The minimum vv-coloring Laplacian energy of a graph

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Abstract. Let $B(G)$ denote the set of all blocks of a graph G. Two vertices are vv-adjacent if they incident on the same block. Then vv-degree of a vertex u, $d_{vv}(u)$ is the number vertices vv-adjacent to the vertex u. In this paper we introduce new kind of graph energy, the minimum vv-coloring Laplacian energy of a graph denoting it as $LE_{cvv}(G)$. It depends both on underlying graph of G and its particular colors on its vertices of G. We studied some of the properties of $LE_{cvv}(G)$ and bounds for $LE_{cvv}(G)$ are established.
Keywords: energy of a graph, Laplacian energy of a graph, Color energy of a graph, vv-coloring of a graph.

1. Introduction

The terminologies and notations used here are as in [10]. By a graph $G(V, E)$ we mean a connected finite simple graph of order p and size q. A vertex $v \in V$ is a cutvertex if $G - \{v\}$ is disconnected. A graph which has no cutvertex is called non-separable. A maximal non-separable subgraph is a block of G. Let $B(G)$ denote the set of all blocks of G with $|B(G)| = m$. The concept of energy of a graph was introduced by I. Gutman [5] in 1978. The eigenvalues of G are the eigenvalues of its adjacency matrix $A(G)$. These eigenvalues arranged in an non-increasing order, will be denoted as $\lambda_1(G), \lambda_2(G) \ldots, \lambda_p(G)$. Then the energy of the graph G is defined as $E(G) = \sum_{i=1}^{p} |\lambda_i(G)|$. In connection with graph energy, energy-like quantities were considered for other matrices such as distance [7], covering [1], incidence [8] and vb-dominating [9]. Gutman and

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Zhou [6] defined the Laplacian energy of a graph G in 2006. Let G be a graph with p vertices and q edges. The Laplacian matrix of the graph G denoted by $L = [l_{ij}]$ is the square matrix of order p where

$$l_{ij} = \begin{cases} -1, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{if } v_i \text{ and } v_j \text{ are not adjacent} \\ d_i, & \text{if } i = j. \end{cases}$$

where d_i is the degree of the vertex v_i. Let $\mu_1, \mu_2, \ldots, \mu_p$ be the eigenvalues of Laplacian matrix $L = [l_{ij}]$, which are called Laplacian eigenvalues of G. Then the Laplacian energy of G is defined as $LE(G) = \sum_{i=1}^{p} |\mu_i(G) - \frac{2q}{p}|$.

A coloring of graph G is a coloring of its vertices such that no two adjacent vertices receive the same color. The minimum number of colors needed for coloring of a graph G is called chromatic number and is denoted by $\chi(G)$. Chandrashekar adiga et al. [2] defined color energy of a graph in 2013. The color matrix of the graph G denoted by $A_L = [a_{ij}]$ is the square matrix of order p where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ -1, & \text{if } v_i \text{ and } v_j \text{ are non-adjacent with } c(v_i) = c(v_j) \\ 0, & \text{otherwise.} \end{cases}$$

where $c(v_i)$ is the color of v_i. Let $\lambda_1, \lambda_2, \ldots, \lambda_p$ be the eigen values of color matrix $A_L = [a_{ij}]$, which are called color eigenvalues of G. Then the color energy of G is defined as $E_c(G) = \sum_{i=1}^{p} |\lambda_i(G)|$.

2. Minimum vv-coloring Laplacian energy of a graph

In 2013, P. G. Bhat et al. [4] defined the vv-adjacency. Two vertices are vv-adjacent if they incident on the same block. vv-degree of a vertex u, $d_{vv}(u)$ is the number vertices vv-adjacent to the vertex u. The point graph $P_G(G)$ is a graph with vertex set same as that of G and any two vertices in $P_G(G)$ are adjacent if, and only if, they are vv-adjacent in G. Number of edges in the point graph is denoted by q_p.

