New investigations on HX-groups and soft groups

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Abstract. The paper presents new results obtained on $HX-$groups introduced by Hongxing in [7] and then investigated by Corsini in his study on HX-hypergroups see [1]. We also present the results obtained by Corsini on the relationship between $HX-$groups and Soft groups. On the other hand we present some new results on these types of groups and define a new objects which we call Soft $HX-$groups. We give many examples to illustrate the notions introduced and explain their usefulness. To conclude we present some topics about the soft $HX-$groups that can be investigated.

Keywords: HX-group, soft groups.

1. Introduction

The notion of $HX-$groups has been introduced by some Chinese mathematicians like Mi Honghai and co [6], Li Honxing [7]. This notion has been revisited by Piergiulio Corsini where he made a link between $HX-$groups and hypergroups. In $[3, 4, 5]$ he constructed some hyperstructures from $HX-$groups. Recently in his paper [1], he determined the hypergroups associated to the $HX-$group $\mathbb{Z}/n\mathbb{Z}$ and associated to the set of square matrices of order 2 with coefficients in $\mathbb{Z}/n\mathbb{Z}$. The notions of $HX-$groups and related topics are of the interest of many researchers worldwide.

The first part of the paper is devoted to the notion of $HX-$groups $\mathcal{G}$ with support a group G introduced by [7]. We investigate the relationship between the structure of $\mathcal{G}$ and G. We establish many interesting results. Among the results, we have been concerned by the extension of a given group morphism to the associated $HX-$structure.

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The second part of the paper is devoted to the soft groups. We establish some results and show that the conditions of the proposition 2.3 in [10] should be improved otherwise the result is not true.

The last part of the paper is concerned with the relationship between the notion of $HX-$groups and the soft groups. We give an example of soft $HX-$group when the support G is of order $p^m q^n$, for some values of the prime integers p and q.

Some works have been cited to enrich the bibliography and to give an overview of the state of the art.

2. Preliminaries

In this section we recall the principal definitions and some well known results related to our subject.

The following classical result will be used continually when the group is of finite order.

Proposition 2.1. *In the table of a finite group each row contains each element of the group exactly once and each column contains each element of the group exactly once.*

Definition 2.2. Let $(G,.)$ be a group and $\mathcal{G} \subset \mathcal{P}(G) \setminus \{\emptyset\}$. $\mathcal{G}$ is a $HX-$group if it is a group for the binary operation $*$ defined by:

$$\forall A, B \in \mathcal{G},\ A * B = \{a.b \mid a \in A,\ b \in B\}.$$

This means that $*$ is internal binary operation on $\mathcal{G}$, associative, $\exists E \in \mathcal{G}$ such that $A * E = E * A = A$, $\forall A \in \mathcal{G}$ and any element of $\mathcal{G}$ has a symmetric with respect to $*$.

In the sequel the group G will be called the support of $\mathcal{G}$.

Definition 2.3 ([4])**.** Let $\mathcal{G}$ be a $HX-$group with support G. If E and e are the identities of the groups $\mathcal{G}$ and G respectively. The group $\mathcal{G}$ is called regular if $e \in E$.

Proposition 2.4 ([7])**.** *The set E is a semigroup of the group G.*

Proposition 2.5 ([7])**.** *Let $\mathcal{G}$ be a $HX-$group with support G. If E is the identity of the group $\mathcal{G}$, then:*

1. $\forall\ A \in \mathcal{G}, |A| = |E|$.
2. $\forall A, B \in \mathcal{G}, A \cap B \neq \emptyset \implies |A \cap B| = |E|$.

Let U be an initial universe set and E be a set of parameters. Let $\mathcal{P}(U)$ be the power set of U , and $A \subset E$.

Definition 2.6. A pair (A, f) is called a soft set over U, where f is a mapping given by
$$f : A \longrightarrow \mathcal{P}(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $a \in A, f(a)$ may be considered as the subset of U of U of type a.

Definition 2.7. A soft set (A, f) over U is called a null soft set, denoted by $f_\emptyset$, if
$$\forall a \in A, f(a) = \emptyset.$$

Definition 2.8. A soft set (A, f) over U is called an absolute soft set, denoted by $\widetilde{f_A}$, if for all $a \in A, f(a) = U$.

