Numerical methods for solving Lane-Emden type differential equations by operational matrix of fractional derivative of modified generalized Laguerre polynomials

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Abstract. The present paper tries to elaborate on the application of operational matrix of derivative of modified generalized Laguerre polynomials for solving Lane–Emden type equations in astrophysics. Moreover, these equations were numerically solved by the help of this operational matrix. Furthermore, some representative instances were presented to indicate the capability, acceptability and logicality of the suggested methods.

Keywords: Lane–Emden type equations; operational matrix of derivative; modified generalized Laguerre polynomials; Caputo derivative; fractional calculus and astrophysics.

1. Introduction

Some of phenomena in mathematical physics and astrophysics are shaped by equations of type Lane-Emden as one of the most important equations in the category of second-order nonlinear ordinary differential equations (ODEs) [1, 2, 3, 4, 5].

Generalizing the idea of n-fold integration and integer-order differentiation leads to "fractional calculus" that has significant applications in diverse areas of engineering sciences and mathematical physics. In fact, fractional calculus as the theory of derivatives and integrals of a given principle or multiplex order can be fascinating and engaging for many researchers, as one of the most effective

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tools in fractional differential equations, to illustrate common characteristics of a variety of processes and materials, whereas such effects are ignored in the classical integer-order models. As the most important effective factor of fractional derivatives, one can consider the fractional derivatives more advantageous than its classical models in modeling electrical and mechanical characteristics of real materials in many fields [11].

As things are, for mathematical modeling of some physical phenomena, one may face the issue of solving varied kinds of fractional differential equations. Then, these equations play main roles in physics and several fields of engineering, as well as, mathematics. Since three decades ago, diverse operators have been investigated in some of papers on fractional calculus such as Erdlyi-Kober operators [13], Riemann-Liouville operators [12], Caputo operators [15], Weyl-Riesz operators [14] and Grnwald-Letnikov operators [16]. Moreover, the present researchers refer the reader to [17] in which the existence of definite solution and multi-positive solutions for nonlinear fractional differential equations are established [18, 19]. Also, in [6], the authors applied Legendre wavelet method for solving differential equations of Lane-Emden type.

It is noteworthy that the present paper is assigned to generalizing the explanation of Lane-Emden equations up to fractional order in the way as provided in the following equation:

$$\begin{gathered} D^{\tau}\omega(\xi)+\tfrac{\theta}{\xi^{\tau-\rho}}D^{\rho}\omega(\xi)+f(\xi,\omega)=g(\xi), \\ 0<\xi\leq 1, \quad 0\leq\theta, \quad 1<\tau\leq 2, \quad 0<\rho\leq 1, \end{gathered}$$

with the initial conditions (IC)

$$\omega(0)=A \quad , \quad \omega'(0)=B,$$

where A, B are constants, $f(\xi,\omega)$ is a real-valued function and $g\in C[0,1]$. As previously mentioned, the present researchers applied operational matrix of fractional derivative of modified generalized Laguerre polynomials (OMFDMGLPs) for solving Lane–Emden type equations. As it is shown, the simplicity of implementation of this method is very simple and the precision of answers is high. To this end, this paper is organized as: Section 2 represents definitions; in this section, the modified generalized Laguerre polynomials (MGLPs) and some attributes of fractional derivative are introduced. The OMDMGLPs of fractional derivative is presented in Section 3. In Section 4, the researchers implemented them on Lane–Emden equation. In Section 5, some models are discussed to illustrate the efficiency and precision of the method. Finally, Section 6 includes a conclusions of the obtained results and findings.

2. Mathematical preliminaries

2.1 Fractional derivative

To recall the requirements of the fractional calculus, the present researchers started with a definition. In the theory of integrals and derivatives of any order, generalization and incorporation of two concepts (i.e., integer-order differentiation and n-fold integration) is called the fractional calculus [20, 21]. Some of mathematicians such as Grunwald-Letnikove and Riemann-Liouville's diversely introduced definitions for fractional integration and differentiation. They are not fruitful in our purpose since, for example, Riemann-Liouville has certain disadvantages in modeling real-world phenomena with fractional differential equations. In fact, the researchers used a changed fractional differential operator D^v proposed in Caputo's work on the theory of viscoelasticity [22].

