The maximal hyperrings of quotients

Hasret Yazarli*
Sivas Cumhuriyet University
Faculty of Science
Department of Mathematics
58140 Sivas
Turkey
hyazarli@cumhuriyet.edu.tr

Damla Yılmaz
Erzurum Technical University
Faculty of Science
Department of Mathematics
Erzurum
Turkey
damla.yilmaz@erzurum.edu.tr

Bijan Davvaz
Yazd University
Department of Mathematics
Yazd
Iran
davvaz@yazd.ac.ir

Abstract. We show that the maximal quotient hyperring $Q_{mr}(R)$ of a semiprime hyperring R can be obtained in a similar way to a maximal quotient ring. In this regard we introduce and study some basic notions of hyperrings such as dense hyperideal, essential hyperideal, singular hyperideal and prove some results satisfying them. Finally, we show that if R is a semiprime hyperring and $Q = Q_{mr}(R)$, then Q is regular (in the sense of von Neumann) if and only if R has a zero right singular hyperideal.

Keywords: hyperring, dense hyperideal, essential hyperideal, good homomorphism.

1. Introduction

Algebraic hyperstructures are a generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements of a set is again an element of the same set, while in an algebraic hyperstructure, the composition of two elements is a non-empty subset of the same set. The theory of hyperstructures was introduced by Marty in 1934 at the 8th Congress of the Scandinavian Mathematicians [8]. Marty defined hypergroups, began to analyze their properties and applied them to groups. Hyperstructures have many applications to several sectors of both pure and applied mathematics [2]. The notion

$*$. Corresponding author

of hyperrings is investigated and studied by Krasner [7], Nakasis [13], Vougiouklis [17], Davvaz [3], Davvaz and his research group [5, 6, 9, 10, 11, 12, 14, 15], and many others. Also, see Davvaz and Leoreanu-Fotea book [4]. A well-known type of a hyperring is called the Krasner hyperring [7]. Krasner hyperring is essentially ring with approximately modified axioms in which addition is hyperoperation, while the multiplication is an operation. This type of hyperrings has been studied by a variety of authors.

The notion of left quotient ring for a ring without right zero divisors was introduced by Utumi [16]. In his paper, Utumi proved that every ring without total right zero divisors has a unique maximal quotient ring. This ring, denoted by $Q^l_{\max}(R)$, is called the maximal left quotient ring of R. The many properties of this ring were investigated in [1].

In this paper, the maximal quotient hyperring $Q = Q_{mr}$ is structured. We define and study dense hyperideal, essential hyperideal and we give the relationship between essential and dense right hyperideals. Furthermore, we study singular hyperideal and some properties of its. Finally, we show that R has zero right singular hyperideal if and only if $Q = Q_{mr}$ is a von Neumann regular hyperring.

2. Preliminaries

In this section we give some definitions and results of hyperstructures which we need to develop our paper.

A mapping $\circ : H \times H \to P^*(H)$ is called a hyperoperation, where $P^*(H)$ is the set of all non-empty subsets of H. An algebraic system $(H, \circ)$ is called a hypergroupoid.

For any two non-empty subsets A and B of H and $x \in H$, we define

$$A \circ B = \bigcup_{\substack{a\in A\\ b\in B}} a \circ b\ ,\ A \circ x = A \circ \{x\} \ \text{ and } x \circ B = \{x\} \circ B.$$

A hyperoperation "$\circ$" is called associative if $a\circ(b\circ c) = (a\circ b)\circ c$ for all $a, b, c \in H$, which means that

$$\bigcup_{u\in b\circ c} a \circ u = \bigcup_{v\in a\circ b} v \circ c.$$

A hypergroupoid with the associative hyperoperation is called a semihypergroup.

A hypergroupoid $(H, \circ)$ is a quasihypergroup, whenever $a \circ H = H = H \circ a$ for all $a \in H$. If $(H, \circ)$ is semihypergroup and quasihypergroup, then $(H, \circ)$ is called a hypergroup.

Definition 2.1 ([4]). *A Krasner hyperring is an algebraic structure* $(R, +, \cdot)$ *which satisfies the following axioms:*

(1) $(R, +)$ *is a canonical hypergroup, i.e.,* *(i) for every* $x, y, z \in R$, $(x + y) + z = x + (y + z)$,

(ii) for every $x, y \in R$, $x + y = y + x$,

(iii) there exists $0 \in R$ *such that* $0 + x = \{x\}$ *for all* $x \in R$,

(iv) for every $x \in R$ *there exists a unique element denoted by* $-x \in R$ *such that* $0 \in x + (-x)$,

(v) for every $x, y, z \in R$, $z \in x + y$ *implies* $y \in -x + z$ *and* $x \in z - y$;

(2) $(R, \cdot)$ *is a semigroup having zero as a bilaterally absorbing element, i.e.,*

(i) for every $x, y, z \in R$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,

(ii) $x \cdot 0 = 0 \cdot x = 0$ *for all* $x \in R$;

(3) *The multiplication is distributive with respect to the hyperoperation* $+$ *, i.e., for every* $x, y, z \in R$, $x \cdot (y + z) = x \cdot y + x \cdot z$ *and* $(x + y) \cdot z = x \cdot z + y \cdot z$.

