On automatic surjectivity of some point spectrum preserving additive maps

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Abstract. Let X and Y be two infinite dimensional complex Banach spaces and let $\Phi : B(X) \longrightarrow B(Y)$ be a point spectrum preserving additive map. We show that if the range of Φ contains the ideal of all finite rank operators of $B(Y)$ then Φ is a Jordan morphism. In the case where X is an infinite dimensional separable Hilbert space the map Φ is surjective.

Keywords: Linear preserver problem, point spectrum, rank one operators, operator algebra.

1. Introduction

The problem of characterizing linear or additive maps on matrix or operator algebras that leave invariant a given subset, function or relation defined on the underlying algebras represents one of the most active research areas; see for example [1], [2], [7], [8], [13], [14], [15] and the references therein. It has been proved by B. Aupetit in [2] that if $\Phi : \mathcal{M}_n(\mathbb{C}) \longrightarrow B$ is a spectrum preserving surjective additive map, (where B is a semi simple algebra of dimension $\leq n^2$) then either $\Phi(x) = uxu^{-1}$ or $\Phi(x) = ux^t u^{-1}$ for every $x \in \mathcal{M}_n(\mathbb{C})$ where u is an invertible element in B. He asked the question: what happens if such map is not surjective?

Many researchers are applied to generalize the result of B. Aupetit in the case of the algebras of linear bounded operators over complex Banach spaces but all theirs maps are supposed to be surjectives.

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In this paper we propose to have the same result of B. Aupetit always in the case of the algebras of linear bounded operators over complex Banach spaces but with weaker hypotheses such that Φ is an additive map preserving only the point spectrum and it's range contains the ideal of all finite rank operators.

Let X and Y are two infinite dimensional complex Banach spaces and $B(X)$ and $B(Y)$ are the algebras of linear bounded operators on X and Y respectively. For an operator $T \in B(X)$ we denote by $\sigma_p(T)$ the point spectrum of T which is the set $\{\lambda \in \mathbb{C}\lambda.I - T$ is not injective$\}$ where I is the identity of $B(X)$. It is said that a map $\Phi : B(X) \longrightarrow B(Y)$ preserves the point spectrum if $\sigma_p(\Phi(T)) = \sigma_p(T)$ for every $T \in B(X)$.

Every rank one operator in $B(X)$ can be written as $x \otimes f$ for some $x \in X - \{0\}$ and $f \in X' - \{0\}$ and defined by $(x \otimes f)y = f(y)x$ for every $y \in X$. The operator $x \otimes f$ is idempotent if and only if $f(x) = 1$ moreover $\sigma_p(x \otimes f) \subset \{0; f(x)\}$ (see [7] and [8]). The operator $x \otimes f + y \otimes g$ is of rank one if and only if either x and y are linearly dependent or f and g are linearly dependent for every $x, y \in X - \{0\}$ and $f, g \in X' - \{0\}$. We denote by $F(X)$ and $F_1(X)$ the ideal of finite rank operators and the set of rank one operators in $B(X)$ respectively. The range of Φ will be denoted by $Im\Phi$ and the operator $\lambda.I$ by λ.

2. Main results

The aim of this work is to prove the following results:

Theorem 2.1. *Let $\Phi : B(X) \longrightarrow B(Y)$ be a point spectrum preserving additive map such that $F(Y) \subset Im\Phi$. Then either*

i) $\Phi(T) = ATA^{-1}$ for every $T \in B(X)$ where $A : X \longrightarrow Y$ is a bounded bijective map; or

ii) $\Phi(T) = BT^*B^{-1}$ for every $T \in B(X)$ where $B : X' \longrightarrow Y$ is a bounded bijective linear map.