Definition 2.1. *The Caputo fractional derivative is marked out as:*

$$D^v f(\xi) = \frac{1}{\Gamma(n-v)} \int_0^{\xi} \frac{f^{(n)}(\omega)}{(\xi-\omega)^{v+1-n}} d\omega, \qquad n-1 < v \le n, n \in \mathbb{N}, \xi > 0$$

In that v is a positive real number as the order of the derivative and n is the smallest integer greater than v.

Note that [23]:

$$(1) \qquad D^v \xi^{\beta} = \begin{cases} 0, & for\ \beta \in \mathbb{N}_0\ and\ \beta < \lceil v \rceil, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-v)} \xi^{\beta-v}, & for\ \beta \in \mathbb{N}_0, \beta \ge \lceil v \rceil\ or\ \beta \notin \mathbb{N}, \beta > \lfloor v \rfloor. \end{cases}$$

In this paper, the symbols $\lceil v \rceil$ and $\lfloor v \rfloor$(the ceiling and the floor functions) stand for the smallest integer greater than or equal to v and the largest integer less than or equal to v, respectively. In addition, the researchers utilized notations $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. It is noteworthy that the differential operator in the sense of Caputo for $v \in \mathbb{N}$ agrees with differential operator of an integer-order in the usual sense. The fractional differentiation in the sense of Caputo is a linear operation, as in the integer-order differentiation:

$$(2) \qquad D^v(\lambda f(\xi) + \mu g(\xi)) = \lambda D^v f(\xi) + \mu D^v g(\xi),$$

where λ and μ are constants.

2.2 MGLPs and properties ([19])

Let $\Lambda = (0, \infty)$ and $w^{(\alpha,\beta)}(\xi) = \xi^{\alpha} e^{-\beta\xi}$ be a weight function on Λ in the usual sense. Now, define:

$$L^2_{w^{(\alpha,\beta)}}(\Lambda) = \{v \mid v\ is\ measurable\ on\ \Lambda\ and\ \|v\|_{w^{(\alpha,\beta)}} < \infty\},$$

with the below inner product and norm:

$$(u, v)_{w^{(\alpha,\beta)}} = \int_{\Lambda} u(\xi) v(\xi) w^{(\alpha,\beta)}(\xi) d\xi, \qquad \|v\|_{w^{(\alpha,\beta)}} = (v, v)^{\frac{1}{2}}_{w^{(\alpha,\beta)}}.$$

Next, let $L_i^{(\alpha,\beta)}(\xi)$ be the MGLPs of degree i for $\alpha > -1$ and $\beta > 0$. Clearly, $L_i^{(\alpha,\beta)}(\xi)$ is explained by:

$$L_i^{(\alpha,\beta)}(\xi) = \frac{1}{i!} \xi^{-\alpha} e^{\beta\xi} \partial_{\xi}^{i} \left(\xi^{i+\alpha} e^{-\beta\xi}\right), \qquad i = 1, 2, \dots$$

For $\alpha > -1$ and $\beta > 0$, it is clear that

$$\partial_\xi L_i^{(\alpha,\beta)}(\xi) = -\beta L_{i-1}^{(\alpha+1,\beta)}(\xi),$$

$$L_{i+1}^{(\alpha,\beta)}(\xi) = \frac{1}{i+1}\left[(2i+\alpha+1-\beta\xi)\, L_i^{(\alpha,\beta)}(\xi) - (i+\alpha)\, L_{i-1}^{(\alpha,\beta)}(\xi)\right], \quad i = 1,2,\ldots,$$

where $L_0^{(\alpha,\beta)}(\xi) = 1$ and $L_1^{(\alpha,\beta)}(\xi) = -\beta\ \xi + \frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)}$.

