On weak δ- McCoy rings

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Abstract. Camillo, Kwak and Lee called a ring R right NC-McCoy if for any nonzero polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j$ over R, $f(x)g(x) = 0$ implies $a_i c \in Nil(R)$ for some $c \in R - \{0\}$ and $0 \leq i \leq m$. For a derivation δ of a ring R, we in this paper introduce the weak δ- McCoy rings. When $\delta = 0$, this coincides with notation of a right NC-McCoy ring. Some properties of this generalization are established and connections of properties of a weak δ-McCoy ring R with $n\times n$ upper triangular $T_n(R, \sigma)$ and its polynomial ring $R[x]$, are investigated.
Keywords: McCoy rings, Skew polynomial ring, weak δ-McCoy rings.

1. Introduction

Throughout this note, all rings are associative with 1. Let R be a ring, δ be a derivation of R, that is δ is an additive map such that $\delta(ab) = \delta(a)b + a\delta(b)$, for all $a, b \in R$. We denote $R[x; \delta]$ the Ore extension whose elements are the polynomials over the ring R, the addition is defined as typical and the multiplication is defined as the relation $xa = ax + \delta(a)$, for any $a \in R$. We use $Nil(R)$, $Nil^*(R)$ and $M_n(R)$ for the nilradical, the upper nilradical (i.e., the sum of all nil two-sided ideals) and the n by n full matrix ring over R respectively.

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Use e_{ij} for the matrix with (i,j)-entry 1 and 0 for $i \neq j$. Let $C_{f(x)}$ denote the set of all coefficients of $f(x) \in R[x]$. By $\mathbb{Z}_n$ we mean the ring of integers module n.

Following Rege and Chhawchharia [1], a ring R is called Armendariz if polynomials $f(x) = a_0+a_1x+\cdots+a_nx^n$ and $g(x) = b_0+b_1x+\cdots+b_mx^m \in R[x]-\{0\}$ satisfy $f(x)g(x) = 0$ then $a_ib_j = 0$ for each i and j. The name "Armendariz ring" is chosen because it is shown [2, Lemma 1] that reduced ring (that is a ring without nonzero nilpotent elements) satisfies the above condition. Nielsen [3], called a ring R *right McCoy* (resp., *left McCoy*) if, for any nonzero polynomials $f(x), g(x)$ over R, $f(x)g(x) = 0$ implies $f(x)c = 0$ (resp., $c'g(x) = 0$) for some $0 \neq c \in R$ (resp., $0 \neq c' \in R$). If a ring R is both left and right McCoy then it is called *McCoy*. By McCoy [4], commutative rings are McCoy rings. Clearly, Armendariz rings are McCoy but the converse does not hold by [1, Remark 4.3]. Following Camillo, Kwak and Lee [5], a ring R is called *right NC-McCoy* whenever, $f(x) = a_0+a_1x+\cdots+a_mx^m, g(x) = b_0+b_1x+\cdots+b_nx^n \in R[x]-\{0\}$ satisfy $f(x)g(x) = 0$ then $a_ic \in Nil(R)$ for some $0 \neq c \in R$. *Left NC-McCoy* rings are defined similarly. If a ring is both left and right NC-McCoy we say that the ring is *NC-McCoy* ring. The authors [10] introduced the notion of a ring with respect to a derivation δ of R. They defined a ring to be δ-McCoy if for any nonzero polynomials $f(x) = \sum_{i=0}^{m} a_ix^i$ and $g(x) = \sum_{j=0}^{n} b_jx^j$ in $R[x;\delta]$, $f(x)g(x) = 0$ implies that there exists $c \in R-\{0\}$ such that $\sum_{l=k}^{m} \binom{l}{k} a_l\delta^{l-k}(c) = 0$ for $k = 0, 1, ..., m$.