Corollary 2.1. *Let X be an infinite dimensional separable Hilbert space and Y be an infinite dimensional Banach space. If $\Phi : B(X) \longrightarrow B(Y)$ be a point spectrum preserving additive map such that $F(Y) \subset Im\Phi$ then, Φ is surjective.*

Lemma 2.1. *Let $A \in B(X)$. We have $\sigma_p(T + A) \subset \sigma_p(T)$ for every $T \in F_1(X)$ if and only if $A = 0$.*

Proof. If $A = 0$, it is clear that $\sigma_p(T + A) \subset \sigma_p(T)$ for every $T \in F_1(X)$.

For the reverse implication we assume that $A \neq 0$. In the case where $A = I$ we take $x \in X$ and $f \in X'$ such that $f(x) = 1$ and $T = x \otimes f$. We have $2 \in \sigma_p(T + A)$ but $\sigma_p(T) \subset \{0; 1\}$. In the case where $A \neq I$ there exist $x \in X$ such that $0 \neq Ax \neq x$. For $y = Ax$ there exists $f \in X'$ such that $f(x) = 1$ and $f(y) \neq 0$. If we take the operator $T = (x - y) \otimes f$ we have $\sigma_p(T) \subset \{0; f(x - y)\}$ and $1 \in \sigma_p(T + A)$ but $f(x - y) \neq 1$. Then $\sigma_p(T + A) \not\subset \sigma_p(T)$.

Lemma 2.2. *let $\Phi : B(X) \longrightarrow B(Y)$ be a point spectrum preserving additive map. Then Φ is injective.*

Proof. Let $A \in B(X)$ such that $\Phi(A) = 0$. We have $\sigma_p(T) = \sigma_p(\Phi(T)) = \sigma_p(\Phi(T) + \Phi(A)) = \sigma_p(T + A)$, for every $T \in F_1(X)$. By the lemma 2.1 we conclude that $A = 0$ and therefore Φ is injective.

Lemma 2.3. *let $\Phi : B(X) \longrightarrow B(Y)$ be a point spectrum preserving additive map such that $F(Y) \subset Im\Phi$. Then $\Phi(\lambda) = \lambda$ for every $\lambda \in \mathbb{C}$.*

Proof. Let $S \in F_1(Y)$ and $T \in B(X)$ such that $\Phi(T) = S$. We have for every $\lambda \in \mathbb{C}$:

$$\begin{aligned} \sigma_p(S + \Phi(\lambda) - \lambda) &= \sigma_p((\Phi(T) + \Phi(\lambda)) - \lambda \\ &= \sigma_p(T + \lambda) - \lambda \\ &= \sigma_p(T) \\ &= \sigma_p(\Phi(T)) \\ &= \sigma_p(S). \end{aligned}$$

We apply the Lemma 2.1 to conclude that $\Phi(\lambda) = \lambda$.

Lemma 2.4. *Let R be non zero operator in $B(X)$. Then the following conditions are equivalent:*
i) R is of rank 1,
ii) $\sigma_p(T + R) \cap \sigma_p(T + 2R) \subset \sigma_p(T)$ for every $T \in F(X)$.

Proof. Let us prove that i) implies ii). Let $R \in B(X)$ be an operator of rank one. Suppose that there exists $T \in F(X)$ such that $\sigma_p(T + R) \cap \sigma_p(T + 2R) \not\subset \sigma_p(T)$. Then there exists $\lambda \in \mathbb{C}$ such that $\lambda - T$ is injective and $\lambda - T - R$ and $\lambda - T - 2R$ are not injective operators. Hence there exist two non zero vectors x and y such that $(\lambda - T)(x) = Rx$ and $(\lambda - T)(y) = 2R(y)$. It is clear that $R(x) \neq 0$ and $R(y) \neq 0$ and the vectors $R(x)$ and $R(y)$ are linearly dependent because R is of rank one. Let α a non zero complex number such that $R(x) = \alpha R(y)$. The vector $z = x - \frac{\alpha}{2}y$ satisfy $z \neq 0$ and $(\lambda - T)(z) = 0$ witch contradicts the fact that $\lambda - T$ is injective.