The set of MGLPs is the $L^2_{w^{(\alpha,\beta)}}(\Lambda)$ -orthogonal system, i.e.

$$\int_0^\infty L_j^{(\alpha,\beta)}(\xi)\, L_k^{(\alpha,\beta)}(\xi) w^{(\alpha,\beta)}(\xi)\, d\xi = h_k \delta_{jk},$$

where δ_{jk} is the Kronecker function and $h_k = \frac{\Gamma(k+\alpha+1)}{\beta^{\alpha+1} k!}$. The MGLPs of degree i on the interval Λ is presented by:

$$L_i^{(\alpha,\beta)}(\xi) = \sum_{k=0}^{i} (-1)^k \frac{\Gamma(i+\alpha+1)\beta^k}{\Gamma(k+\alpha+1)(i-k)!k!} \xi^k, \qquad i = 0,1,\ldots, \tag{3}$$

where $L_i^{(\alpha,\beta)}(0) = \frac{\Gamma(i+\alpha+1)}{\Gamma(\alpha+1)\Gamma(i+1)}$.

The special value:

$$D^q L_i^{(\alpha,\beta)}(0) = \frac{(-1)^q \beta^q \Gamma(i+\alpha+1)}{(i-q)!\,\Gamma(q+\alpha+1)}, \qquad i \geq q$$

can be of significant application later.

2.3 Operational matrix of fractional derivative of MGLPs in Caputo sense ([24])

Let $u \in L^2_{w^{(\alpha,\beta)}}(\Lambda)$, then $u(\xi)$ may be defined based on MGLP as:

$$u(\xi) = \sum_{\mathrm{j}=0}^{\infty} a_{\mathrm{j}} \mathrm{L}_{\mathrm{j}}^{(\xi,\beta)}(\xi),$$

$$a_{\mathrm{j}} = \frac{1}{\mathrm{h_k}} \int_0^\infty \mathrm{u}(\xi)\mathrm{L}_{\mathrm{j}}^{(\alpha,\beta)}(\xi)\mathrm{w}^{(\alpha,\beta)}(\xi)\, \mathrm{d}\xi, \quad \mathrm{j} = 0,1,\ldots.$$

In specific uses, the MGLPs up to degree $N+1$ are noticed. Then, the present researchers have:

$$u_N(\xi) = a_{\mathrm{j}} \mathrm{L}_{\mathrm{j}}^{(\alpha,\beta)}(\xi) = \mathrm{C}^{\mathrm{T}} \mathbf{\Phi}(\xi),$$

where the MGLPs coefficient vector C and the MGLPs vector $\mathbf{\Phi}(\xi)$ are presented by:

$$C^T = [c_0, c_1, , c_N], \quad \mathbf{\Phi}(\xi) = [\mathrm{L}_0^{(\alpha,\beta)}(\xi), \mathrm{L}_1^{(\alpha,\beta)}(\xi), \ldots, \ \mathrm{L}_{\mathrm{N}}^{(\alpha,\beta)}(\xi)]^T,$$

Then, the derivative of the vector $\mathbf{\Phi}(\xi)$ can be uttered by the follows:

$$\frac{\mathrm{d}\mathbf{\Phi}(\xi)}{\mathrm{d}\xi} = \mathrm{D}^{(1)}\mathbf{\Phi}(\xi), \tag{4}$$

where $\mathrm{D}^{(1)}$ is the $(N+1)\times(N+1)$ operational matrix of derivative given by:

$$\mathrm{D}^{(1)} = -\beta \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 0 \end{pmatrix}. \tag{5}$$

By using Eq. (4), it is clear that:

$$\frac{d^n\mathbf{\Phi}(\xi)}{d\xi^n} = \left(D^{(1)}\right)^n \mathbf{\Phi}(\xi), \tag{6}$$

where $n \in \mathbb{N}$ and the superscript in $\mathrm{D}^{(1)}$ give the meaning to matrix powers. Therefore,

$$D^{(n)} = \left(D^{(1)}\right)^n, \qquad n = 1, 2, \ldots.$$

Lemma 2.1. *Let $\mathrm{L}_{\mathrm{i}}^{(\xi,\beta)}(\xi)$ be a MGLPs. Then*

$$D^v L_j^{(\alpha,\beta)}(\xi) = 0, \qquad i = 0, 1, \ldots, \lceil v \rceil - 1, v > 0.$$

Proof. By utilizing Eqs. (1) and (2) in Eq. (3), the lemma can be proved [24].