Definition 2.9. The union of two soft sets (A, f), (B, g) over a common universe U is the soft set $h : E \longrightarrow \mathcal{P}(U)$ defined by

$$\forall x \in E, h(x) = \begin{cases} f(x), & \text{if } x \in A \setminus B \\ g(x), & \text{if } x \in B \setminus A \\ f(x) \cup g(x), & \text{if } x \in A \cap B \end{cases}.$$

Definition 2.10. The intersection of two soft sets (A, f), (B, g) over a common universe U the soft set $h : E \longrightarrow \mathcal{P}(U)$ defined by

$$\forall x \in E, h(x) = f(x) \cap g(x).$$

Definition 2.11. For a soft set (A, f) over U, the relative complement of (A^c, f^c) is defined by $f^c : E \longrightarrow \mathcal{P}(U)$, where $\forall a \in E, f^c(a) = U \setminus f(a)$.

Definition 2.12. A pair (A, f) is called a soft subgroup over a group U, where f is a mapping given by
$$f : A \longrightarrow \mathcal{P}(U)$$
if for all $x \in A$; $f(x)$ is a subgroup of U.

3. Main results

3.1 HX-groups

Proposition 3.1. *Let $\mathcal{G}$ be a $HX-$group. If E is its identity such that $|E| < \infty$, then E is a subgroup and therefore $\mathcal{G}$ is regular.*

Proof. To prove the proposition it suffices to prove that $e \in E$.

Let us set $E = \{a_1, a_2, \cdots, a_n\}$ since $E * E = E$, then for a fixed index $j \in \{1, 2, \cdots, n\}$, there exists $k \in \{1, 2, \cdots, n\}$ such that $a_j.a_k = a_j$ and then $a_j^{-1}.a_j.a_k = a_j^{-1}.a_j \iff a_k = e$. □

The following result is a consequence of the above Proposition and Proposition 2.5.

Corollary 3.2. *Let $\mathcal{G}$ be a $HX-$group of support G and E its identity.*

1. *If E is finite, two elements of $\mathcal{G}$ are disjoint and none of them excepted E is a subgroup of G.*
2. *If E is infinite, the intersection of two elements of $\mathcal{G}$ is also infinite.*

Proposition 3.3. *Let $\mathcal{G}$ be a $HX-$group. If E is finite, then $\forall a \in E,\ a^{-1} \in E$.*

Proof. Let us set $E = \{a_1, a_2, \cdots, a_n\}$ and let $a_i \in E$. Since $E * E = E$ and $e \in E$ then there exists $k \in \{1, 2, \cdots, n\}$ such that $a_i.a_k = e$. By the uniqueness of the symmetric element of a_i in G, even the group is not abelian. □

Corollary 3.4. *Let $\mathcal{G}$ be a $HX-$group. If E is finite, then it is a subgroup of G.*

Proof. the proof is a consequence of the above propositions and of the condition $E * E = E$. □

Example 3.5. Let $G = \mathbb{Z}_4$, the set $\mathcal{G} = \{\{0, 2\}, \{1, 3\}\}$ is a $HX-$group, its identity is the set $\{0, 2\}$ and the symmetric of $\{1, 3\}$ is $\{1, 3\}$ itself.

Lemma 3.6. *Let $\mathcal{G}$ be a $HX-$group. If G is finite and G has no element of order 2, then $|E|$ is of odd order.*

Proof. Since E is a subgroup of G, by Cauchy's theorem its order divides the order of G. If the order of E is even then 2 divides $|G|$ and then it contains an element of order 2, contradiction. □

Lemma 3.7. *Let $\mathcal{G}$ be a $HX-$group. If $E = \{e\}$, then $\forall A \in \mathcal{G},\ |A| = 1$.*

Proof. Suppose that there exists in $\mathcal{G}$ an element A such that $|A| \geq 2$. Let a_1, a_2 two distinct elements of A. As $\mathcal{G}$ is a group $A^{-1} \in \mathcal{G}$, so it is non empty, it contains an element $c \in G$. On the other hand $A * A^{-1} = \{e\}$ so $a_1.c = e = a_2.c$ and then $a_1 = a_2$ contradiction. □

Example 3.8. From the example above and from the lemma, we can deduce that the only structure of $HX-$group on $\mathbb{Z}_4$ are

$$\mathcal{G}_1 = \{\{0\}, \{1\}, \{2\}, \{3\}\},$$
$$\mathcal{G}_2 = \{\{0\}, \{2\}, \},$$
$$\mathcal{G}_3 = \{\{0, 2\}, \{1, 3\}\}.$$

Proposition 3.9. *Let $\mathcal{G}$ be a $HX-$group. If $E = \{e\}$ and H is a subgroup of G, then $\mathcal{H} = \{\{a\} \mid a \in H\}$ is a subgroup of $\mathcal{G}$.*

Proof. Although the proof is trivial, we give it as an illustration of our study.