We prove now that ii) implies i). Assume that ii) is satisfied and $rankR > 1$. Suppose that there exists $x \in X$ such that x, Rx and R^2x are linearly independent. Define the operator S by:

$$\begin{aligned} Sx &= 3x - Rx \\ SRx &= 3Rx - 2R^2x \\ SR^2x &= x \\ Sz &= 0 \text{ pour } z \in L \end{aligned}$$

where L is a closed complement of the span of $\{x, Rx, R^2x\}$. We have $S \in B(X)$ and $3 \in \sigma_p(S+R) \cap \sigma_p(S+2R)$ but $3 \notin \sigma_p(S)$. Thus x, Rx, and R^2x are linearly dependent for every $x \in X$. So there exists a quadratic polynomial p such that $p(R) = 0$. This polynomial has one of the forms $p(t) = (t - \alpha)(t - \beta)$ or

$p(t)=(t-\alpha)^2$ or $p(t)=t(t-\alpha)$ or $p(t)=t^2$ where $\alpha\neq 0\neq\beta\neq\alpha$. By the standard decomposition of algebraic operators and since $rankR>1$ we can affirm that R has a finite dimensional invariant subspace W such that, R/W has a matrix representation

$$\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix} \text{ or } \begin{pmatrix}\alpha & 1\\ 0 & \alpha\end{pmatrix} \text{ or } \begin{pmatrix}\alpha & 0 & 0\\ 0 & \alpha & 0\\ 0 & 0 & 0\end{pmatrix} \text{ or } \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\oplus\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},$$

respectively. We consider a complement Z of W in X and a operator T such that $T/Z=0$ and T/W has matrix representation

$$\begin{pmatrix}-3\alpha/2 & \beta/2\\ \alpha/2 & -3\beta/2\end{pmatrix} \text{ or } \begin{pmatrix}1 & -1\\ \alpha^2 & 1\end{pmatrix} \text{ or } \begin{pmatrix}0 & 0 & 0\\ 0 & 2\alpha & 2\alpha\\ 0 & -2\alpha & -2\alpha\end{pmatrix}$$
$$\text{or } \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\oplus\begin{pmatrix}0 & 0\\ 2 & 0\end{pmatrix},$$

respectively. We have $T\in F(X)$ and there exists $\lambda\in\sigma_p(T+R)\cap\sigma_p(T+2R)$ but $\lambda\notin\sigma_p(T)$, namely $\lambda=0$ or $\lambda=\alpha+1$ or $\lambda=2\alpha$ or $\lambda=\sqrt{2}$.

Lemma 2.5. *Let $\Phi: B(X)\longrightarrow B(Y)$ be a point spectrum preserving additive map such that $F(Y)\subset Im\Phi$ and $R\in B(X)$. Then $\Phi(R)$ is of rank 1 if and only if R is of rank 1.*

Proof. Let $R\in B(X)$. If R is of rank one we apply the Lemma 2.4 to have $\sigma_p(T+R)\cap\sigma_p(T+2R)\subset\sigma_p(T)$ for every $T\in F(X)$. Since Φ is additive and preserves the point spectrum we have $\sigma_p(\Phi(T)+\Phi(R))\cap\sigma_p(\Phi(T)+2\Phi(R))\subset\sigma_p(\Phi(T))$ for every $T\in F(X)$. By the hypothesis $F(Y)\subset Im\Phi$ we have $\sigma_p(S+\Phi(R))\cap\sigma_p(S+2\Phi(R))\subset\sigma_p(S)$ for every $S\in F(Y)$. We apply the Lemma 2.4 to conclude that $\Phi(R)$ is of rank one.

To prove the reverse case we assume that $\Phi(R)$ is of rank one and $T\in F(X)$. Since T is a finite sum of some rank one operators we have $\Phi(T)\in F(Y)$. we apply the Lemma 2.4 to get $\sigma_p(S+\Phi(R))\cap\sigma_p(S+2\Phi(R))\subset\sigma_p(S)$ for every $S\in F(Y)$. We take $S=\Phi(T)$ to have $\sigma_p(\Phi(T)+\Phi(R))\cap\sigma_p(\Phi(T)+2\Phi(R))\subset\sigma_p(\Phi(T))$. Since Φ preserves the point spectrum we have $\sigma_p(T+R)\cap\sigma_p(T+2R)\subset\sigma_p(T)$ and by the Lemma 2.4 we establish that R is of rank one.