We are motivated to introduce the notion of a *weak δ-McCoy* ring with respect to a derivation δ of R. This notion extends NC-McCoy rings. We do this by considering the NC-McCoy condition on polynomials in $R[x;\delta]$ instead of $R[x]$. This provides us with an opportunity to study NC-McCoy rings in a general setting, and several known results on NC-McCoy rings are obtained as corollaries.

2. Main results

We begin this section by the following definition and also we study properties of weak δ-McCoy rings.

Definition 2.1. *Let δ be a derivation of a ring R. The ring R is called* weak δ-McCoy *if for any nonzero polynomials $f(x) = \sum_{i=0}^{m} a_ix^i$ and $g(x) = \sum_{j=0}^{n} b_jx^j$ in $R[x;\delta]$, $f(x)g(x) = 0$, implies that there exists $c \in R-\{0\}$ such that $\sum_{l=k}^{m} \binom{l}{k} a_l\delta^{l-k}(c) \in Nil(R)$ for $k = 0, 1, ..., m$.*

It is clear that a ring R is right NC-McCoy if R is weak 0-McCoy, where 0 is the zero mapping. Also, it is clear that δ-McCoy rings are weak δ-McCoy, but the converse is not always true by the following example.

Example 2.2. Let $R = T_2(\mathbb{Z}_2)$ and the derivation $\delta : R \to R$ given by $\delta(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. If $f(x) = \sum_{i=0}^{m} A_ix^i$ and $g(x) = \sum_{j=0}^{n} B_jx^j$ be nonzero polynomials in $R[x;\delta]$ such that $f(x)g(x)=0$, then

$$\sum_{l=k}^{m}\binom{l}{k}A_l\delta^{l-k}(e_{12})=\sum_{l=k}^{m}\binom{l}{k}A_le_{12}\in Nil(R)$$

for $k=0,1,...,m$. Therefore, R is weak δ-McCoy. But R is not δ-McCoy ring, because $(e_{11}+e_{12}+e_{12}x)(e_{12}+e_{22}+e_{12}x)=0$ in $R[x;\delta]$, and if $(e_{11}+e_{12}+e_{12}x)C=0$ for some $C\in R$, then $C=0$.

Let R_k be a ring, for each $k\in I$, δ_k a derivation of R_k and $R=\prod_{k\in I}R_k$. Then the map $\delta: R\to R$ defined by $\delta((a_k))=(\delta_k(a_k))$ is a derivation of R.

Proposition 2.3. *Let R_k be a ring with a derivation δ_k, where $k\in I$. If R_k is weak δ_k- McCoy, for each $k\in I$ then $R=\prod_{k\in I}R_k$ is weak δ-McCoy.*

Proof. Suppose that each R_k is weak δ_k- McCoy, for each $k\in I$ and $R=\prod_{k\in I}R_k$. Let $f(x)g(x)=0$ for some polynomials $f(x)=\sum_{i=0}^{m}a_ix^i$ and $g(x)=\sum_{j=0}^{n}b_jx^j\in R[x;\delta]\backslash\{0\}$, where $a_i=(a_i^{(k)})$ and $b_j=(b_j^{(k)})$ are elements of the product ring R. Define $f_k(x)=\sum_{i=0}^{m}a_i^{(k)}x^i$ and $g_k(x)=\sum_{j=0}^{n}b_j^{(k)}x^j\in R[x;\delta_k]$. Since $f_k(x)g_k(x)=0$ and R_k is weak δ_k-McCoy ring, there exists $0\neq s_k\in R_k$ such that $\sum_{l=t}^{m}\binom{l}{t}a_l^{(k)}\delta_k^{l-t}(s_k)\in Nil(R_k)$. Thus,

$$\sum_{l=t}^{m}\binom{l}{t}(a_l^{(1)},\cdots,a_l^{(k)},\cdots)\delta^{l-t}(0,\cdots,s_k,0,\cdots)=$$

$$(0,\cdots,\sum_{l=t}^{m}\binom{l}{t}a_l^{(k)}\delta^{l-t}(s_k),0,\cdots)\in Nil(R).$$

Therefore, R is weak δ-McCoy. □

The converse of the above Proposition does not hold by the following example.