1. Since $e \in H$, then $\{e\} \in \mathcal{H}$.
2. Let $\{a\}, \{b\} \in \mathcal{H}$, since H is a subgroup, $a, b \in H$ then $ab \in H$ and so $\{a\} * \{b\} = \{ab\} \in \mathcal{H}$.
3. Let $\{a\} \in \mathcal{H}$, then the element $a \in H$ so its symmetric a^{-1} is also an element of H. But $\{a\}^{-1} = \{a^{-1}\} \in \mathcal{H}$.

□

The converse of the above proposition is also true. More exactly;

Proposition 3.10. *Let $\mathcal{G}$ be a $HX-$group. If $E = \{e\}$, H a subset of G and $\mathcal{H} = \{\{a\} \mid a \in H \subset G\}$. If $\mathcal{H}$ is a subgroup of$\mathcal{G}$, the set H is then a subgroup of G.*

Proof. Note that since $E = \{e\}$, then for any $a \in G, \{a\}^{-1} = \{a^{-1}\}$.

1. Since $\mathcal{H}$ is a subgroup and $E = \{e\}$ then $e \in H$.
2. Let $a, b \in H$ then $\{a\}, \{b\}^{-1} \in \mathcal{H}$ and so $\{ab^{-1}\} = \{a\} * \{b\}^{-1} \in \mathcal{H}$. Finally $ab^{-1} \in H$.

□

Definition 3.11. A subgroup H of a group G is said to be closed with respect to the elements of the group $\mathcal{G}$ if for any expression of $x \in H$ as a product $x = a.b,\ a \in A,\ b \in B$ implies $a, b \in H$.

Proposition 3.12. *Let $\mathcal{G}$ be a $HX-$group with support G. If the elements of $\mathcal{G}$ form a cover of G and order of G is prime then $\mathcal{G} = \{G\}$ or $\mathcal{G} = \{\{a\} \mid a \in G\}$.*

Proof. From the proposition 2.5, the elements of $\mathcal{G}$ are disjoint and since they form a cover of G then $|G| = n|E|$ where $n = |\mathcal{G}|$. But $|G|$ is prime then $n \in \{1, |G|\}$ and the result follows. □

Proposition 3.13. *Let $\mathcal{G}$ be a $HX-$group. For any subgroup H of G closed with respect to the elements of the group $\mathcal{G}$ such that $E \subset H$ and $H \cap A \in \mathcal{G}$, $\forall A \in \mathcal{G}$, the set $\mathcal{H} = \{H \cap A \mid A \in \mathcal{G}\}$ is a subgroup of $\mathcal{G}$.*

Proposition 3.14 ([7])**.** *Let H be a subgroup of a group G, E an idempotent subset of G (i.e. $E^2 = E$). If for all $a \in H$, $aE = Ea$, then the set $\mathcal{G} = \{aE \mid a \in H\}$ is a $HX-$group with support the group G.*

In the proof it is shown that $H/kerf \approx \mathcal{G}$ where $f : H \longrightarrow \mathcal{G},\ a \longmapsto aE$. We can then deduce the following corollary.

Corollary 3.15. *If E is a subgroup of G such that $H \cap E = \{e\}$, then $H \approx \mathcal{G}$.*

Proof. To proof the corollary it suffices to prove that f is injective. The conditions $H \cap E = \{e\}$ and $aE = E$ imply that $a = e$ and the mapping is injective. □

Example 3.16. Suppose that $|G| = p^m q^n$, p, q distinct prime integers. Let E, H be respectively a p sylow and a q sylow subgroup of G. We have $E^2 = E$ and $E \cap H = \{e\}$. If $p^\alpha \equiv 1[q] \implies \alpha = 0$ then the $HX-$group $\mathcal{G} = \{aE \mid a \in H\}$ is isomorphic to the group H. This holds for example when $|G| = 5^2 \times 7^2$, $|E| = 7^2$ and $|H| = 5^2$.

Proof. 1. Since $E = H \cap E$ and $E \in \mathcal{G}$, then $E \in \mathcal{H}$.

2. Suppose that $A, B \in \mathcal{H}$, then there exist $A_1, B_1 \in \mathcal{G}$ such that $A = H \cap A_1$, $B = H \cap B_1$. It is easy to show that $A * B \subset H \cap (A_1 * B_1)$.

On the other hand, suppose that $x \in H \cap (A_1 * B_1)$ then there exist $a \in A_1$, $b \in B_1$ such that $x = a.b$. Since H is closed with respect to the elements of the group $\mathcal{G}$, the elements a, b are in H so $a.b \in (H \cap A_1) * (H \cap B_1)$ and then $H \cap (A_1 * B_1) \subset (H \cap A_1) * (H \cap B_1) = A * B$. The set $\mathcal{H}$ is then closed for $*$.