Motivated by the definition of coloring and Laplacian energy of the graph G, we define the following.

vv-coloring of graph G is the coloring of vertices such that no two vertices in the same block receives same color. Then vv-chromatic number of a graph G denoted by $\chi_{cvv}(G)$ is the minimum number of colors needed for vv-coloring the graph G. In this paper, we consider the vv-colored graph. The vv-colored

Laplacian matrix of G is the $p \times p$ matrix $L_{cvv}(G) = [l_{ij}]$, where

$$l_{ij} = \begin{cases} -1, & \text{if } v_i \text{ and } v_j \text{ are vv-adjacent} \\ 1, & \text{if } v_i \text{ and } v_j \text{ are not vv-adjacent but } c(v_i) = c(v_j) \\ d_{vv}(v_i), & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$$

where $c(v_i)$ is the color of v_i. The characteristic polynomial of $L_{cvv}(G)$ is denoted by $f_p(G, \mu) = \det(\mu I - L_{cvv}(G))$.

The minimum vv-coloring Laplacian eigenvalues of the graph G are the eigenvalues of $L_{cvv}(G)$. Since $L_{cvv}(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_p$. The minimum vv-coloring Laplacian energy of G is then defined as $LE_{cvv}(G) = \sum_{i=1}^{p} |\mu_i(G) - \frac{2q_p}{p}|$. In this paper we discuss some basic properties of minimum vv-coloring Laplacian energy of the graph $LE_{cvv}(G)$ and derive an upper and lower bound for $LE_{cvv}(G)$. The minimum vv-coloring Laplacian matrix is independent of the internal structure of the blocks of the graph.

3. Some basic properties of minimum vv-coloring Laplacian energy of a graph

First we compute the minimum vv-coloring Laplacian energy of the graph shown in Figure 1.

Example 3.1.

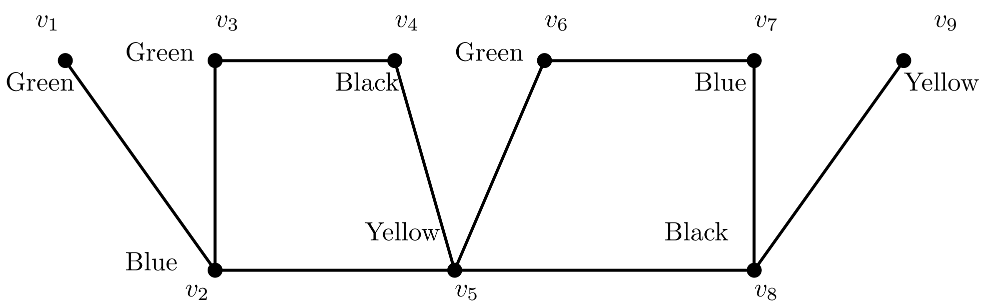

Figure 1: Graph G

Let G be the graph with 9 vertices which are vv-colored using colors black, blue, yellow and green (see Figure 1). Then

$$L_{vv}(G)=\begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0\\ -1 & 4 & -1 & -1 & -1 & 0 & 1 & 0 & 0\\ 1 & -1 & 3 & -1 & -1 & 1 & 0 & 0 & 0\\ 0 & -1 & -1 & 3 & -1 & 0 & 0 & 1 & 0\\ 0 & -1 & -1 & -1 & 6 & -1 & -1 & -1 & 1\\ 1 & 0 & 1 & 0 & -1 & 3 & -1 & -1 & 0\\ 0 & 1 & 0 & 0 & -1 & -1 & 3 & -1 & 0\\ 0 & 0 & 0 & 1 & -1 & -1 & -1 & 4 & -1\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

The characteristic polynomial of $L_{cvv}(G)$ is $\mu^9-28\mu^8+319\mu^7-1903\mu^6+6366\mu^5-11849\mu^4+11373\mu^3-4613\mu^2+245\mu+105$. Then the vv-colored eigenvalues are $-0.1130, 0.2778, 0.9229, 1.3257, 3.1845, 3.5438, 5.5648, 5.8480$ and 7.4455. Therefore the minimum vv-coloring Laplacian energy of the graph G is $LE_{cvv}(G)=20.0619$. Note that the minimum vv-coloring Laplacian energy of the graph G depends on its color set.