Theorem 2.1 ([24])**.** *Let $\emptyset(\xi)$ be the MGLPs vector explained in Eq. (3.3) and notice $v > 0$, then*

$$D^v\mathbf{\Phi}(\xi) \simeq D^{(v)}\mathbf{\Phi}(\xi), \tag{7}$$

where $D^{(v)}$ is the $(N+1)\times(N+1)$ OMDMGLPs of fractional derivatives of order v in Caputo sense, expressed as follows:

$$D^{(v)} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ \Omega_v(\lceil v \rceil, 0) & \Omega_v(\lceil v \rceil, 1) & \Omega_v(\lceil v \rceil, 2) & \cdots & \Omega_v(\lceil v \rceil, N) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \Omega_v(i, 0) & \Omega_v(i, 1) & \Omega_v(i, 2) & \cdots & \Omega_v(i, N) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \Omega_v(N, 0) & \Omega_v(N, 1) & \Omega_v(N, 2) & \cdots & \Omega_v(N, N) \end{pmatrix}, \tag{8}$$

where

$$\Omega_v(i,j) = \sum_{k=\lceil v\rceil}^{i}\sum_{l=0}^{j}\frac{(-1)^{k+l}\beta^v j!\Gamma(i+\alpha+1)\Gamma(k-v+\alpha+l+1)}{(i-k)!(j-l)!\Gamma(k-v+1)\Gamma(k+\alpha+1)\Gamma(\alpha+l+1)}.$$

Notice that in $D^{(v)}$, the first $\lceil v\rceil$ rows are all zero.

Remark 2.1. In the case of $v = n \in \mathbb{N}$, Theorem 2.1 presents the same result as Eq. (6).

3. Applications of the OMFDMGLPs in Caputo sense

3.1 Solution of Lane-Emden type equations

Consider the Lane–Emden equation of the following form [25, 26]:

$$\omega''(\xi) + \frac{\theta}{\xi}\omega'(\xi) + f(\xi,\omega) = g(\xi), \qquad \theta,\xi \geq 0, \tag{9}$$

with IC:

$$\omega(0) = a, \quad \omega'(0) = 0.$$

By approximating $\omega(\xi)$, $f(\xi,\omega)$ and $g(\xi)$ using MGLPs, we have:

$$\omega(\xi) \approx \sum_{i=0}^{N} c_i p_i(\xi) = C^T\boldsymbol{\Phi}(\xi). \tag{10}$$

$$f(\xi,\omega) \approx f\left(\xi,\ C^T\boldsymbol{\Phi}(\xi)\right) = H^T\boldsymbol{\Phi}(\xi), \qquad g(\xi) \approx \sum_{i=0}^{N} c_i \mathrm{p_i}(\xi) = G^T\boldsymbol{\Phi}(\xi),$$

where the unknowns are $C = [c_0,\ldots,c_N]^T$ and $H = [h_0,\ldots,h_N]^T$. By applying (5), Eq. (9) can be as

$$C^T\boldsymbol{D}^{(2)}\boldsymbol{\Phi}(\xi) + \frac{\theta}{\xi}C^T\boldsymbol{D}^{(1)}\boldsymbol{\Phi}(\xi) + H^T\boldsymbol{\Phi}(\xi) \approx G^T\boldsymbol{\Phi}(\xi), \tag{11}$$

$$\omega(0) = C^T\boldsymbol{\Phi}(0) = d_0, \qquad \omega'(0) = C^T\boldsymbol{D}^{(1)}\boldsymbol{\Phi}(0) = d_1. \tag{12}$$