3. Let now $A \in \mathcal{H}$, then there exists $A_1 \in \mathcal{G}$ such that $A = H \cap A_1$. Let us prove that $A^{-1} = H \cap A_1^{-1}$.

Let $x \in A^{-1}$ then for all $y \in A$, $x.y \in E \subset H$. From $A \subset H$, we can deduce that $x \in H$. On the other hand since $H \cap A_1 \subset A_1$, then $(H \cap A_1)^{-1} \subset (A_1)^{-1}$ and so $x \in (A_1)^{-1}$. We can conclude that $A^{-1} \subset H \cap A_1^{-1}$.

Now let $x \in H \cap A_1^{-1}$ and $y \in A = H \cap A_1$. We then have $x, y \in H$, $x \in A_1^{-1}$ and $y \in A_1$ so $x.y \in H$ and $x.y \in E$ and then $x \in (H \cap A_1)^{-1} = A^{-1}$. Conclusion $H \cap A_1^{-1} \subset A^{-1}$.

□

Definition 3.17. A subgroup H of a group G is said to be strongly normal with respect to the $HX-$group $\mathcal{G}$ if for all $K \in \mathcal{G}$, for all $a \in G$, $x \in H \cap K \implies a.x.a^{-1} \in H \cap K$, .

Proposition 3.18. *Let the subgroup H as in the above proposition and suppose that there exists at least an element $K' \in \mathcal{G}$ such that $H \subset K'$ and for any element $K \in \mathcal{G}$ if $K \subset H \cap K'$, there exists $K'' \in \mathcal{G}$ such that $K = H \cap K''$. If it is normal then so is the subgroup $\mathcal{H}$.*

Proof. Let $A \in \mathcal{G}$ and $B \in \mathcal{H}$. and let us prove that $A * B * A^{-1} \in \mathcal{H}$. Take $x \in B, a \in A$ and $b \in A^{-1}$. The element $a.x = x'.a$ for some $x' \in H \cap K$ since H is strongly normal with respect to $\mathcal{G}$. Now the element $a.b \in A * A^{-1} = E$, then $a.x.b = x'.a.b \in B * E = B$. Consequently $A * B * A^{-1} \subset B$.

Let $x \in B$, $a \in A$, then for any $a \in A$ and $b \in A^{-1}$ we have $x = a.(a^{-1}.x.b^{-1})b$. But it is clear as shown above that $a^{-1}.x.b^{-1} \in B$. We can conclude that $A * B * A^{-1} \supset B$ and then $A * B * A^{-1} = B$ and $\mathcal{H}$ is normal in $\mathcal{G}$. □

Now, we want to extend a group morphism to the associated $HX-$group structures. More exactly we have the following

Proposition 3.19. *Let $f : G_1 \longrightarrow G_2$ be a group morphism. The mapping $\widetilde{f} : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ defined by: $\widetilde{f}(A) = \{f(x) \mid x \in A\}$ is a group morphism, where $\mathcal{G}_1,\ \mathcal{G}_2$ are $HX-$groups with support G_1 and G_2 respectively.*

Proof. The proof is trivial. More exactly, let $A_1, A_2 \in \mathcal{G}$. We have

$$\begin{aligned}\widetilde{f}(A_1 * A_2) &= \{f(x) \mid x \in A_1 * A_2\} = \{f(x_1.x_2) \mid x_1 \in A_1, x_2 \in A_2\}\\ &= \{f(x_1).f(x_2) \mid x_1 \in A_1, x_2 \in A_2\}\\ &= \{f(x) \mid x \in A_1\} * \{f(x) \mid x \in A_2\} = \widetilde{f}(A_1) * \widetilde{f}(A_2).\end{aligned}$$

□

Proposition 3.20. *Let $f : G_1 \longrightarrow G_2$ be a group morphism and $\widetilde{f} : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ as defined above. we have the relation $\widetilde{f}(kerf) \subset ker\widetilde{f}$. The equality holds when $E = \{e\}$.*

Proof. Let $A = kerf$ then $ker\widetilde{f}(A) = \{f(x) \mid x \in A\} = \{e\} \subset E = ker\widetilde{f}$. The second assertion follows from the inclusion. □

Proposition 3.21. *Let $f : G_1 \longrightarrow G_2$ be a group morphism and $\widetilde{f} : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ as defined above. If E_1 is finite and f is one-to-one then $\widetilde{f}$ is also one-to-one.*

