order ε.

Putting (22) in (20) and ignoring the bar over $\rho(a_i), \eta(a_i), \kappa(a_i), \beta(e_j), \theta(e_j)$ we obtain

$$\begin{aligned} (\bar{x}^0)' &= -\nu \bar{y}^0 + \varepsilon f_{01}(\bar{x}^0, \bar{y}^0, \bar{z}^0, t, \varepsilon, \rho(a_i), \eta(a_i), \kappa(a_i), \theta(e_j)), \\ (\bar{y}^0)' &= \nu \bar{x}^0(1 + \bar{y}^0) + \frac{\nu \Delta_{03} \bar{x}^0 \bar{z}^0}{\Delta_{02} + \Delta_{03}} + (\nu)^2(\rho(a_i) + \eta(a_i) + \kappa(a_i) + \theta(e_j)) \\ &\quad \times \cos(2\pi t + T)\left(1 + \bar{y}^0 + \frac{\Delta_{03}\bar{z}^0}{\Delta_{02} + \Delta_{03}}\right) \\ &\quad + \varepsilon f_{02}(\bar{x}^0, \bar{y}^0, \bar{z}^0, t, \varepsilon, \rho(a_i), \eta(a_i), \kappa(a_i), \theta(e_j)), \\ \varepsilon(\bar{z}^0)' &= -(\Delta_{02} + \Delta_{03})\bar{z}^0 + \varepsilon f_{03}(\bar{x}^0, \bar{y}^0, \bar{z}^0, t, \varepsilon, \rho(a_i), \eta(a_i), \kappa(a_i), \theta(e_j)), \end{aligned} \tag{24}$$

where

$$\begin{aligned} & f_{01}(\bar{x}^0, \bar{y}^0, \bar{z}^0, t + T, \pi, \varphi^0, \varepsilon) = -\bar{x}^0\left(\Theta - \frac{\Delta_{02}\Delta_{03}}{(\Delta_{02} + \Delta_{03})^2}\right) \\ & - \bar{x}\left(\bar{y}^0 - \frac{\Delta_{03}\bar{z}^0}{\Delta_{02} + \Delta_{03})}\right)\left(1 - \frac{\Delta_{02}\Delta_{03}}{(\Delta_{02} + \Delta_{03})^2}\right) + 0(|\varepsilon| + \varphi^0|), \\ & f_{02}(\bar{x}^0, \bar{y}^0, \bar{z}^0, t+T, \pi, \varphi^0, \varepsilon) = \frac{\nu^2\Delta_{03}\bar{z}^0}{(\Delta_{02}+\Delta_{03})^2}\left(1+\bar{y}^0 - \frac{\Delta_{03}\bar{z}^0}{\Delta_{02}+\Delta_{03})}\right) + 0(|\varepsilon| + \varphi^0|), \\ & f_{03}(\bar{x}^0, \bar{y}^0, \bar{z}^0, t+T, \pi, \varphi^0, \varepsilon) = -\Delta_{02}\nu^{-1}\bar{x}^0\left(1+\bar{y}^0 - \frac{\Delta_{03}\bar{z}^0}{\Delta_{02}+\Delta_{03})}\right) + 0(|\varepsilon| + |\varphi^0|), \end{aligned}$$

where $\varphi^0 = \rho(a_i), \eta(a_i), \kappa(a_i), \theta(e_j)$ We will treat ε and φ^0 as small parameters in (24). Setting $\varepsilon = \varphi^0 = 0$ in (24) we obtain the reduced equations

$$\begin{aligned} (\bar{x}^0)' &= -\nu \bar{y}^0, \\ (\bar{y}^0)' &= \nu \bar{x}^0(1 + \bar{y}^0), \\ \bar{z}^0 &= 0. \end{aligned} \tag{25}$$

Now,we show that the system (25) is conservative with first integral given by

$$V = (x^0)^2 + 2y^0 - 2\ln|1 + y^0|.$$

A system is conservative if there a function $V(x^0, y^0)$ such that $\frac{dV}{dt} = 0$ along the solution curves of x^0 and y^0.