Lemma 2.6. *Let $\Phi: B(X)\longrightarrow B(Y)$ be a point spectrum preserving additive map such that $F(Y)\subset Im\Phi$. Then Φ is 1-homogeneous over $F_1(X)$ (i.e: $\Phi(\lambda T)=\lambda\Phi(T)$ for every $\lambda\in\mathbb{C}$ and $T\in F_1(X)$).*

Proof. Every rank one operator can be written of the form $x\otimes f$ where $0\neq x\in X$ and $0\neq f\in X'$. So it suffices to proof that $\Phi(\alpha x\otimes f)=\alpha\Phi(x\otimes f)$ for every $\alpha\in\mathbb{C}$, $x\in X-\{0\}$ and $f\in X'-\{0\}$.

If $\alpha=0$ or $\alpha=-1$ we have $\Phi(\alpha x\otimes f)=\alpha\Phi(x\otimes f)$ because Φ is additive.

If $\alpha \neq 0$ and $\alpha \neq -1$ we start by the case where $f(x) \neq 0$ in which case we can take $f(x) = 1$. By the Lemma 2.5 the operator $\Phi(x \otimes f)$ is of rank one so, there exist $u \in Y$ and $\varphi \in Y^{'}$ such that $\Phi(x \otimes f) = u \otimes \varphi$. Since Φ preserves the point spectrum we have $\varphi(u) = 1$. Let $v \in Y$ and $\psi \in Y^{'}$ such that $\varphi(v) = 0$ and $\psi(u) = 0$. The fact that $F(Y) \subset Im\Phi$ implies that there exist two vectors w and z in X and two linearly forms h and k in $X^{'}$ such that:

$$\Phi(w \otimes h) = v \otimes \varphi \text{ and } \Phi(z \otimes k) = u \otimes \psi.$$

The operator $\alpha x \otimes f$ is of rank one and by the Lemma 2.5 there exist $u_\alpha \in Y$ and $\varphi_\alpha \in Y^{'}$ such that $\Phi(\alpha x \otimes f) = u_\alpha \otimes \varphi_\alpha$. $\Phi(\alpha x \otimes f + x \otimes f) = u_\alpha \otimes \varphi_\alpha + u \otimes \varphi$ is an operator of rank one which implies that either the vectors u_α and u are linearly dependent or the forms φ_α and φ are linearly dependent. Suppose that φ_α and φ are linearly dependent, by absorbing a constant in the first term of the tensor product, we can take $\varphi_\alpha = \varphi$. We have $\Phi(x \otimes f + z \otimes k) = u \otimes \varphi + u \otimes \psi$ who is an operator of rank one and the Lemma 2.5 shows that $x \otimes f + z \otimes k$ is of rank one so, $\alpha x \otimes f + z \otimes k$ is also of rank one. By the Lemma 2.5 we have $\Phi(\alpha x \otimes f + z \otimes k) = u_\alpha \otimes \varphi + u \otimes \psi$ an operator of rank one. Since $\varphi(u) = 1$ and $\psi(u) = 0$ we can affirm that φ and ψ are linearly independent therefore, u_α and u are linearly dependent. Consequently $u_\alpha = \beta u$ for some $\beta \in \mathbb{C}^*$. Thus $\Phi(\alpha x \otimes f) = \beta u \otimes \varphi$ and $\alpha = \beta$ because Φ preserves the point spectrum.

