Exponential stabilization of semi-linear wave equation

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Abstract. In this article, we establish the stabilization of a class of second order semi-linear hyperbolic systems obtained by nonlinear feedback using the observability of the corresponding uncontrolable systems. We obtain the well-posedness of the semi-linear system by standard argument of Ball ([3]), Our technique of proof relies on an appropriate decomposition of the solution, and the energy method. Our result generalizes an earlier one by Haraux [5] who studied the same type of problem for linear systems, and by Louis Tebou [9] for de case of semi-linear systems. Application of our result are provided.
Keywords: semi-linear systems, nonlinear feedback stabilization, decay estimate.

1. Introduction and statements of main results

Control and stability of distributed parameter systems can be reformulated as problems of analysis of nonlinear (semi-linear, bi-linear, ...) systems in many real problems, see [2, 5, 6, 7, 10, 11, 12, 13] for instance. Within this broad heading there are many different concepts for example exponential stabilization.

Let Ω be an open bounded subset of $\mathbb{R}^n$ with regular boundary $\partial\Omega$. let H be a Hilbert space and A be an unbounded coercive operator on H with $A = A^*$. Also let $B : H \to H$ be a bounded non-negative linear operator. Denote $< .,. >$, the scalar product on H, and by $|\ |_H$ the corresponding norm on H, also we denote $\|\ \|_V$ the norms on V with $V = D(A^{\frac{1}{2}})$ respectively such that for every $v \in V$, set $\|v\|_V = |A^{\frac{1}{2}}v|_H$ and $Z = V \times H$ is the state space. In this paper we are concerned with the question of feedback stabilization of the

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following distributed semi-linear control system:

$$\begin{cases} y_{tt} = -Ay + Ny + uBy_t & \text{in} \quad Q \\ y(x,0) = y_0(x), y_t(x,0) = y_1(x) & \text{in} \quad \Omega \\ y(\xi,t) = 0 & \text{on} \quad \Sigma, \end{cases} \tag{1}$$

where $Q = \Omega\times]0,+\infty[$, $\Sigma = \partial\Omega\times]0,+\infty[$ and A generates a C_0-semigroup $(S(t))_{t\geq 0}$ on Hilbert H and N is a nonlinear dissipative operator. For well-posed linear systems, we refer to [8]. In this work we take nonlinear state-feedback for (1), that is, $u(t) = -f_\rho(y_t(t))$, where

$$f_\rho(y) = \rho\frac{\langle y, By\rangle}{1+|\langle y, By\rangle|}, \; \rho > 0. \tag{2}$$

or

$$\begin{cases} f_\rho(y) = \rho\frac{\langle y,By\rangle}{\|y\|^2}, \; \rho > 0, & y \neq 0, \\ f_\rho(y) = 0, & y = 0. \end{cases} \tag{3}$$

Then, we obtain the following closed-loop system

$$\begin{cases} y_{tt} = -Ay + Ny - f_\rho By_t & \text{in} \quad Q \\ y(x,0) = y_0(x), y_t(x,0) = y_1(x) & \text{in} \quad \Omega \\ y(\xi,t) = 0 & \text{on} \quad \Sigma. \end{cases} \tag{4}$$

We first consider the well-posedness of (5), which is, we prove that (5) admits a unique mild solution $y(t, x_0)$ for all $x_0 \in Z$.

2. Well-posedness problem

In this section, we show the well-posedness of the system

$$\begin{cases} y_{tt} = -Ay + Ny - f_\rho By_t & \text{in} \quad Q \\ y(x,0) = y_0(x), y_t(x,0) = y_1(x) & \text{in} \quad \Omega \\ y(\xi,t) = 0 & \text{on} \quad \Sigma, \end{cases} \tag{5}$$

where A generates a C_0-semigroup $(S(t))_{t\geq 0}$ on Hilbert H, and $N(\cdot) : H \to H$ is a locally Lipschitz, that is, there exists a positive constant L such that

$$\|N(x) - N(y)\| \leq L\|x - y\|, \tag{6}$$

for all $x, y \in H$, and $N(0) = 0$. Since A is a strictly positive operator, then there exists the best positive constant such that:

$$\|v\|_V^2 = |(A)^{1/2}v|_H^2 \geq \mu^2\|v\|_H^2. \tag{7}$$

Theorem 2.1. *Assume that A generates a C_0-semigroup $(S(t))_{t\geq 0}$ on Hilbert H, and that $N(\cdot) : H \to H$ is a locally Lipschitz. Then, for any $z_0 = (y_0, y_1) \in Z$.It can be shown that a function $Z \in C([0,T], Z), \forall T > 0$. (5) has a mild solution if and only if z satisfies the variation formula ([2])*

$$z(t) = T(t)z_0 + \int_0^t T(t-s)g(z(s))ds, \forall t \in [0,T]. \tag{8}$$

Where g and $T(t), t > 0$ are given in proof.