The following elementary facts follow easily from the axioms: $-(-x) = x$ and $-(x + y) = -x - y$, where $-A = \{-a \mid a \in A\}$. In definition, for simplicity of notations we write sometimes xy instead of $x \cdot y$ and in (iii), $0+x = x$ instead of $0 + x = \{x\}$.

In a hyperring R, if there exists an element $1 \in R$ such that $1a = a1 = a$ for every $a \in A$, then the element 1 is called the identity element of the hyperring R. If $ab = ba$ for every $a, b \in R$ then the hyperring R is called a commutative hyperring.

A hyperring R is called a hyperdomain if R does not have zero divisors. In other words, for $x, y \in R$ if $xy = 0$ then either $x = 0$ or $y = 0$.

A Krasner hyperring is called a Krasner hyperfield, if $(R\backslash \{0\}, \cdot)$ is a group.

Throughout this paper, by a hyperring we mean that Krasner hyperring.

Let R be a hyperring. A non-empty subset S of R is called a subhyperring of R, if $x - y \subseteq S$ and $xy \in S$ for all $x, y \in S$.

A subhyperring I of a hyperring R is a left (resp. right) hyperideal of R if $ra \in I$ (resp. $ar \in I$) for all $r \in R$, $a \in I$. A hyperideal of R is both a left and a right hyperideal.

Lemma 2.1 ([4])**.** *A non-empty subset* A *of a hyperrring* R *is a left (right) hyperideal if and only if¿*

(1) a , $b \in A$ *implies* $a - b \subseteq A$,

(2) $a \in A$, $r \in R$ *imply* $ra \in A$ $(ar \in A)$.

Let A and B be non-empty subsets of a hyperring R

$$A + B = \{x \mid x \in a + b \text{ for some } a \in A \text{ , } b \in B\}$$

and

$$AB = \left\{ x \mid x \in \sum_{i=1}^{n} a_i bi \text{ , } a_i \in A \text{ , } b_i \in B \text{ , } n \in \mathbb{Z}^+ \right\}.$$

If A and B are hyperideals of R, then $A + B$ and AB are also hyperideals of R.

Definition 2.2 ([18]). *The hyperideal I said to be the direct sum of its hyperideals J and K, denoted by $I = J \oplus K$ if for every element $x \in I$ there exist unique elements a, b such that $a \in J$, $b \in K$ and $x \in a + b$.*

A hyperideal P of a hyperring R is called a prime hyperideal, if for hyperideals I and J of R, satisfying $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. A hyperideal I of a hyperring R is called semiprime hyperideal if for any hyperideal H of R, satisfying $H^2 \subseteq R$ implies $H \subseteq I$. Prime hyperideals are surely semiprime hyperideals. A hyperring R is called a prime hyperring if $aRb = 0$ for all $a, b \in R$ implies $a = 0$ or $b = 0$. Also, R is called semiprime if $aRa = 0$ implies that $a = 0$. Clearly, every prime hyperring is a semiprime hyperring but the converse is not always true.

Definition 2.3 ([4]). *Let R_1 and R_2 be hyperrings. A mapping φ from R_1 into R_2 is said to be a good (strong) homomorphism if for all $a, b \in R_1$,*

$$\varphi(a+b) = \varphi(a) + \varphi(b) \ , \ \varphi(ab) = \varphi(a)\varphi(b) \ \textit{and} \ \varphi(0) = 0.$$

A good homomorphism φ is an isomorphism if φ is one to one and onto. If there exists isomorphism between hyperrings R_1 and R_2, we write $R_1 \cong R_2$.

Since R_1 is a hyperring, $0 \in a - a$ for all $a \in R_1$, then we have $\varphi(0) \in \varphi(a) + \varphi(-a)$ or $0 \in \varphi(a) + \varphi(-a)$ which implies that $\varphi(-a) \in -\varphi(a) + 0$, therefore $\varphi(-a) = -\varphi(a)$ for all $a \in R_1$. Moreover, if φ is a good homomorphism from R_1 into R_2, then the kernel of φ is the set $\{x \in R_1 \mid \varphi(x) = 0\}$. It is trivial that $\ker\varphi$ is a hyperideal of R_1 and $\Im(\varphi) = \{\varphi(r) \mid r \in R\}$ is a subhyperring of R_2.

Corollary 2.1. *Let φ be a good homomorphism from R_1 into R_2. Then φ is one to one if and only if* $\ker\varphi = \{0\}$.

Let R be a hyperring. A canonical hypergroup $(M, +)$ together with the map $\cdot : R \times M \to M$ is called a left hypermodule over R if for all $r_1, r_2 \in R$, $m_1, m_2 \in M$ the following axioms hold:

(1) $r_1(m_1 + m_2) = r_1m_1 + r_1m_2$,

(2) $(r_1 + r_2)m_1 = r_1m_1 + r_2m_1$,

(3) $(r_1r_2)m_1 = r_1(r_2m_1)$,

(4) $0_Rm_1 = 0_M$.