Example 2.4. Let R be any ring which is not weak 0-McCoy and consider $S=R\times\mathbb{Z}_4$. S is always weak 0-McCoy since if $f(t)\in S[t]$ is any polynomial, then one can take $c=(0,\bar{2})$ and then $f(t)c\in Nil(S[t])$.

Let I be an ideal and δ be a derivation of R. If $\delta(I)\subseteq I$, then $\overline{\delta}: R/I\to R/I$ defined by $\overline{\delta}(\overline{a})=\delta(a)+I$ for $a\in R$, is a derivation of the factor ring R/I, where $\overline{a}=a+I$. Now we have the following proposition.

Proposition 2.5. *Let δ be a derivation of a ring R and I be an ideal of R. If $\delta(I)\subseteq I$, $I\subseteq Nil(R)$ and R/I is weak $\overline{\delta}$-McCoy, then R is weak δ-McCoy.*

Proof. Let $f(x)=\sum_{i=0}^{m}a_ix^i$ and $g(x)=\sum_{j=0}^{n}b_jx^j\in R[x;\delta]\backslash\{0\}$ such that $f(x)g(x)=0$. Then $(\sum_{i=0}^{m}\overline{a_i}x^i)(\sum_{j=0}^{n}\overline{b_j}x^j)=0$ in R/I. Thus there exists some positive integer n such that $(\sum_{l=k}^{m}\binom{l}{k}\overline{a}^l\overline{\delta}^{l-k}(\overline{s}))^n=0$ for some $s\in R-I$. Therefore $(\sum_{l=k}^{m}\binom{l}{k}a^l\delta^{l-k}(s))^n\in I\subseteq Nil(R)$. Thus, R is weak δ-McCoy. □

Now, we turn our attention to relationship between the weak δ-McCoy property of a ring R and its polynomial ring $R[x]$. Let δ be a derivation of a ring R. The map $\bar{\delta} : R[x] \to R[x]$ defined by $\bar{\delta}(\sum_{i=0}^{m} a_i x^i) = \sum_{i=0}^{m} \delta(a_i)x^i$ is a derivation of the polynomial ring $R[x]$, and clearly this map extends δ.

We say that R satisfies condition $(*)$, if for any $a, b \in R$ and $i \geq 0$, $ab \in Nil(R)$ implies that $a\delta^i(b) \in Nil(R)$.

Theorem 2.6. (1) *For a ring R and any derivation δ of R, if $R[x]$ is weak $\bar{\delta}$-McCoy, then R is weak δ-McCoy.*

(2) *Let R be a ring satisfying the condition $(*)$ and $Nil(R[x])$ is a subring of $R[x]$. If R is a weak δ-McCoy ring, then $R[x]$ is weak $\overline{\delta}$-McCoy.*

Proof. (1) Let $f(x)g(x) = 0$ for nonzero polynomials $f(x) = a_0 + a_1x + \cdots + a_mx^m$ and $g(x) = b_0 + b_1x + \cdots + b_nx^n$ in $R[x, \delta]$. Set $f(y) = a_0 + a_1y + \cdots + a_my^m$ and $g(y) = b_0 + b_1y + \cdots + b_ny^n \in (R[x])[y, \bar{\delta}]$, where $R([x, \delta])[y]$ is the Ore extension of polynomials with an indeterminate y over $R[x]$. Then $f(y)$ and $g(y)$ are nonzero since $f(x)$ and $g(x)$ are nonzero. Moreover, $f(y)g(y) = 0$. So there exists a nonzero polynomial $c(x) = c_0 + c_1x + \cdots + c_tx^t \in R[x]$ such that $\sum_{l=k}^{m} \binom{l}{k} a_l \overline{\delta}^{l-k}(c(x)) \in Nil(R[x])$ for $k = 0, \cdots, m$. Then $(\sum_{i=0}^{t} \sum_{l=k}^{m} \binom{l}{k} a_l \delta^{l-k}(c_i))x^i \in Nil(R[x])$ for $k = 0, \cdots, m$. Since $c(x)$ is nonzero, there exists an integer p such that $c_p \neq 0$. Then $\sum_{l=k}^{m} \binom{l}{k} a_l \delta^{l-k}(c_p) \in Nil(R)$ for $k = 0, \cdots, m$. Therefore, R is weak δ- McCoy.