Proof of Theorem 2.1. We have Z form a Hilbert space under the inner product

$$(< v_1, w_1 >, < v_2, w_2 >)_Z =< v_1, v_2 >_V + < w_1, w_2 >_H .$$

The equation 1 can be written in the form

$$z_t = \tilde{A}z + g(z), z_0 = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \tag{9}$$

$$\tilde{A} : D(\tilde{A}) \longrightarrow Z \text{ with } \tilde{A} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}, \tag{10}$$

where

$$D(\tilde{A}) = \{(x,y) \in V \times H, -Ax \in H, y \in V\}, \tag{11}$$

and

$$z(t) = \begin{pmatrix} y(t) \\ y_t(t) \end{pmatrix}, g(z(t)) = \begin{pmatrix} 0 \\ Ny(t) - f_\rho(y_t(t))By_t(t) \end{pmatrix}.$$

Our assumptions imply that $\tilde{A}$ generates a C_0-semigroup of linear contraction $(T(t))_{t\geq 0}$ on Hilbert Z, and that $g : Z \to Z$ is locally lipschitz.Then according to Ball ([2]) we have the inial data $z(0) = z_0$, that z is a weak solution of (5) , if and only if z satisfies (8).

Theorem 2.2. *A standard argument shows that for a given $z_0 \in Z$, there exists a unique solution $z(t) \in Z$ to system (8). Moreover, if $z_0 \in D(\tilde{A})$ then $z \in C(0,\infty; D(\tilde{A}) \cap C^1(0,\infty; Z)$.*

Proof of Theorem 2.2. Since $\tilde{A}$ is m-dissipative see [1] it generates a contraction semigroup on Z, denoted by $(T(t))$. Then $\tilde{A}$ is also the generator of a strongly continuous semigroup on Z and using Ball and Slemrod ([3]), which completes the proof.

3. Exponential stabilization using observability

In this section we consider the following semi-linear equations:

$$(12)\qquad \begin{cases} y_{tt} = -Ay + Ny - f_\rho B y_t & \text{in} \quad Q \\ y(x,0) = y_0(x), y_t(x,0) = y_1(x) & \text{in} \quad \Omega \\ y(\xi, t) = 0 & \text{on} \quad \Sigma, \end{cases}$$

and

$$(13)\qquad \begin{cases} \phi_{tt} = -A\phi + N\phi & \text{in} \quad Q \\ \phi(x,0) = \phi_0(x), \phi_t(x,0) = \phi_1(x) & \text{in} \quad \Omega \\ \phi(\xi, t) = 0 & \text{on} \quad \Sigma. \end{cases}$$

Our result concerns the equivalence between the observability of system 13 and stabilization of the equilibrium state of 12. We will construct a nonlinear control serving primarly to bring the nonlinearity in the system generated by $\psi = y - \phi$ to zero, and secondly to ensure the exponential decay of the system energy. More precisely, we will prove the following results:

Theorem 3.1. *We prove that,(a) implies (b)such that:*

(a) *Assume that there is a time T and constant $C > 0$ such that*

$$\int_0^T | < B\phi_t, \phi_t > |_H dt \geq C \parallel (\phi_0, \phi_1) \parallel_Z^2 \ \ \forall (\phi_0, \phi_1) \in Z,$$

where $\phi \in C(]0, +\infty[, H) \cap C^1(]0, +\infty[, V)$ is the mild solution of 13.

(b) *There exist $M > 0$, and $\sigma > 0$ such that for every $(y_0, y_1) \in Z$, the solution of 12 with $y(0) = y_0$ and $y_t(0) = y_1$ satisfies:*

$$\|y(t)\|_V^2 + |y_t|_H^2 \leq M \exp(-\sigma t)(\|y_0\|_V^2 + |y_1|_H^2).$$

Proof of Theorem 3.1. Firstly, if $u(t) = 0$ the idea (see [5]) is to achieve the equivalent between observability and stabilization concerned the case of linear systems. Secondly, if $u(t) \neq 0$, we prove that $(a) \Rightarrow (b)$. Then, we consider the solution of (13) with $\phi(0) = y_0$ and $\phi_t(0) = y_1$ and let us consider the following energy functional

$$(14)\qquad E_\phi(t) = \frac{1}{2}(\|\phi(t)\|_V^2 + |\phi_t(t)|_H^2) + F(\phi(t)),$$

where

$$F(u) = \int_0^u N(s) ds,$$

and set $\psi = y - \phi$, then we obtain the following system

$$(15)\qquad \begin{cases} \psi_{tt} = -A\psi - f_\rho B y_t - N(y) + N(\phi) & \text{in} \quad Q \\ \psi(x,0) = 0, \dfrac{\partial \psi}{\partial t}(x,0) = 0 & \text{in} \quad \Omega \\ \psi(\xi, t) = 0 & \text{on} \quad \Sigma. \end{cases}$$

We consider

$$(16)\qquad E_\psi(t) = \frac{1}{2}(\|\psi(t)\|_V^2 + |\psi_t(t)|_H^2)$$

multiplying (15) by ψ and integrating over Ω we have

$$(17)\qquad \frac{dE_\psi}{dt} = \int_\Omega -(N(y) - N(\phi))\psi_t dx - \int_\Omega < f_\rho B y_t, \psi_t > dx.$$