Eqs. (11) and (12) present two linear equations. Since the total unknowns for vector C in Eq. (10) is $(N+1)$, the present researchers collocate Eq. (11) in $(N-2)$ points ξ_p in the interval $[0,1]$ as

$$\xi_p = \frac{2p-1}{2(n+1)}, \quad p = 1,2,\ldots,n-2\ .$$

Then, we will have:

$$C^T\boldsymbol{D}^{(2)}\boldsymbol{\Phi}(\xi_i) + \frac{\theta}{\xi_i}C^T\boldsymbol{D}^{(1)}\boldsymbol{\Phi}(\xi_i) + H^T\boldsymbol{\Phi}(\xi_i) \approx G^T\boldsymbol{\Phi}(\xi_i). \tag{13}$$

For $i = 1,2,\ldots,N-2$, now, the Eqs. (12) and (13) generate a system of $(N+1)$ nonlinear system of equations that can be solved by using known methods.

3.2 Solution of fractional differential equations of Lane–Emden type

Consider the fractional differential equations of Lane–Emden type

$$D^{\tau}\omega(\xi)+\frac{\theta}{\xi^{\tau-\rho}}D^{\rho}\omega(\xi)+f(\xi,\omega)=g(\xi),0<\xi\leq 1,\theta\geq 0,1<\tau\leq 2,0<\rho\leq 1, \tag{14}$$

with the IC:

$$\omega\left(0\right)=A\quad,\quad\omega'\left(0\right)=B, \tag{15}$$

where $A,\ B$ are constants, $f(\xi,\omega)$ is a continuous real-valued function and $g\left(\xi\right)\in C\left[0,1\right].$

Approximating $\omega(\xi),\ f(\xi,\omega)$ and $g(\xi)$ by MGLP, we have:

$$\omega(\xi)\approx\sum_{i=0}^{N}c_i p_i\left(\xi\right)=C^T\boldsymbol{\Phi}\left(\xi\right),$$

$$f\left(\xi,\omega\right)\approx f\left(\xi,\ C^T\mathbf{\Phi}\left(\xi\right)\right)=H^T\mathbf{\Phi}\left(\xi\right),\quad g\left(\xi\right)\approx\sum_{i=0}^{N}c_i\mathrm{p_i}\left(\xi\right)=G^T\mathbf{\Phi}\left(\xi\right), \tag{16}$$

where the unknowns are $C=\left[c_0,\dots,c_N\right]^T$and $H=\left[h_0,\dots,h_N\right]^T$. By using (7), Eq. (14) can be as

$$C^T\boldsymbol{D}^{(\tau)}\mathbf{\Phi}\left(\xi\right)+\frac{\theta}{\xi}C^T\boldsymbol{D}^{(\rho)}\mathbf{\Phi}\left(\xi\right)+H^T\mathbf{\Phi}\left(\xi\right)\approx G^T\mathbf{\Phi}\left(\xi\right). \tag{17}$$

The IC (15) is given by:

$$\omega\left(0\right)=C^T\mathbf{\Phi}\left(0\right)=d_2,\qquad\omega'\left(0\right)=C^T\boldsymbol{D}^{(1)}\mathbf{\Phi}\left(0\right)=d_3. \tag{18}$$

Eqs. (17) and (18) present two linear system of equations. Since the total unknowns for vector C in Eq. (16) is $(N+1)$, the researchers collocate Eq. (17) in $(N-2)$ points ξ_i in the interval $[0,1]$ as,

$$\xi_p=\frac{2p-1}{2(n+1)},\qquad p=1,2,\dots,n-2\ .$$

Then, the present researchers will have:

$$C^T\boldsymbol{D}^{(\tau)}\mathbf{\Phi}\left(\xi_i\right)+\frac{\theta}{\xi_i}C^T\boldsymbol{D}^{(\rho)}\mathbf{\Phi}\left(\xi_i\right)+H^T\mathbf{\Phi}\left(\xi_i\right)\approx G^T\mathbf{\Phi}\left(\xi_i\right). \tag{19}$$

For $i=1,2,\dots,N-2$; now, Eqs. (18) and (19) generate a system of $(N+1)$ nonlinear equations that can be sove by known methods.