Theorem 3.1. *If $\mu_1(G),\mu_2(G)\dots,\mu_p(G)$ are the eigenvalues of $L_{cvv}(G)$, then*

$$\sum_{i=1}^{p}\mu_i=2q_p, \tag{1}$$

$$\sum_{i=1}^{p}\mu_i^2=2(q_p+V_s)+\sum_{i=1}^{p}\left(d_{vv}(v_i)\right)^2, \tag{2}$$

where q_p is the number of edges in the point graph of G and V_s is the number of pairs of vertices receive the same color.

Proof. (1) We know that the sum of the eigenvalues of $L_{cvv}(G)$ is equal to trace of $L_{cvv}(G)$. Therefore $\sum_{i=1}^{p}\mu_i=\sum_{i=1}^{p}a_{ii}=\sum_{i=1}^{p}d_{vv}(v_i)=2q_p$.

(2) The sum of the squares of the Laplacian eigenvalues of $L_{cvv}(G)$ is just the trace of $L_{cvv}(G)^2$.

Therefore

$$\begin{aligned}\sum_{i=1}^{p}\mu_i^2&=\sum_{i=1}^{p}\sum_{j=1}^{p}l_{ij}l_{ji}\\ &=2\sum_{i<j}(l_{ij})^2+\sum_{i=1}^{p}(l_{ii})^2\\ &=2(q_p+V_s)+\sum_{i=1}^{p}\left(d_{vv}(v_i)\right)^2.\end{aligned}$$

□

Observation 3.1 ([4])**.** For any graph G with p vertices

$$\sum_{i=1}^{p} d_{vv}(v_i) = 2q_p.$$

Theorem 3.2. *If $\mu_1(G)$ is the largest eigenvalue of the minimum vv-coloring Laplacian matrix $L_{cvv}(G)$, then*

$$\mu_1(G) \geq \frac{2V_s}{p}. \tag{3}$$

Proof. Let Y be any non zero vector, then we have
$\mu_1\left(L_{cvv}(G)\right) = \begin{matrix} \max \\ Y \neq 0 \end{matrix} \left[\frac{Y' L_{cvv}(G) Y}{Y'Y}\right]$ (see [3]),
$\mu_1(L_{cvv}(G)) \geq \left[\frac{J' L_{cvv}(G) J}{J'J}\right] = \frac{2V_s - 2q_p + \sum_{i=1}^{p} d_{vv}(i)}{p} = \frac{2V_s}{p}$, where J is the all one's vector.

$$\text{Let } N = q_p + V_s + \frac{1}{2}\sum_{i=1}^{p} \left(d_{vv}(v_i)\right)^2, \tag{4}$$

where q_p is the number of edges in the point graph of G and V_s is the number of pairs of vertices receive the same color. □

Theorem 3.3. *Let G_1 and G_2 be two vv-colored graphs with p vertices, q_p and $q_{p'}$ be the number of edges in the point graph of G_1 and G_2 respectively and V_s and $V_{s'}$ be the number of pairs of vertices in G_1 and G_2 receives same color respectively. If $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_p$ and $\mu_1' \geq \mu_2' \geq \cdots \geq \mu_p'$ are the Laplacian eigenvalues of G_1 and G_2 respectively. Then*

$$\sum_{i=1}^{p} \mu_i \mu_i' \leq 2\sqrt{(N(G_1)N(G_2)}, \tag{5}$$

where N is the expression defined as in (4)*.*

Proof. Using Cauchy-Schwarz inequality,

$$\left(\sum_{i=1}^{p} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{p} a_i^2\right)\left(\sum_{i=1}^{p} b_i^2\right).$$

Taking $a_i = \mu_i, b_i = \mu_i'$, we get

$$\begin{aligned}\left(\sum_{i=1}^{p} \mu_i \mu_i'\right)^2 &\leq \left(\sum_{i=1}^{p} \mu_i^2\right)\left(\sum_{i=1}^{p} (\mu_i')^2\right), \text{[From the Theorem 3.1]} \\ &= 2N(G_1)2N(G_2).\end{aligned}$$

□

4. Some bounds for minimum vv-coloring Laplacian energy of a graph

Theorem 4.1. *Let G be a graph with p vertices, and let q_p be the number of edges in the point graph of G. Then*

$$LE_{cvv}(G) \leq \sqrt{2Np} + 2q_p, \tag{6}$$

where N is the expression defined as in (4).