Proof. Let us denote by E_1, E_2 the identities of $\mathcal{G}_1,\ \mathcal{G}_2$ respectively and suppose that $E_1 = \{a_1, a_2, ..., a_n\}$. Since $\widetilde{f}(E_1) = E_2$ and ff is one-to-one, necessary $E_2 = \{b_1, b_2, ..., b_n\}$ with distinct elements. Let $A \in \mathcal{G}$ such that $\widetilde{f}(A) = E_2$. If there exists $c \in A$ such that $c \notin E_1$ and $f(c) \in E_2$ then there exist two different element $c, a_i \in G$ such that $c \neq a_i,\ f(c) = f(a_i) = b_j$ contradiction. So $A \subset E_1$ and then $A = E_1$. □

Proposition 3.22. *Let $f : G_1 \longrightarrow G_2$ be a group morphism and $\widetilde{f} : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ as defined above. If $G_2 \subset \bigcup_{A \in \mathcal{G}_2} A$ and $\widetilde{f}$ is onto, then f is also onto.*

Proof. Let $y \in G_2$. Since the elements of $\mathcal{G}_2$ form a cover of G_2, then there exists $B \in \mathcal{G}_2$ such that $y \in B$. The mapping $\widetilde{f}$ is onto, then there exists $A \in \mathcal{G}_1$ such that $\widetilde{f}(A) = B$ and then $y \in A = \{f(x) \mid x \in A\}$ so $y = f(x)$ for some $x \in A$ and the mapping f is onto. □

Proposition 3.23. *Let $f : G_1 \longrightarrow G_2$ be a group morphism and $\widetilde{f} : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ as defined above. If the inverse image of any element of $\mathcal{G}_2$ is an element of $\mathcal{G}_1$ and f is onto then $\widetilde{f}$ is also onto.*

Proof. Let $B \in \mathcal{G}_2$. Since f is onto then $B \subset f(G_1)$ and then any element of B is the image of an element of G_1, so there exists $A \subset G_1$ such that $f(A) = B$. We deduce that A is the inverse image of B which is an element of $\mathcal{G}_2$ so it is an element of $\mathcal{G}_1$ and the proof follows. □

3.2 Soft groups

This section is devoted to some algebraic notions of Soft set theory which are different from others. We introduce and define some soft operations and also we establish a few interesting results in this context. However any soft (A, f) can be viewed as the soft set (E, f), where if $x \notin A, f(x) = \emptyset$, so all the soft sets will be denoted in the form (E, f) or simply f_E.

Let us start this subsection by an example of soft group.

Example 3.24. Let G be the additive group $(\mathbb{Z}/6\mathbb{Z}, +)$ for example and $E = \{1, 2, 3, 6\}$. We can parameterize the set of subgroups of $(\mathbb{Z}/6\mathbb{Z}, +)$ by their cardinality. More exactly let

$$\begin{array}{cccc} f: & E & \longrightarrow & \mathcal{P}(\mathbb{Z}/6\mathbb{Z}) \\ & 1 & \longmapsto & 6\mathbb{Z}/6\mathbb{Z} \\ & 2 & \longmapsto & 3\mathbb{Z}/6\mathbb{Z} \\ & 3 & \longmapsto & 2\mathbb{Z}/6\mathbb{Z} \\ & 6 & \longmapsto & \mathbb{Z}/6\mathbb{Z} \end{array}$$

the soft set (E, f) is a soft group.

Definition 3.25. Let $*$ be a binary operation on U. If f_E and g_E are two soft sets over U, then we define $f_E \star g_E$ as a soft set h_E defined by : $h : E \longrightarrow \mathcal{P}(U)$ such that $h(a) = \{x * y : x \in f(a) \ \ and \ \ y \in g(a)\}$.

We denote the collection of all soft sets over U with domain E by the symbol $S_E(U)$.

Proposition 3.26 ([2])**.** *The above operation $\star$ is a binary operation on $S_E(U)$.*

The following example shows that the Proposition 2.3 [10] is false, the problem is that the author at the end of his proof, affirm that $F_E \circ F_E^{-1} = I_E$ which is false as in the example. We adopt in this example the same notations as in [].

Example 3.27. Let $E = \{1, 2\}$ and U be the group $(\mathbb{Z}/3\mathbb{Z}, +)$ if $f_E(1) = \{0, 1\}$ then $f_E^{-1}(1) = \{0, 2\}$ so $(f_E(1) \circ f_E^{-1})(1) = \{x + y \mid x \in f_E(1) \ and \ -y \in f_E(1)\} = \{0, 1, 2\} \neq \{0\} = I_E(1)$.