4. Illustrative examples

To illustrate the acceptability, logicality, and effectiveness of the suggested method, we solve some Lane–Emden type equations presented in Eqs. (9) and (14) numerically. It is noteworthy that all of the computations have been performed using Mathematica 9.

Example 4.1. Notice the following nonlinear Lane-Emden equation [8]:

$$\omega''(\xi) + \frac{2}{\xi}\omega'(\xi) + \omega^m = 0, \qquad 0 < \xi < 1,$$

The IC are $\omega(0) = 1, \quad \omega'(0) = 0$. For $m = 5$, this example has the exact solution $\omega(\xi) = (1 + \frac{\xi^2}{3})^{-\frac{1}{2}}$. The results of this example are tabulated in Table 1 and Fig. 1 for N= 8. In addition, the absolute errors diagram is shown in Fig. 2.

For $m = 3$, this example has the following form:

$$\omega''(\xi) + \frac{2}{\xi}\omega'(\xi) + \omega^3 = 0, \qquad 0 < \xi < 1,$$

with the IC

$$\omega(0) = 1, \;\; \omega'(0) = 0.$$

Table 2 shows the approximation of $\omega(\xi)$ for the present method and exact values given by Horedt [29].

Table 1. Numerical results and exact solution for Example 4.1

ξ_i	$N = 8,\ \alpha = \beta = 1/2$	$N = 8,\ \alpha = \beta = 1$	Exact solution
0.0	1.	1.	1.
0.2	0.9933992680853407	0.9933992678561196	0.9933992677987828
0.4	0.9743547050475172	0.9743547050142785	0.9743547036924463
0.6	0.9449111844072604	0.9449111843377054	0.944911182523068
0.8	0.9078413117090349	0.9078413116788155	0.9078412990032037
1.0	0.8660300001529322	0.8660299995409204	0.8660254037844386

Table 2. Comparison between the numerical solutions and error obtained by the proposed methods and its numerical values [7].

ξ_i	Method of [8]	Present method	Exact value [29]	Error of [8]	Error at N=10
0.0	1.0000000	1.0000000	1.0000000	0.00	0.00
0.1	0.99833720	0.99833582	0.9983358	$1.40e-06$	$2.00e-08$
0.5	0.95984209	0.95983906	0.9598391	$2.99e-06$	$4.00e-08$
1.0	0.85505959	0.85505754	0.8550576	$1.99e-06$	$6.00e-08$

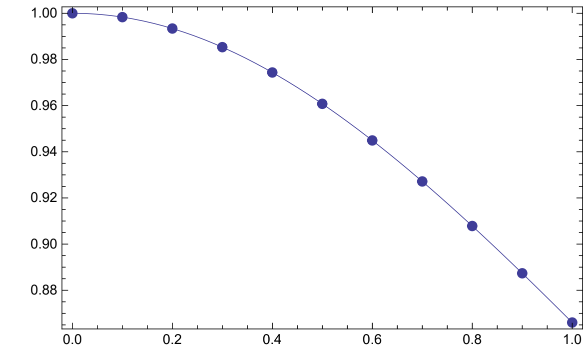

Figure 1: — Exact solution, . . Approximate solution.