In the case where u_α and u are linearly dependent, by absorbing a constant in the second term of the tensor product, we can take $u_\alpha = u$. The operator $\Phi(x \otimes f + w \otimes h) = u \times \varphi + v \otimes \varphi$ is of rank one and by the Lemma 2.5 we establish that $x \otimes f + w \otimes h$ is of rank one thus $\alpha x \otimes f + w \otimes h$ is also of rank one. We apply the Lemma 2.5 to show that $\Phi(\alpha x \otimes f + w \otimes h) = u \otimes \varphi_\alpha + v \otimes \varphi$ is rank one operator. Since $\varphi(u) = 1$ and $\varphi(v) = 0$ we can said that the vectors u and v are linearly independent therefore φ_α and φ are linearly dependent that is to say $\varphi_\alpha = \gamma\varphi$ for some $\gamma \in \mathbb{C}^*$. We have $\Phi(\alpha x \otimes f) = \gamma u \otimes \varphi$ and $\alpha = \gamma$ because Φ preserves the point spectrum. Finally $\Phi(\alpha x \otimes f) = \alpha\Phi(x \otimes f)$.

Now, if $f(x) = 0$ we can find $g \in X^{'}$ such that $g(x) = 1$. We have $\Phi(\alpha x \otimes f) = \Phi(\alpha x \otimes (f + g)) - \Phi(\alpha x \otimes g) = \alpha\Phi(x \otimes f)$.

3. Proof of main results

Proof of Theorem 2.1. The proof of this theorem follows the same plan of A. Jafarian and A.R. Sourour [1] in two steps but with different approaches especially in the second step.

In the first step we show that either there exist two bijective linear maps $A : X \longrightarrow Y$ and $C : X^{'} \longrightarrow Y^{'}$ such that:

$$\Phi(x \otimes f) = Ax \otimes Cf, \tag{1}$$

for every $x \in X$ and $f \in X^{'}$ or there exist two bijective linear maps $B : X^{'} \longrightarrow Y$ and $D : X \longrightarrow Y^{'}$such that:

$$\Phi(x \otimes f) = Bf \otimes Dx, \tag{2}$$

for every $x \in X$ and $f \in X'$.

In the second step we show that either $\Phi(T) = ATA^{-1}$ or $\Phi(T) = BT^*B^{-1}$ for every $T \in B(X)$.

Step 1: The Lemmas 2.2, 2.5 and 2.6 which are specifics to our situation show that Φ is injective , $\Phi(F_1(X)) = F_1(Y)$ and Φ is 1−homogeneous over $F_1(X)$ thus allow us to be in the same hypotheses of A.Jafarian and A.R.Sourour in [1] to affirm step 1. All that remains is to demonstrate that A, B, C and D are bounded. Let us begin with A and C, which are given by (1). Choose an operator T of finite rank in $B(X)$ and $\lambda \in \mathbb{C}^*$ with $\lambda \notin \sigma_p(T)$. For $x \in X - \{0\}$ and $f \in X' - \{0\}$ we have $\lambda \in \sigma_p(T + x \otimes f)$ if and only if $f((\lambda - T)^{-1}x) = 1$. The fact that Φ preserves the point spectrum we have $f((\lambda - T)^{-1}x) = Cf((\lambda - \Phi(T))^{-1}Ax)$. The closed graph Theorem shows that A and C are bounded. In the same way we can show that B and D are bounded.

Step 2: To establish this step A.Jafarian and A.R.Sourour have exploited with advantage the analytic properties of the resolvent which is related to the spectrum thing which can not be used here since we only consider the point spectrum and we give an algebraic proof.

First case : Let us put in the situation (1) and show that $\Phi(T) = ATA^{-1}$ for every $T \in B(X)$. Fix $0 \neq x \in X$ and $0 \neq f \in X'$.