(2) Assume that R is weak δ-McCoy and $f(y)g(y) = 0$ for nonzero polynomials $f(y) = f_0 + f_1y + \cdots + f_my^m$ and $g(y) = g_0 + g_1y + \cdots + g_ny^n$ in $(R[x])\,[y]$. Take the positive integer t with $t = \sum_{i=0}^{m} deg(f_i) + \sum_{j=0}^{n} deg(g_j)$ where the degree of the zero polynomial is taken to be 0. Then $f(x^t)$ and $g(x^t)$ are nonzero polynomials in $R[x]$ and $f(x^t)g(x^t) = 0$, since the set of coefficients of $f(x^t)$ and $g(x^t)$ coincides with the set of coefficients of the f_i's and g_j's. Let $degf(x^t) = s$. Since R is weak δ-McCoy, there exists a nonzero $c \in R$ such that $\sum_{l=k}^{s} \binom{l}{k} a_l \delta^{l-k}(c) \in Nil(R)$ for $0 \leq k \leq s$. Therefore,

$$
\begin{aligned}
&a_s c \in Nil(R),\\
&a_{s-1}c + sa_s\delta(c) \in Nil(R),\\
&\vdots\\
&a_0c + a_1\delta(c) + a_2\delta^2(c) + \cdots + a_s\delta^s(c) \in Nil(R).
\end{aligned}
$$

Since $Nil(R[x])$ is a subring of $R[x]$, and R satisfies the condition $(*)$, then $ac \in Nil(R)$ for any $a \in C_{f_{i(x)}}$ and so $\sum_{l=k}^{m} \binom{l}{k} f_l \overline{\delta}^{l-k}(c) \in Nil(R)[x]$. On the other hand, for any $a \in Nil(R)$ and nonnegative integer t, ax^t is nilpotent. Thus $ax^t \in Nil(R[x])$, and so $Nil(R)[x] \subseteq Nil(R[x])$ as the latter is closed under addition. Thus $R[x]$ is weak $\overline{\delta}$-McCoy. □

Theorem 2.7. *Let R be a ring, e a central idempotent of R and δ be a derivation of R with $\delta(e) = 0$ for every $e^2 = e \in R$. Then R is weak δ- McCoy if and only if eR is weak δ- McCoy.*

Proof. Assume that R is a weak δ- McCoy ring. Consider $f(x)=\sum_{i=1}^{m} ea_i x^i$ and $g(x)=\sum_{i=1}^{n} eb_j x^j \in eR[x] \subseteq R[x]$ such that $f(x)g(x)=0$. Since R is weak δ- McCoy, there exists $c \in R$ such that, $\sum_{l=k}^{m}\binom{l}{k} ea_l \delta^{l-k}(c) \in Nil(R)$ therefore, $\sum_{l=k}^{m}\binom{l}{k} ea_l \delta^{l-k}(ec) \in Nil(R)$. Consequently, eR is weak δ- McCoy. Conversely, let eR be a weak δ- McCoy, consider $f(x)=a_0+a_1x+\cdots+a_mx^m$ and $g(x)=b_0+b_1x+\cdots+b_nx^n$ in $R[x;\delta]$ with $f(x)g(x)=0$. Let $f_1(x)=ef(x)$ and $g_1(x)=eg(x)$. Then $f_1(x)g_1(x)=0$, since eR is weak δ -McCoy, there exists c_1 in R such that $\sum_{l=k}^{m}\binom{l}{k} ea_l \delta^{l-k}(ec_1) \in Nil(R)$ for $0 \le k \le m$. If $c=ec_1$ then $\sum_{l=k}^{m}\binom{l}{k} a_l \delta^{l-k}(c) \in Nil(R)$ for $0 \le k \le m$. Thus R is weak δ-McCoy. □