Thanks to 6, and the Cauchy-Schwarz inequality, one derives from $0 \leq t \leq T$

$$(18)\qquad E_\psi(t) \leq \int_0^T L\|\psi\|_H |\psi_t|_H ds + \int_0^T \|f_\rho B y_t\|_H |\psi_t|_H ds$$

then

$$(19)\qquad E_\psi(t) \leq \frac{L}{\delta} \int_0^T \|\psi\|_V \|\psi_t\|_H ds + \frac{1}{2}\left(\int_0^T \|f_\rho B y_t\|_H^2 + \|\psi_t\|_H^2 ds\right)$$

from the boundedness of the operator B and 3 we obtain

$$(20)\quad E_\psi(t) \leq \frac{L}{\mu} \int_0^T E_\psi(s) ds + \frac{\rho}{2}\left(\int_0^T K^2 | < B y_t, y_t > |_H ds\right) + \int_0^T E_\psi(s) ds.$$

Applying Gronwall lemma we derive:

$$(21)\qquad E_\psi(t) \leq K' \exp(\mu_1) \int_0^T | < B y_t, y_t > |_H ds$$

with $\mu_1 = \dfrac{TL + T\mu}{\mu}$, and $K' = \rho.K^2$.

Now we set

$$(22)\qquad E_y(t) = \frac{1}{2}(\|y(t)\|_H^2 + \|y_t(t)\|_V^2) + F(y(t))$$

and we observe that

$$E_\phi(0) = E_y(0),$$

using the observability estimate provided by (a) we have

$$(23)\qquad E_y(0) \leq C \int_0^T | < B\phi_t, \phi_t > |_H ds$$

then for $C^{'} > 2C$ we have

$$E_y(0) \leq C^{'} \int_0^T | < B\psi_t, \psi_t > |_H ds + C^{'} \int_0^T | < By_t, y_t > |_H ds. \tag{24}$$

Thanks to 21 and the boundedness of the operator B gives the existance of $K^{''} > 0$ such that

$$E_y(0) \leq K^{''} \int_0^T | < By_t, y_t > |_H ds \tag{25}$$

we multiply (15) by ψ_t and integrate over Ω we have

$$E_y(T) - E_y(0) = -\int_0^T | < By_t, y_t >_H | ds \tag{26}$$

then E_y is a non-increasing function of the time variable. we deduce that

$$E(0) \leq \beta E_y(T) - \beta E_y(0),$$

and the semigroup property gives

$$\|y(t)\|_V^2 + |y_t|_H^2 \leq M \exp(-\sigma t)(\|y_0\|_V^2 + |y_1|_H^2)$$

with

$$M = \frac{\beta}{1+\beta} \text{ and } \sigma = \frac{1}{T} \log(\frac{\beta+1}{\beta}),$$

which gives the exponential stabilization.

4. Application to wave equation

We consider the following semi-linear equation and $\Omega =]0,1[$:

$$\begin{cases} \dfrac{\partial^2 y(x,t)}{\partial t^2} = \Delta y(x,t) - y(x,t)(\beta + \sin(|y(x,t)|)) \\ +v(t)\dfrac{\partial y(t)}{\partial t}, (x,t) \in \ \Omega \times]0, +\infty[, \\ y(x,0) = y_0, y_t(x,0) = y_1, \qquad x \in \Omega, \\ y(0,t) = y(1,t) = 0, y_t(0,t) = y_t(1,t) = 0, \qquad t > 0, \end{cases} \tag{27}$$

we take the state space $Z = V \times H = H_0^1(\Omega) \times L^2(\Omega)$ and the operators B and A are defined by $B = I$, $A = -\Delta$ which A generates a C_0-semigroup of linear contraction where $\Delta y(t) = \frac{\partial^2 y(t)}{\partial x^2}$ with $\mathcal{D}(A) = (H^2(\Omega) \cap H_0^1(\Omega))$. In [4], we consider the operator $A_1 z = \Delta z$ in $H = L^2(\Omega)$, (Ω is a bounded open set in $\mathcal{R}^n$, $n \geq 1$) and domain

$$\mathcal{D}(A_1) = \{z \in L^2(\Omega) | \ z \text{ is absolutely continuous}, \Delta z \in L^2(\Omega) \text{ et } z = 0 \text{ sur } \partial\Omega\}.$$

We have $\overline{\mathcal{D}(A_1)} = L^2(\Omega)$, $A_1^* = A_1$ and $\mathcal{R}e(\langle A_1 z, z\rangle) = -\|\nabla z\|^2 \leq 0$. Then A_1 is dissipative.

5. Conclusion

In this work, we have developed the partial exponential stabilization using the proof of non uniform observability estimates for some semi-linear problems with superlinear nonlinearities. The idea of the non-linearity cost of the optimal stabilization is very interesting and constitutes a new issue in the applications. In addition. Various questions remain open for instance, the case of stabilization of semi-linear systems in Banach space, a confrontation to more realistic situations remain done. This leads us to consider the stabilization problem for stochastic semi-linear systems.

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