In the following proposition which is easy to prove, we introduce a new definition of composition of soft sets such that the set of all soft sets over a group, becomes a group for this composition.

Proposition 3.28. *If $(U, *)$ is a group then $(S_E(U), \star)$ is also a group where $\star$ is defined by*

$$(f \star g)(a) = \begin{cases} \{x * y \mid x \in f(a) \ and \ -y \in g(a)\}, & if \ \ g \neq f^{-1} \\ \{e\}, & if \ \ g = f^{-1} \end{cases}.$$

Corollary 3.29. *Suppose that $(U,*)$ is a group and H is a subgroup of $(U,*)$, then the set*

$$\mathcal{H} = \{f : E \longrightarrow \mathcal{P}(H), \text{ the elements of } S_E(U) \text{ with values in } \mathcal{P}(H)\}$$

is a subgroup of $(S_E(U),\star)$.

The result follows directly from the Proposition 4, however we propose the following direct proof.

Proof.
1. H contains at least the neutral element e of U then $\{e\} \in \mathcal{P}(H)$ so the mapping $\mathcal{O} : E \longrightarrow \mathcal{P}(H),\ x \longmapsto \{e\}$ is an element of $\mathcal{H}$.
2. Let $f, g \in \mathcal{H}$ then for all $a \in E$, $(f \star g)(a) = \{x * y \mid x \in f(a),\ y \in g(a)\}$. But $x \in f(a),\ y \in g(a) \implies x \in H$ *and* $y \in H$ so the element $x * y \in H$ and then $\{x * y \mid x \in f(a),\ y \in g(a)\} \subset H$.
3. Let $f \in \mathcal{H}$ then for all $a \in E$, $(f^{-1})(a) = \{x^{-1} \mid x \in f(a)\} \subset H$ since the inverse x^{-1} of any x is also in H. The mapping f^{-1} is an element of $\mathcal{H}$.

□

Example 3.30. 1. Let $E = \mathbb{R}^*$ and $U = GL_n(\mathbb{R})$. For any $a \in \mathbb{R}^*$ let $f(a) = \{A \in U \mid detA = a\}$. The pair (E, f) is a soft set.

2. Suppose that H is a normal subgroup of $(GL_n(\mathbb{R}), \times)$ if the corresponding set $\mathcal{H}$ of the previous proposition is such that $\forall f \in \mathcal{H}$ and $\forall a \in \mathbb{R}^*, f(a)$ is closed by similarity, then it is a normal subgroup of $S_{\mathbb{R}^*}(GL_n(\mathbb{R}))$. Indeed suppose that $f \in \mathcal{H}, g \in GL_n(\mathbb{R})$ then for $a \in \mathbb{R}^*$, $(f \circ g)(a) = \{A \times P \mid A \in f(a), P \in g(a)\}$. But in $GL_n(\mathbb{R})$ if $A \times P = P \times A'$ then $A' = P^{-1} \times A \times P$ which still an element of $f(a)$ by our hypothesis on $\mathcal{H}$ and since H is normal, then $(f \circ g)(a) = \{A \times P \mid A \in f(a), P \in g(a)\} = (g \circ f)(a) = \{P \times A \mid A \in f(a), P \in g(a)\}$ for all $g \in GL_n(\mathbb{R})$ and $\mathcal{H}$ is then normal.

Proposition 3.31. *Suppose that $(U,*)$ is a group and H is a normal subgroup of $(U,*)$, if the set $\mathcal{H} = \{f : E \longrightarrow \mathcal{P}(H)\}$ is such that, for all $a \in E$ and for all $x \in H$ there exists $f \in \mathcal{H}$ such that $x \in f(a)$, then $\mathcal{H}$ is also a normal subgroup.*

Proof. If $f \in \mathcal{H}$ and $g \in S_E(U)$, then $(f \circ g)(a) = \{x * y \mid x \in f(a) \text{ and } y \in g(a)\}$. For any $x \in f(a) \subset H$ since H is normal then there exists $x' \in H$ such that $x * y = y * x'$. By our hypothesis as $x' \in H$ and $a \in E$ then there exists $f' \in \mathcal{H}$ such that $x' \in f'(a)$ so the set $(f \star g)(a)$ is included in the set $(g \star f')(a)$ for any $a \in E$ and then $\mathcal{H} \star g \subset g \star \mathcal{H}$, for all $g \in S_E(U)$. The proof of the other inclusion is similar. Finally we get

$$\forall g \in S_E(U), g \star \mathcal{H} = \mathcal{H} \star g$$

and the subgroup is then normal. □

Proposition 3.32. *Suppose that we are in the conditions of Proposition 3.30, then the following conditions are equivalent:*

1. $f \equiv g[\mathcal{H}]$,
2. $\forall a \in E,\ \forall x \in f(a),\ \forall y \in g(a), y \equiv x[H]$.