Let δ be a derivation of a ring R and $M_n(R)$ be the $n \times n$ matrix over ring R and $\overline{\delta} : M_n(R) \to M_n(R)$ defined by $\overline{\delta}((a_{ij}))=(\delta(a_{ij}))$. From Proposition 2.13 we may suspect that every $n \times n$ matrix ring over a ring R is weak $\overline{\delta}$-McCoy for any derivation δ on R. But the following example erase the possibility.

Example 2.8. Let R be a reduced ring with a derivation δ. Consider nonzero polynomials $f(x)=e_{11}+e_{12}x+e_{21}x^2+e_{22}x^3$ and $g(x)=-(e_{21}+e_{22})+(e_{11}+e_{12})x$ in $M_2(R)[x]$ with $f(x)g(x)=0$. Assume to the contrary $M_2(R)$ is weak $\overline{\delta}$-McCoy. Then there exists nonzero $C=(c_{ij}) \in M_2(R)$ such that

$$
\begin{aligned}
&(e_{22}C)^{n_1}=0,\\
&(e_{21}C+3e_{22}\overline{\delta}(C))^{n_2}=0,\\
&(e_{12}C+2e_{21}\overline{\delta}(C)+3e_{22}\overline{\delta}(C))^{n_3}=0,\\
&(e_{11}C+e_{12}\overline{\delta}(C)+e_{21}\overline{\delta}^2(C)+e_{22}\overline{\delta}^3(C))^{n_4}=0
\end{aligned}
$$

for some positive integers n_1, n_2, n_3, n_4, and so $c_{ij}=0$ for any i, j by a simple computation, since R is reduced. This implies $C=0$; which is a contradiction. Thus $M_2(R)$ is not weak $\overline{\delta}$-McCoy.

The next example shows that there exists a weak δ-McCoy ring R such that $R/J(R)$ is not weak $\overline{\delta}$-McCoy, where $J(R)$ is the radical Jacobson of R..

Example 2.9. Let R denote the localization of the ring $\mathbb{Z}$ of integers at the prime ideal $\langle 3\rangle$. Consider the quaternions $\mathbf{Q}$ over R, that is a free R-module with basis $\{1,\ i,\ j,\ k\}$ and multiplication satisfying $i^2=j^2=k^2=-1,\ ij=k=-ji$. Then $\mathbf{Q}$ is a noncommutative domain with $J(\mathbf{Q})=3\mathbf{Q}$, and so is weak δ-McCoy. But $\mathbf{Q}/J(\mathbf{Q})$ is isomorphic to the 2-by-2 full matrix ring over $\mathbb{Z}_3$ and is not weak δ-McCoy by Example 2.8.

Although Example 2.8 shows that if R is a reduced ring, then $M_2(R)$ is not weak δ-McCoy, but we have the following.

Theorem 2.10. *Let R be a ring with derivation δ such that $\delta(Nil(R)) \subseteq Nil(R)$. Then:*

(1) *If R contains a nonzero nil one-sided ideal, then R is a weak δ-McCoy ring.*

(2) *Every ring R with $Nil^*(R) \neq 0$ is a weak δ-McCoy ring.*

(3) *If R contains a nonzero central nilpotent element, then the matrix ring over R ($M_n(R)$) is a weak $\overline{\delta}$-McCoy ring for $n \geq 2$.*

Proof. (1) If I is a nil one-sided ideal of R, then c in definition can be any nonzero element of I. Part (2) and (3) are trivial consequence of part (1). □

By using the same argument in the proof of [5, Proposition 11], we have the following.