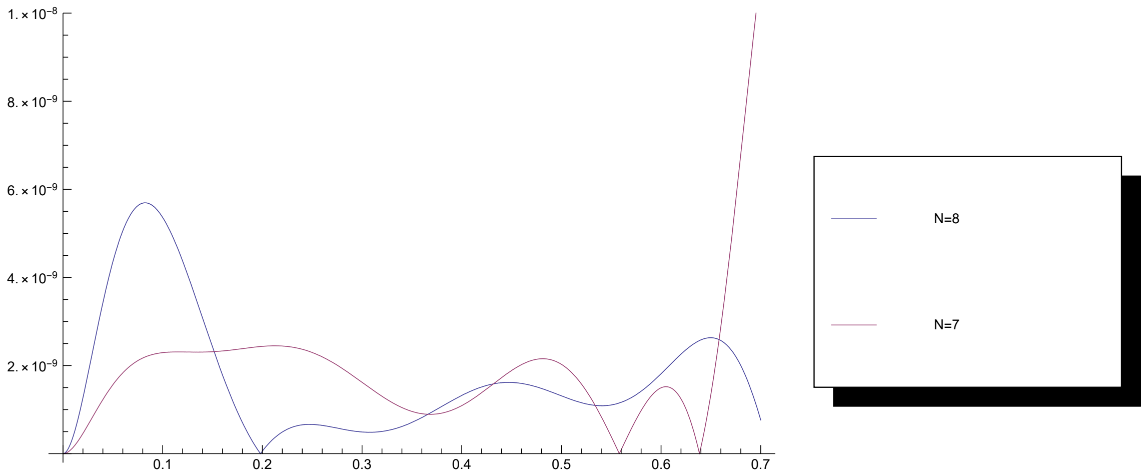

Figure 2: Graph of absolute errors for N = 7 and N = 8.

Example 4.2. The isothermal gas spheres are modeled by Davis (see [31]) as follows:

$$\omega''(\xi) + \frac{2}{\xi}\omega'(\xi) + e^{\omega(\xi)} = 0, \quad \xi \geq 0,$$

with the IC:

$$\omega\,(0) = 0, \quad \omega'\,(0) = 0.$$

The researchers clarified this instance by $N = 10$, which the results are tabulated in Table 3.

Table 3. Comparison of $\omega(\xi)$, with the present method and series of solutions offered by Wazwaz [27] and numerical values in [7] and [28], for isothermal gas sphere equation ($\alpha = \beta = 1$).

ξ_i	Method of [7]	Method of [27]	Method of [28]	Present method	Error of [7]	Error
0.0	0. 000000000	0. 000000000	0. 000000000	0. 000000000	0.00	0.00
0.1	−0. 0016664188	-0. 0016658339	-0. 0016655333	-0. 00166583385	5.85e -07	1.00e -10
0.2	-0. 0066539713	-0. 0066533671	−0. 0066533333	-0. 00665336709	6.04e -07	1.00e -11
0.5	-0. 0411545150	-0. 0411539568	-0. 04114583333	-0. 0411539572	5.58e -07	4.00e -10
1.0	-0. 1588281737	-0. 1588273537	-0. 1583333333	-0. 15882767809	8.20e -07	3.24e -07

Example 4.3. Consider the following equation [30]:

$$D^{\tau}\omega(\xi) + \frac{\theta}{\xi^{\tau-\rho}} D^{\rho}\omega(\xi) + \frac{1}{\xi^{\tau-2}}\omega(\xi) =$$

$$\left(2\left(\frac{\Gamma(3-\rho)+\theta\left(\Gamma(3-\tau)\right)}{\Gamma(3-\rho)\Gamma(3-\tau)} + \frac{\xi^2}{2}\right) - 6\xi\left(\frac{\Gamma(4-\rho)+\theta\left(\Gamma(4-\tau)\right)}{\Gamma(4-\rho)\Gamma(4-\tau)} + \frac{\xi^2}{6}\right)\right)\xi^{2-\tau},$$

where the exact solution is $\omega(\xi) = \xi^2 - \xi^3$ for $\tau = \frac{3}{2}, \rho = 1, \theta = 2$, and the IC are as follows:

$$\omega(0) = 0, \quad \omega'(0) = 0.$$

To compare the numerical results obtained by the proposed method with the exact solution, see Table 4.

Table 4: Comparing the numerical results and the exact solution for Example 4.3.