Proof. Let $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_p$ be the Laplacian eigenvalues of $L_{cvv}(G)$.

Using Cauchy-Schwarz inequality,

$$\left(\sum_{i=1}^{p} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{p} a_i^2\right)\left(\sum_{i=1}^{p} b_i^2\right)$$

choose $a_i = 1$ and $b_i = |\mu_i|$,

$$\left(\sum_{i=1}^{p} |\mu_i|\right)^2 \leq \left(\sum_{i=1}^{p} 1\right)\left(\sum_{i=1}^{p} |\mu_i|^2\right) = p2N, \text{[From the Theorem 3.1]}.$$
$$\sum_{i=1}^{p} |\mu_i| \leq \sqrt{2Np}.$$

Using Triangular inequality $|a - b| \leq |a| + |b|$, we have

$$\left|\mu_i - \frac{2q_p}{p}\right| \leq |\mu_i| + \frac{2q_p}{p},$$
$$\sum_{i=1}^{p}\left|\mu_i - \frac{2q_p}{p}\right| \leq \sum_{i=1}^{p} |\mu_i| + \sum_{i=1}^{p} \frac{2q_p}{p},$$
$$LE_{cvv}(G) \leq \sum_{i=1}^{p} |\mu_i| + 2q_p$$
$$\leq \sqrt{2Np} + 2q_p.$$

□

Theorem 4.2. *Let G be a graph with p vertices, and let q_p be the number of edges in the point graph of G. Then*

$$LE_{cvv}(G) \leq \sqrt{2Np - 4q_p^2}, \tag{7}$$

where N is the expression defined as in (4).

Proof. Let $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_p$ be the Laplacian eigenvalues of $L_{cvv}(G)$.

Using Cauchy-Schwarz inequality,

$$\left(\sum_{i=1}^{p} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{p} a_i^2\right)\left(\sum_{i=1}^{p} b_i^2\right)$$

choose $a_i = 1$ and $b_i = |\mu_i - \frac{2q_p}{p}|$,

$$\begin{aligned}
\left(\sum_{i=1}^{p}\left|\mu_i - \frac{2q_p}{p}\right|\right)^2 &\leq \left(\sum_{i=1}^{p} 1\right)\left(\sum_{i=1}^{p}\left|\mu_i - \frac{2q_p}{p}\right|^2\right),\\
(LE_{cvv}(G))^2 &\leq p\left(\sum_{i=1}^{p}\mu_i^2 + \sum_{i=1}^{p}\frac{4q_p^2}{p^2} - \frac{4q_p}{p}\sum_{i=1}^{p}\mu_i\right)\\
&= p\left(2N + \frac{4q_p^2}{p} - \frac{4q_p}{p}2q_p\right), \text{ [From the Theorem 3.1]}\\
&= 2Np + 4q_p^2 - 8q_p^2.
\end{aligned}$$

Therefore

$$LE_{cvv}(G) \leq \sqrt{2Np - 4q_p^2}.$$

□

Theorem 4.3. *Let G be a graph with p vertices. If D is the determinant of $L_{cvv}(G)$, then*

$$LE_{cvv}(G) \geq \sqrt{2N + p(p-1)D^{\frac{2}{p}} - 2q_p}, \tag{8}$$

where N is the expression defined as in (4).

Proof.

$$\begin{aligned}
\left(\sum_{i=1}^{p}|\mu_i|\right)^2 &= \left(\sum_{i=1}^{p}|\mu_i|\right)\left(\sum_{j=1}^{p}|\mu_j|\right)\\
&= \sum_{i=1}^{p}|\mu_i|^2 + \sum_{i\neq j}|\mu_i||\mu_j|,\\
\sum_{i\neq j}|\mu_i||\mu_j| &= \left(\sum_{i=1}^{p}|\mu_i|\right)^2 - \sum_{i=1}^{p}|\mu_i|^2,\\
\frac{1}{p(p-1)}\sum_{i\neq j}|\mu_i||\mu_j| &\geq \left(\prod_{i\neq j}|\mu_i||\mu_j|\right)^{\frac{1}{p(p-1)}},\\
\sum_{i\neq j}|\mu_i||\mu_j| &\geq p(p-1)\left(\prod_{i\neq j}|\mu_i||\mu_j|\right)^{\frac{1}{p(p-1)}}.
\end{aligned}$$