Corollary 2.11. *Let R be a ring of bounded index with a nonzero nil two-sided ideal of bounded index. If δ is a derivation of R such that $\delta(Nil(R)) \subseteq Nil(R)$ and $(NilR)[x] = Nil(R[x])$ then both R and $R[x]$ are weak δ-McCoy $\big($weak $\overline{\delta} - McCoy\big)$.*

Proof. Let I be a nonzero nil two-sided ideal of R. Since $0 \neq I \subseteq N^*(R)$, $N^*(R)$ containes a nonzero two-sided nilpotent ideal N of R by [[8], Lemma 5]. Then $N[x]$ is nonzero two-sided nilpotent ideal of $R[x]$. Since $\overline{\delta}(Nil(R[x])) \subseteq Nil(R[x])$ this implies that both R and $R[x]$ are weak δ- McCoy (weak $\overline{\delta} - McCoy$) rings, by Theorem 2.10. □

In [12], a ring R is called NI if $Nil^*(R) = Nil(R)$. Note that R is NI if and only if $Nil(R)$ forms a two sided ideal if and only if $R/Nil(R)$ is reduced. Any NI ring with a derivation δ such that $\delta(Nil(R)) \subseteq Nil(R)$ is weak δ-McCoy by Proposition 2.5. But the converse does not hold by the following example.

Example 2.12. Let R be a ring with derivation δ and nonzero central nilpotent element c such that $\delta(Nil(R)) \subseteq Nil(R)$. Then $M_n(R)$ ($n \geq 2$) is a weak $\overline{\delta}$-McCoy ring by Proposition 2.10. However $M_n(R)$ can not be an NI ring as can be seen by the two nilpotent matrix units e_{12} and e_{21}.

Let R be a ring and σ denotes an endomorphism of R such that $\sigma(1) = 1$. In [9], the authors introduced *skew triangular matrix ring* a set of all triangular matrices with addition point-wise and a new multiplication subject to condition $e_{ij}r = \sigma^{j-i}(r)e_{ij}$. Therefore, $(a_{ij})(b_{ij}) = (c_{ij})$, where $c_{ij} = a_{ii}b_{ij} + a_{i,i+1}\sigma(b_{i+1,j}) + ... + a_{ij}\sigma^{j-i}(b_{jj})$, for each $i \leq j$ and denoted it by $T_n(R, \sigma)$. The derivation δ of R is extended to $\overline{\delta} : T_n(R, \sigma) \to T_n(R, \sigma)$ defined by $\overline{\delta}((a_{ij})) = (\delta(a_{ij}))$.

One can see that the map $\overline{\sigma} : R[x; \delta] \to R[x; \delta]$ defined by $\overline{\sigma}(\sum_{i=0}^{m} a_i x^i) = \sum_{i=0}^{m} \sigma(a_i)x^i$ is an endomorphism of the polynomial ring $R[x; \delta]$. Also the derivation δ of R is extended to $\overline{\delta} : T_n(R, \sigma) \to T_n(R, \sigma)$ defined by $\overline{\delta}((a_{ij})) = (\delta(a_{ij}))$,.

Proposition 2.13. *Let R be a ring, σ an endomorphism and δ a derivation of R such that $\delta\sigma = \sigma\delta$. Then $T_n(R, \sigma)$ is a weak $\overline{\delta}$-McCoy ring for $n \geq 2$.*

Proof. Let $f(x) = A_0 + A_1x + \cdots + A_px^p$ and $g(x) = B_0 + B_1x + \cdots + B_qx^q$ be elements of $T_n(R, \sigma)[x; \overline{\delta}]$ satisfying $f(x)g(x) = 0$. Then

$$\left(\sum_{l=k}^{p} \binom{l}{k} A_l \overline{\delta}^{(l-k)}(e_{1n})\right)^2 = 0$$

and the proof is complete. □

Proposition 2.14. *Let δ be a derivation of a ring R. Let S be a ring and $\phi : R \to S$ be a ring isomorphism. Then R is weak δ-McCoy if and only if S is weak $\phi\delta\phi^{-1}$-McCoy.*