Proof. 1. Suppose that $f \equiv g[\mathcal{H}]$ then for all $a \in E,\ (f \star g^{-1})(a) \subset H$ and so $\{x * y^{-1} \mid x \in f(a), y \in g(a)\} \subset H$. the conclusion follows.

2. Suppose now that $\forall a \in E,\ \forall x \in f(a),\ \forall y \in g(a), y \equiv x[H]$, then $(f \star g^{-1})(a) = \{x * y^{-1} \mid x \in f(a), y \in g(a)\}$ is certainly included in H since any element $x * y^{-1} \in H,\ \forall x \in f(a),\ \forall y \in g(a)$ and then $f \equiv g[\mathcal{H}]$.

□

Proposition 3.33. *If (A, f) and (B, g) are soft groups such that (A, f) is normal, then:*

1. (A, f^{-1}) *is a soft group,*
2. $(A \cap B, f \star g)$ *is a soft group if and only if* $A \cap B \neq \emptyset$.

Proposition 3.34. *If the group U is of prime order then the only soft group of $S_E(U)$ are:*

1. $f : E \longrightarrow S_E(U); x \longmapsto f(x) = U$ *and*
2. $I : E \longrightarrow S_E(U); x \longmapsto f(x) = \{e\}$.

Proof. The proof follows from the classical theorem on groups, that is, the order of any subgroup divides the order of the group. □

3.3 Soft HX-group

The origin of this section is the paper [1] of Professor P. Corsini, where he studied certain types of HX-hypergroup.

Definition 3.35. Let $(G, *)$ be a group a soft HX-group is a soft set $f \in S_E(G)$ such that $f(E)$ is a group under the binary operation:

$$f(x) * f(y) = \{a * b,\ a \in f(x), b \in f(y)\}.$$

Definition 3.36. A soft set $f \in S_E(G)$ is said almost-surjective if for any subset $H \in \mathcal{P}^*(G)$ there exists $a \in E$ such that $f(a) \subset H$.

Lemma 3.37. *Let $f \in S_E(G)$ be an almost-surjective soft HX-group, on the set E, $\circ$ defined by:*

$$\forall x, y \in E, x \circ y = \{a \in E \mid f(a) \subset f(x) * f(y)\},$$

where $f(x) * f(y) = \{a * b,\ a \in f(x),\ b \in f(y)$, is a hyperoperation on E.

Proof. Since $\forall x \in E,\ f(x) \neq \emptyset$, then $f(x) * f(y) \neq \emptyset$ and is an element of $\mathcal{P}^*(G)$. f is almost-surjective then there is $a \in E$ such that $f(a) \subset f(x) * f(y)$ and then $x \circ y$ is a non empty subset of E and $\circ$ is then well defined. □

Lemma 3.38. *If the neutral element $e \in f(a),\ a \in E$, then for all $x \in E,\ x \in x \circ a$.*

Proof. If $e \in f(a)$ then $f(x) = e * f(x) \subset f(a) * f(x)$ and the conclusion follows. □

Proposition 3.39. *If $f \in S_E(G)$ is a surjective soft HX-group and $e \in f(x)$ for all $x \in E$, the set $(E, \circ)$ is a hypergroup.*

Proof. First if $x \in E$ and $F \subset E$ then we define

$$x \circ F = \bigcup_{a \in F} x \circ a.$$

1. Let $x, y, z \in G$ and $a \in x \circ (y \circ z)$. then there exists $b \in (y \circ z)$ such that $a \in x \circ b$ so $f(a) \subset f(x) * f(b)$. On the other hand $f(b) \subset f(y) * f(z)$ so $f(a) \subset f(x) * (f(y) * f(z)) = (f(x) * f(y)) * f(z)$. Since f is surjective there exists $c \in E$ such that $f(c) = f(x) * f(y)$ and then $f(a) \subset f(c) * f(z)$ which implies that $a \in c \circ z$. The equality $f(c) = f(x) * f(y)$ implies that $c \in x \circ y$ and then

$$a \in \bigcup_{\alpha \in x \circ y} \alpha \circ z.$$

And then $x \circ (y \circ z) \subset (x \circ y) \circ z$. The other inclusion is similar.

2. Let $a \in E$, then if $b \in E$, from the lemma $f(b) \subset f(a) * f(b)$ so $b \in a \circ b \subset \bigcup_{\alpha \in E} a \circ \alpha = a \circ E$ and the reproduction law follows.