Thus

$$\left(\sum_{i=1}^{p}|\mu_i|\right)^2-\sum_{i=1}^{p}|\mu_i|^2\geq p(p-1)\left(\prod_{i=1}^{p}|\mu_i|^{2(p-1)}\right)^{\frac{1}{p(p-1)}},$$

$$\left(\sum_{i=1}^{p}|\mu_i|\right)^2-2N\geq p(p-1)\left(\prod_{i=1}^{p}|\mu_i|\right)^{\frac{2}{p}},$$

$$\left(\sum_{i=1}^{p}|\mu_i|\right)^2\geq 2N+p(p-1)D^{\frac{2}{p}},$$

$$\left(\sum_{i=1}^{p}|\mu_i|\right)\geq\sqrt{2N+p(p-1)D^{\frac{2}{p}}}.$$

Using Triangular inequality $|a|-|b|\leq|a-b|$,

$$|\mu_i|-\left|\frac{2q_p}{p}\right|\leq\left|\mu_i-\frac{2q_p}{p}\right|,\quad\forall i=1,2,\dots p,$$

$$\sum_{i=1}^{p}|\mu_i|-\sum_{i=1}^{p}\frac{2q_p}{p}\leq\sum_{i=1}^{p}\left|\mu_i-\frac{2q_p}{p}\right|,$$

$$\sum_{i=1}^{p}|\mu_i|-2q_p\leq LE_{cvv}(G),$$

$$LE_{cvv}(G)\geq\sum_{i=1}^{p}|\mu_i|-2q_p\geq\sqrt{2N+p(p-1)D^{\frac{2}{p}}}-2q_p.$$

□

Theorem 4.4. *Let G be a graph with p vertices, and let q_p be the number of edges in the point graph of G. If D is the determinant of $L_{cvv}(G)$, then*

$$LE_{vv}(G)\leq\sqrt{2N(p-1)+pD^{\frac{2}{p}}}+2q_p, \tag{9}$$

where N is the expression defined as in (4).

Proof. Using Kober's inequality,

$$p\left[\frac{1}{p}\sum_{i=1}^{p}a_i-\left(\prod_{i=1}^{p}a_i\right)^{\frac{1}{p}}\right]\leq p\sum_{i=1}^{p}a_i-\left(\sum_{i=1}^{p}\sqrt{a_i}\right)^2.$$

Putting $a_i=|\mu_i|^2$

$$p\left[\frac{1}{p}\sum_{i=1}^{p}|\mu_i|^2-\left(\prod_{i=1}^{p}|\mu_i|^2\right)^{\frac{1}{p}}\right]\leq p\sum_{i=1}^{p}|\mu_i|^2-\left(\sum_{i=1}^{p}|\mu_i|\right)^2,$$

$$2N-pD^{\frac{2}{p}}\leq p(2N)-\left(\sum_{i=1}^{p}|\mu_i|\right)^2.$$

Thus

$$\left(\sum_{i=1}^{p}|\mu_i|\right) \leq \sqrt{2N(p-1)+pD^{\frac{2}{p}}}.$$

Now, using triangular inequality,

$$\left|\mu_i - \frac{2q_p}{p}\right| \leq |\mu_i| + |\frac{2q_p}{p}|,$$

$$\sum_{i=1}^{p}\left|\mu_i - \frac{2q_p}{p}\right| \leq \sum_{i=1}^{p}|\mu_i| + \sum_{i=1}^{p}\frac{2q_p}{p},$$

$$LE_{cvv}(G) \leq \sum_{i=1}^{p}|\mu_i| + 2q_p.$$

Therefore

$$LE_{cvv}(G) \leq \sqrt{2N(p-1)+pD^{\frac{2}{p}}} + 2q_p.$$

□

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