Proof. Let $\alpha' = \varphi\alpha\varphi^{-1}$ and $\delta' = \varphi\delta\varphi^{-1}$. Since $\delta'(ab) = \varphi\delta(\varphi^{-1}(a)\ \varphi^{-1}(b)) = \varphi((\delta\varphi^{-1}(a)\varphi^{-1}(b) + \varphi^{-1}(a)(\delta\varphi^{-1}(b))) = \delta'(a)b + a\delta'(b)$, then δ' is a derivation of S. Suppose $a' = \varphi(a)$, for each $a \in R$. Therefore $p(x) = \sum_{i=0}^{m} a_ix^i$ and $q(x) = \sum_{j=0}^{n} b_jx^j$ are nonzero in $R[x; \delta]$ if and only if $p'(x) = \sum_{i=0}^{m} a'_ix^i$ and $q'(x) = \sum_{j=0}^{n} b'_jx^j$ are nonzero in $S[x; \delta']$. On the other hand, $p(x)q(x) = 0$ iff $\sum_{l=0}^{k}\sum_{i=l}^{m}\binom{i}{l}\varphi(a_i\delta^{i-l}(b_{k-l})) = 0$ iff $\sum_{l=0}^{k}\sum_{i=l}^{m}\binom{i}{l}a'_i\varphi(\varphi^{-1}\varphi\delta^{i-l}\varphi^{-1}\varphi(b_{k-l})) = 0$ iff $\sum_{l=0}^{k}\sum_{i=l}^{m}\binom{i}{l}a'_i\delta'^{i-l}(b'_{k-l}) = 0$ iff $p(x)q(x) = 0$ for $0 \le k \le m + n$. Also for some nonzero $c \in R$, $\sum_{l=k}^{m}\binom{l}{k}a_l\delta^{l-k}(c) \in Nil(R)$ iff for some positive integer n, $(\sum_{l=k}^{m}\binom{l}{k}a_l\delta^{l-k}(c))^n = 0$ iff $(\sum_{l=k}^{m}\binom{l}{k}\varphi(a_l)\varphi\delta^{l-k}\varphi^{-1}\varphi(c))^n = 0$ iff $(\sum_{l=k}^{m}\binom{l}{k}a'_l\delta'^{l-k}(c'))^n = 0$ iff $\sum_{l=k}^{m}\binom{l}{k}a'_l\delta'^{l-k}(c') \in Nil(S)$, for $0 \neq c' = \varphi(c) \in S$. Thus R is weak δ-McCoy if and only if S is weak $\varphi\delta\varphi^{-1}$-McCoy. □

Following Cohn [6], a ring R is called reversible if $ab = 0$ implies that $ba = 0$. Clearly, reduced rings are reversible. Moreover for any derivation δ, R is said to be δ-compatible if for each $a, b \in R$, $ab = 0$ implies that $a\delta(b) = 0$. The following lemma is appeared in [7].

Lemma 2.15. *Let R be a δ-compatible ring. If $ab = 0$, then $a\delta^m(b) = 0 = \delta^m(a)b$, for all positive integer m.*

Theorem 2.16. *Let R be a reversible ring. If R is δ-compatible then R is weak δ-McCoy.*