□

Example 3.40 ([1])**.** Set $n = mq \in \mathbb{N}$ and define on the additive group $\mathbb{Z}/n\mathbb{Z}$ the family:

$A_0 = \{0, m, 2m, ..., (q-1)m\}$
$A_1 = \{1, 1+m, 1+2m, ..., 1+(q-1)m\}$
.
.
$A_{m-1} = \{m-1, 2m-1, 1+2m, ..., qm-1\}$.

If we define on the set $\mathcal{G} = \{A_0, A_1, ..., A_{m-1}\}$ the binary operation $\circ$ by:

$$A_i \circ A_j = \begin{cases} A_{i+j}, & \text{if } i+j \leq m-1 \\ A_{(m-1)-(i+j)}, & \text{if } i+j \geq m \end{cases},$$

$(\mathcal{G}, \circ)$ is a $HX-$group.

Now if we set $E = \mathbb{Z}/(m-1)\mathbb{Z}$ and $f(i) = A_i$, then $f(E) = \mathcal{G}$ and it defines a soft $HX-$group.

Remark 3.41. From the above example we note that if a prime p divides a natural n on $\mathbb{Z}/n\mathbb{Z}$, there exists a cyclic soft $HX-$group, defined as above by $f : \{0, 1, ..., p-1\} \longrightarrow \mathcal{G} = \{A_0, A_1, ..., A_{p-1}\}$. So there exists A_i which generates the group $\mathcal{G}$. What can we deduce for the mapping f in this case?

Since $\mathcal{G}$ is cyclic of prime order it is generated by A_1 for example and then $f(j) = A_1 + A_1 + ... + A_1,$ *j times* and so f is a group morphism.

From the above example we can deduce the following result.

Proposition 3.42. *Let G be a cyclic group of order $n = mq$ generated by a. Let for $i \leq m-1$, $A_i = \{a^i, a^{i+m}, ..., a^{(q-1)m+i}\}$. The set*

$$\mathcal{G} = \{A_i,\ 0 \leq i \leq m-1\},$$

endowed with the following binary operation $\circ$ is a $HX-$group.

$$A_i \circ A_j = \begin{cases} A_{i+j}, & \text{if } i+j \leq m-1 \\ A_{(m-1)-(i+j)}, & \text{if } i+j \geq m \end{cases}.$$

The structure $f : \mathbb{Z}/mq\mathbb{Z} \longrightarrow \mathcal{G}, i \longmapsto A_i$ is a soft $HX-$group.

Proof. The group G is then isomorphic to the group $\mathbb{Z}/mq\mathbb{Z}$. It suffices then to note that under this isomorphism the images of the sets A_i in the above example are exactly the sets A_i of the proposition. □

Proposition 3.43. *There exists a natural action of the group $\mathbb{Z}/m\mathbb{Z}$ on the group $\mathcal{G}$ of the above proposition given by $(g, A_i) \longmapsto A_{i+g}$ where the addition is performed modulo m.*

Lemma 3.44. *Let $f \in S_E(G)$ be a soft HX-group with support the group G. If $\mathcal{E}$ denotes the identity of $f(E)$, for any element $x \in E$,*

$$(f(x))^{-1} = \{a \in G \mid a.b \in \mathcal{E} \text{ and } b \in f(x)\}.$$

Proof. Let $c \in (f(x))^{-1}$ and $b \in f(x)$ then $c.b \in (f(x))^{-1} * f(x) = \mathcal{E}$. So $c \in \{a \in G \mid a.b \in \mathcal{E} \text{ and } b \in f(x)\}$.

Now let $c \in \{a \in G \mid a.b \in \mathcal{E} \text{ and } b \in f(x)\}$. So $c.b \in \mathcal{E}$. From the uniqueness of the symmetric of b as an element of G and as $e \in \mathcal{E}$, $\exists e_1 \in \mathcal{E}$ such that $c.b = e_1$ and then $c = e_1.b^{-1} \in \mathcal{E} * (f(x))^{-1} = (f(x))^{-1}$. □

Proposition 3.45. *Suppose that $|G| = p^m q^n$, p, q distinct prime integers. Let E, H be respectively a p sylow and a q sylow subgroup of G. We have $E^2 = E$ and $E \cap H = \{e\}$. If $p^\alpha \equiv 1[q] \implies \alpha = 0$ then the $HX-$group $\mathcal{G} = \{aE \mid a \in H\}$ is isomorphic to the group H and the mapping $f : H \longrightarrow \mathcal{G},\ a \longmapsto aE$ is a soft $HX-$group.*

Proof. The proof is trivial from the Example 3.15. □

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