$$\frac{dV}{dt} = \frac{dV}{dx^0}\frac{dx^0}{dt} + \frac{dV}{dy^0}\frac{dy^0}{dt},$$
$$\frac{dV}{dt} = 2x^0(-\nu y^0) + \left[2 - \frac{2}{1+y^0}\right](\nu x^0(1+y^0)),$$
$$\frac{dV}{dt} = -2x^0\nu y^0 + 2\nu x^0(1+y^0) - 2\nu x^0 = 0.$$

It is also interesting to observe that if we write $u = \ln(1 + y^0)$ and use (4.2.41) to write a second order differential equation for u we obtain

$$y^0 = e^u - 1, u' = \frac{\nu x^0(1+y^0)}{1+y^0}, u' = \nu x^0, u'' + \nu^2(e^u - 1) = 0.$$

The essential feature the case when $\varepsilon = \varphi_h^0 = 0$ using (25) is that the origin is a center surrounded by periodic orbit with periods ranging between $\frac{2\pi}{\nu}$ near the origin to $+\infty$. In particular, for every integer $n, \frac{2\pi}{\nu} < \nu < \infty$, there is a periodic solution of (25), $(\bar{x}_n^0(t), \bar{y}_n^0(t))$ of at least period n. These solutions of the reduced equation may be excited by period one forcing at the correct amplitude. Hence, we are led to expect n-periodic solution of (23) near $\Gamma_{n1}^0 = (x^0, y^0, z^0) : (x^0, y^0, z^0) = (\bar{x}_n^0(t), \bar{y}_n^0(t), 0), 0 \le t \le n)$ at least for a suitable small values of ε and φ^0. Similar argument holds for ε, $\beta(e_j), \theta(e_j)$ and $\phi_r(e_j)$ in the rodent population. Thus, we have the following results:

Theorem 2. *Let $\bar{x}_n^0(t), \bar{y}_n^0(t) = \bar{x}_n^0(t+n), \bar{y}_n^0(t+n)$ denote a periodic solution of the reduced equation (25), where $n > \frac{2\pi}{\nu}$. Let*

$$\psi_2(a_i)\alpha_2(a_i) \equiv \nu^2 \int_0^n \bar{y}^0(t)\cos(2\pi t + T)dt \ne 0, \psi_2(a_i)\alpha_2(a_i) = \frac{-2r_1}{\nu}.$$

(Area interior to Γ_{n1}) and for $\epsilon_h(a_i)\varepsilon[0, n), |\varepsilon| \ll 1, |\varphi^0| \ll 1$, let

$$\begin{aligned}\mathbf{B}(\epsilon_h(a_i), \varepsilon, \varphi^0) &= \psi_1(a_i)\alpha_2(a_i)\varepsilon + \psi_2(a_i)\alpha_2(a_i)\varphi^0 \cos 2\pi\epsilon_h(a_i)\\ &\quad + 0(|\varepsilon| + |\varphi^0|)|)^2.\end{aligned} \tag{26}$$

If $(\bar{x^{0}}'_n, \bar{y^{0}}'_n)$ spans the n- periodic of the variational equations of (25) about $(\bar{x}'_n, \bar{y^{0}}'_n)$, and $(\epsilon_h(a_i), \varepsilon, \varphi^0 = 0$, then equation (24) has an n-periodic solution $(\bar{x}^0, \bar{y}^0, \bar{z}^0)$ given by

$$\begin{aligned}\bar{x}^0(t) &= \bar{x}_n^0(t + \epsilon_h(a_i) + 0(|\varepsilon| + (1 + |\varphi^0|)),\\ \bar{y}^0(t) &= \bar{x}_n^0(t + \epsilon_h(a_i) + 0(|\varepsilon| + |\varphi^0|),\\ \bar{z}^0(t) &= -\frac{\varepsilon\Delta_{02}\bar{y^0}'(t + \epsilon_h(a_i))}{\nu^2(\Delta_{02} + \Delta_{03})} + 0(|\varepsilon| + |\varphi^0|)^2.\end{aligned} \tag{27}$$