Proof. Let $f(x) = \sum_{i=0}^{m} a_ix^i$ and $g(x) = \sum_{j=0}^{n} b_jx^j$ be nonzero polynomials in $R[x; \delta]$ such that $f(x)g(x) = 0$. We can assume $g(x)$ has minimum degree that satisfies $f(x)g(x) = 0$ and $b_1 \neq 0$. We show that $a_ib_j = 0$, for each i and j, and this implies $\sum_{l=k}^{m}\binom{l}{k}a_l\delta^{l-k}(b_1) = 0 \in Nil(R)$ and so R is weak δ-McCoy. Since $f(x)g(x) = 0$ and R is reversible, we have $a_mb_n = 0 = b_na_m$. So $b_nx^na_m = 0$, since R is δ-compatible. On the other hand, $f(x)g(x)a_m = f(x)(\sum_{j=0}^{n} b_jx^j)a_m = 0$. Thus $f(x)(b_0+...+b_{n-1}x^{m-1})a_m = 0$. Since the degree of $g(x)$ is minimum, we have $(b_0+...+b_{n-1}x^{n-1})a_m = 0$. So $b_ja_m = a_mb_j = 0$, for each $0 \le j \le n-1$, since R is reversible and δ-compatible. Hence $a_mx^mb_j = 0$, for $0 \le j \le n$, since R is δ-compatible. So $(a_0 + ... + a_{m-1}x^{m-1})g(x) = 0$, and hence $a_{m-1}b_n = 0$. Therefore, $a_{m-1}b_n = b_n a_{m-1} = 0$. On the other hand, we have $f(x)g(x)a_{m-1} = 0$. Hence $f(x)(b_0 + ... + b_{n-1}x^{n-1})a_{m-1} = 0$, since $b_n x^n a_{m-1} = 0$. Therefore we have $(b_0 + ... + b_{n-1}x^{n-1})a_{m-1} = 0$, since the degree of $g(x)$ is minimum, and so according to above $a_{m-1}b_j = b_j a_{m-1} = 0$, for all j. Continuing in this way, we get $a_i b_j = 0$, for each i and j, and the result follows. □

Let S denote a multiplicatively closed subset of a ring R consisting of central regular elements. Let RS^{-1} be the localization of R at S. Then each derivation δ of R, extends to RS^{-1}, by setting $\overline{\delta}(rc^{-1}) = r\delta(c)^{-1}$, for each $r, c \in R$, with c regular. Now we have the following:

Theorem 2.17. *For a ring R and derivation δ of R, if R is weak δ-McCoy then RS^{-1} is weak $\overline{\delta}$-McCoy.*

Proof. Let $f(x) = \sum_{i=0}^{m} a_i c_i^{-1} x^i$ and $g(x) = \sum_{j=0}^{n} b_j d_j^{-1} x^j \in RS^{-1}[x;\overline{\delta}]$ such that $f(x)g(x) = 0$. Let $a_i c_i^{-1} = c^{-1}a'_i$ and $b_i d_i^{-1} = d^{-1}b'_j$ with c , d regular elements in R. So $f'(x)g'(x) = 0$ such that $f'(x) = \sum_{i=0}^{m} a'_i x^i$ and $g'(x) = \sum_{j=0}^{n} b'_j x^j$. Since R is weak δ -McCoy, there exists $0 \neq r \in R$ such that $\sum_{l=k}^{m} \binom{l}{k} a'_l \delta^{l-k}(r) \in Nil(R)$. Hence $\sum_{l=k}^{m} \binom{l}{k} a_l c_l^{-1} \overline{\delta}^{l-k}(r) \in Nil(RS^{-1})$ and so RS^{-1} is weak $\overline{\delta}$-McCoy. □

Corollary 2.18. *Let $R[x,\delta]$ be a weak δ-McCoy ring. Then $R[x;x^{-1},\delta]$ is a weak δ-McCoy ring.*

Proof. It is directly follows from Proposition 2.17. Let $S = \{1, x, x^2, ...\}$, then clearly S is a multiplicatively closed subset of $R[x,\delta]$ and $R[x,x^{-1},\delta] = S^{-1}R[x,\delta]$. □

Acknowledgement

This paper is supported by Islamic Azad University Central Tehran Branch (IAUCTB). The authors want to thank the authority of IAUCTB for their support to complete this research.

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Accepted: 26.04.2017