ξ_i	$N = 15,\ \alpha = 0, \beta = 1$	$N = 15,\ \alpha = 1/2, \beta = 1/2$	*Exact solution*
0.0	$1.033413307865862\times10^{-24}$	$3.7065625075664017\times10^{-24}$	0.
0.2	$1.487269876187146\times10^{-22}$	$6.110365821755565\times10^{-23}$	0.032
0.4	$3.435493502332913\times10^{-22}$	$1.6510946972972456\times10^{-22}$	0.096
0.6	$4.147499545115917\times10^{-22}$	$2.274652005311722\times10^{-22}$	0.144
0.8	$3.197835497234143\times10^{-22}$	$1.8155964929517072\times10^{-22}$	0.128

Example 4.4. Deem the equation [30]:

$$D^{\tau}\omega(\xi) + \frac{\theta}{\xi^{\tau-\rho}} D^{\rho}\omega(\xi) + \frac{1}{\xi^{\tau-2}}\omega(\xi) =$$

$$\left(6\xi\left(\frac{\Gamma(4-\rho)+\theta(\Gamma(4-\tau))}{\Gamma(4-\rho)\Gamma(4-\tau)}+\frac{\xi^2}{6}\right)-2\left(\frac{\Gamma(3-\rho)+\theta(\Gamma(3-\tau))}{\Gamma(3-\rho)\Gamma(3-\tau)}+\frac{\xi^2}{2}\right)\right)\xi^{2-\tau},$$

with the IC

$$\omega(0)=0\ ,\quad \omega'(0)=0,$$

and the exact solution $\omega(\xi)=\xi^3-\xi^2$, where $\tau=\frac{3}{2}, \rho=\frac{1}{2},\ \theta=2$. To see absolute error for $N=10$, one can refer to Table 5.

Table 5: Absolute error for Example 4.4.

ξ_i	Error for $N=10,\ \alpha=0$ and $\beta=1$
0.0	1.93271×10^{-27}
0.2	0.032
0.4	0.096
0.6	0.144
0.8	0.128
1	6.01587×10^{-18}

Example 4.5. Consider the following equation

$$D^{\tau}\omega(\xi)+\frac{2}{\xi}\omega'(\xi)+\omega(\xi)^5=0.$$

The exact solution of this example for $\tau=2$ is $\omega(\xi)=(1+\frac{\xi^2}{3})^{-\frac{1}{2}}$. Fig. 3 illustrates approximation of $\omega(\xi)$ with $\tau=1.3$, $\tau=1.6$, $\tau=1.8$, and $\tau=2$, where the IC are $\omega(0)=1,\ \ \omega'(0)=0$. According to Fig. 3, it is clear that the proposed method can be suitable method for solving these equations.

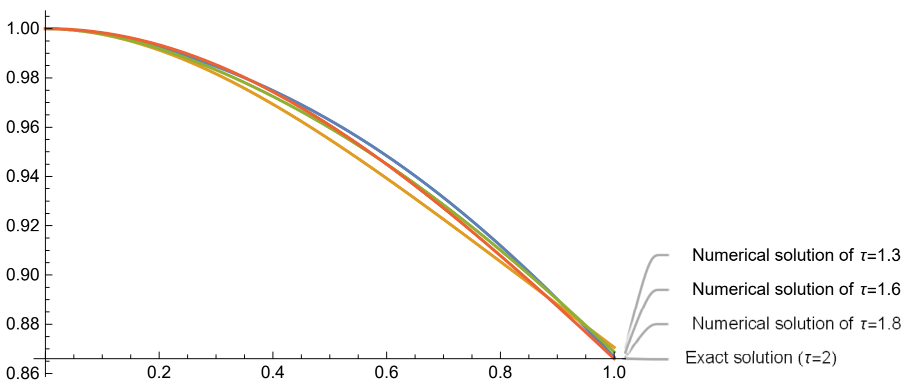

Figure 3: Numerical solutions for different values of τ for Example 4.5 with $N=15$.

5. Conclusions

In this paper, we have presented numerical methods for solving nonlinear singular differential equations of Lane-Emden type. Using the modified generalized Laguerre polynomials and the operational matrix of derivative of these polynomials, we converted these equations to a nonlinear system of equations that can be solved by the known methods. The numerical results show that the proposed method can be suitable method for solving these equations.

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