Table 2. Description of the model parameters

Definition	Parameter	Value
Recruitment term of humans	$\Lambda_h(a_i)$	0.038
Effective transmission rate in susceptible humans by infected rodents	$\rho(a_i)$	0.6
Interacting rate of reservoirs	$\sigma_1(e_j)$	0.8
Effective transmission rate in susceptible humans by infected humans	$\eta(a_i)$	0.6
Interacting rate of humans	$\sigma_2(a_i)$	0.56
Treatment rate of exposed humans	$\alpha_1(a_i)$	0.05
Treatment rate of infected humans	$\alpha_2(a_i)$	0.9
Effective contact rate between Lassa virus and humans or reservoirs	$\theta(e_j)$	0.022
Rate of inoculation	$\kappa(a_i)$	0.018
Progression rate of humans from the exposed state to the infectious state	$\epsilon_h(a_i)$	0.85
Diagnostic factor of exposed humans	$\gamma(a_i)$	0.9
proportion of effective treatment of infected humans	$\psi(a_i)$	0.45
Natural death rate of humans	$\mu_h(a_i)$	0.02
Progression rate of reservoirs from the exposed state to infectious state	$\epsilon_r(e_j))$	0.85
Effective transmission rate in susceptible reservoirs by infected reservoirs	$\beta(e_j)$	0.75
Disease induced death rate of humans	$\delta_h(e_j)$	0.2
Mortality of reservoirs due to hunting	$\delta_r(e_j)$	0.3
Natural death rate reservoirs	$\mu_r(e_j)$	0.6
Recruitment term of reservoirs	$\Lambda_r(e_j)$	0.56
Seasonal variation of rodent factor	$\phi_r(e_j)$	Assumed to vary

4. Numerical results

The model (1)-(7) is simulated using parameters in Table 1 to illustrate some of the theoretical results established in this study. Figures 1, and 2 show the impact of time harmonic disturbance for rodent birth and Lassa fever transmission to humans. It is noticed that rodent birth and Lassa fever transmission varies seasonally, depending on the influence of climatic variables.

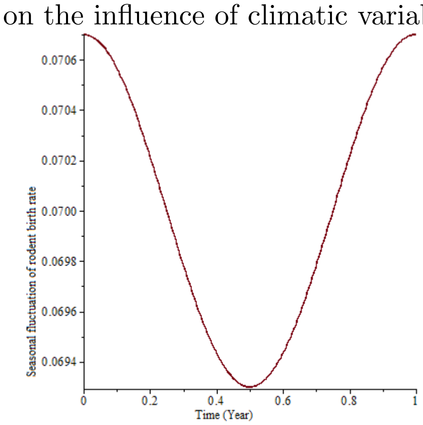

Figure 1: Seasonal fluctuation of rodents birth rate

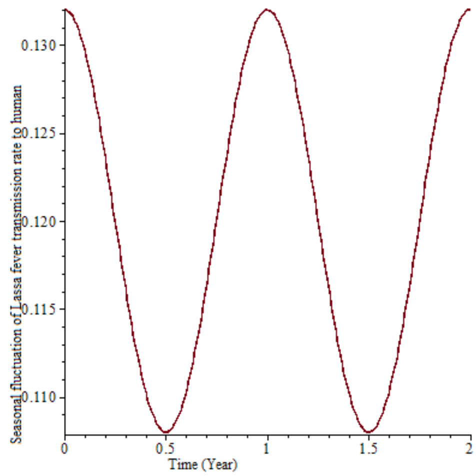

Figure 2: Seasonal fluctuation of Lassa fever transmission rate to humans

5. Conclusion

Seasonal change in the incidence of infectious diseases is a common phenomenon in both temperate and tropical climates. However, the mechanisms responsible for seasonal disease incidence, and the epidemiological consequences of seasonality, are poorly understood with rare exception. This study highlighted a compartmental modelling for the transmission dynamics of Lassa fever considering time harmonic disturbance for rodent (seasonal variation of rodent birth and transmission rate) with focus on two populations: Human and vector. The model was analysed for endemic equilibrium, the basic reproduction number $R_0(a,t)$ is obtained, and local stability of the non-trivial equilibrium was determined. Applying perturbation methods, stable subharmonic bifurcation solutions of period n are proved to coexist simultaneously and infinite number of stable subharmonic bifurcation solutions were established. Numerical simulations are carried out to investigate the behaviour of the system. This study suggests that intervention strategies needs to be deployed especially during the wet season. Also, every effort must be put in place by all concerned to prevent the virus infection by reducing the basic reproduction number, $R_0(a,t)$ until Lassa fever eradication is achieved.

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