If T is injective we have $T(x) = y \neq 0$. Since Φ preserves the point spectrum we have,

$$f(x) = 1 \text{ if and only if } Cf(Ax) = 1. \tag{3}$$

We have $0 \in \sigma_p(T - y \otimes f) = \sigma_p(\Phi(T) - Ay \otimes Cf)$ so, there exists $z \in Y$ such that $\Phi(z) = Ay \otimes Cf(z)$ that is to say that $\Phi(T)(z) = Cf(z)Ay$. Show that z and Ax are linearly dependent in other wise there exists $g \in Y'$ such that $g(Ax) = 1$ and $g(z) = 0$. Since C is bijective there exists $f \in X'$ such that $Cf = g$. We have $Cf(Ax) = 1$ and by (3) we get $f(x) = 1$. Therefore $\Phi(T)(z) = Cf(z)Ay = g(z)Ay = 0$ from there $0 \in \sigma_p(\Phi(T)) = \sigma_p(T)$ which is absurd. Therefore z and Ax are linearly dependent and consequently we have for some suitable $\lambda \in \mathbb{C}^*$, $\Phi(T)(\lambda Ax) = Cf(\lambda Ax)Ay = \lambda Ay = \lambda ATx$. So $\Phi(T)A = AT$ and $\Phi(T) = ATA^{-1}$.

If T is not injective there exists $\lambda \in \mathbb{C}$ such that $T + \lambda$ is injective. The Lemma 2.4 indicate that $\Phi(\lambda) = \lambda$ and consequently $\Phi(T) = ATA^{-1}$.

Second case : Let us put in situation (2) and show that $\Phi(T) = BT^*B^{-1}$ for every $T \in B(X)$. Fix $T \in B(X)$.

If T is injective we study the case where the range of T is dense in X and the case where is not. In the case where the range of T is dense in X we find $f \in X' - \{0\}$ and $x \in X$ such that $f(Tx) = 1$. For $y = Tx$ we have $0 \in \sigma_p(T - y \otimes T^*f) = \sigma_p(\Phi(T) - B(T^*f) \otimes Dy)$ and there exists $z \in Y$ such that:

$$\Phi(T)(z) = Dy(z)B(T^*f). \tag{4}$$

Show that z and Bf are linearly dependent in other wise there exists $g \in Y'$ such that $g(z) = 0$ and $g(Bf) = 1$. Since the map D is bijective there exists $y \in Y$ such that $D(y) = g$ which implies that $D(y)(Bf) = 1$. Since Φ preserves the point spectrum we have $f(y) = 1$. The formula (4) implies that $\Phi(T)(z) = Dy(z)B(T^*f) = g(z)B(T^*f) = 0$ and consequently $0 \in \sigma_p(\Phi(T)) = \sigma_p(T)$ which is absurd. So, z and Bf are linearly dependent and for a suitable $\lambda \in \mathbb{C}^*$ we have $\Phi(T)(\lambda Bf) = D(y)(\lambda Bf)(BT^*f)$ which implies that $\Phi(T)(Bf) = BT^*f$ for every $f \in X'$ and consequently $\Phi(T) = BT^*B^{-1}$. In the case where the range of T is not dense in X we can find a real number α such that $T + \alpha$ is surjective and we apply the forgoing and the fact that $\Phi(\alpha) = \alpha$ to deduce that $\Phi(T) = BT^*B^{-1}$.

If T is not injective there exists a non zero real number λ such that $T + \lambda$ is injective. Since $\Phi(\lambda) = \lambda$ we can affirm that $\Phi(T) = BT^*B^{-1}$. Finally $\Phi(T) = BT^*B^{-1}$ for every $T \in B(X)$.

Proof of Corollary 2.1. The Theorem 2.1 shows that either $\Phi(T) = ATA^{-1}$ or $\Phi(T) = BT^*B^{-1}$ for every $T \in B(X)$ where $A : X \longrightarrow Y$ and $B : X' \longrightarrow Y$ are bijective linear bounded maps. The fact that Φ preserves the point spectrum and $\Phi(T) = BT^*B^{-1}$ show that, T is injective if and only if T^* is injective which is not true if X is a separable Hilbert space of infinite dimension and T is a unilateral shift operator (see [12] problem 82). So the case $\Phi(T) = BT^*B^{-1}$ can not take place and we have $\Phi(T) = ATA^{-1}$ for every $T \in B(X)$. Consequently Φ is surjective.

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