A subhypermodule N of M is a subhypergroup of M which is closed under multiplication by elements of R.

Definition 2.4. *Let M and N be two R-hypermodules. A function $f : M \to N$ that satisfies the conditions:*

(1) $f(x + y) \subseteq f(x) + f(y)$,

(2) $f(xr) = f(x)r$ *, for all $r \in R$ and all $x, y \in M$*

is called to be a right R-homomorphism from M into N.

In Definition 2.4, if the equality holds, then f is called a good (strong) right R-homomorphism.

3. Result

Definition 3.1. *A right hyperideal I of R is dense if given any $0 \neq r_1 \in R$, $r_2 \in R$ there exists $r \in R$ such that $r_1 r \neq 0$ and $r_2 r \in I$. One defines a dense left hyperideal in an analogous fashion. The set of all dense right hyperideal of R will be denoted by $D = D(R)$.*

Example 3.1. The set $R = \{e, a, b, c\}$ with

$+$	e	a	b	c
e	e	a	b	c
a	a	$\{e,a\}$	c	$\{b,c\}$
b	b	c	$\{e,b\}$	$\{a,c\}$
c	c	$\{b,c\}$	$\{a,c\}$	R

$\cdot$	e	a	b	c
e	e	e	e	e
a	e	a	a	a
b	e	a	b	a
c	e	a	b	b

is a semiprime hyperring. It ıs easy to see that $I = \{e, a\}$ is a dense right hyperideal of R. But $J = \{e, b\}$ is not dense right hyperideal of R.

For any subhypermodule J of a right R -hypermodule M and any subset $S \subseteq M$ we set

$$(S : J)_R = \{x \in R \mid Sx \subseteq J\}.$$

When the context is clear we will simply write $(S : J)$. In particular, $(M : 0) = \{x \in R \mid mx = 0$, for all $m \in M\}$ is called the right annihilator of M and is denoted by $r(M)$. The left annihilator $l(M)$ is similarly defined.

Remark 3.1. $(S : J)_R$ is a right hyperideal of R.

Proposition 3.1. *Let R be a semiprime hyperring. If I , J , $S \in D(R)$ and $f : I \to R$ is a good homomorphism of right R-hypermodules, then:*

(i) $f^{-1}(J) = \{a \in I \mid f(a) \in J\} \in D(R)$;

(ii) $(a : J) \in D(R)$ for all $a \in R$;

(iii) $I \cap J \in D(R)$;

(iv) If K is a right hyperideal of R and $I \subseteq K$, then $K \in D(R)$;

(v) $l(I) = 0 = r(I)$;

(vi) If K is a right hyperideal of R and $(a : K) \in D(R)$ for all $a \in I$, then $K \in D(R)$;

(vii) If L is a right hyperideal of R and $g : L \to R$ is a good homomorphism of right S -hypermodules, then g is a good homomorphism of right R -hypermodules;

(viii) $IJ \in D(R)$.

Proof. (i) $f^{-1}(J)$ is a right hyperideal of R. The proof is similar to ordinary algebra. Now, we show that $f^{-1}(J)$ is a dense hyperideal of Rhyperideal of R, $r_1 r^{'} \neq 0$ and $r_2 r^{'} \in I$ for some $r^{'} \in R$. Similarly, since J is a dense right hyperideal of R, $(r_1 r^{'}) r^{''} \neq 0$ and $f(r_2 r^{'}) r^{''} \in J$ for some $r^{''} \in R$. Setting $r = r^{'} r^{''}$ we conclude that $r_1 r \neq 0$ and $r_2 r \in f^{-1}(J)$. Thus $f^{-1}(J)$ is a dense right hyperideal of R.

(ii) Let $l_a : R \to R$, $x \mapsto ax$. Then

$$(a : J) = \{x \in R \mid ax \in J\} = \{x \in R \mid l_a(x) \in J\} = l_a^{-1}(J).$$

Let $x, y \in (a : J)$, hence $ax, ay \in J$. For some $z \in x-y$ we have $az \in a(x-y) = ax - ay$. Thus $az \in J$ and so $z \in (a : J)$. Therefore, $x - y \subseteq (a : J)$. Let $r \in R$ and $x \in (a : J)$, hence $ax \in J$. Then $a(xr) = (ax)r \in J$. Since $a(xr) \in J$, $xr \in (a : J)$. Let $r_1 \neq 0$, $r_2 \in R$. Hence $r_1 \neq 0$, $ar_2 \in R$. Since J is a dense right hyperideal of R, $r_1r \neq 0$ and $(ar_2)r \in J$ for some $r \in R$. Then $r_1r \neq 0$ and $r_2r \in (a : J)$, i.e.; $(a : J) \in D(R)$ for all $a \in R$.

(iii) If $i : I \to R$ is the inclusion map, then $i^{-1}(J) = I \cap J$. If the option (i) is applied, the proof ends.

(iv) Let K is a right hyperideal of R, $I \subseteq K$ and $r_1 \neq 0$, $r_2 \in R$. Since I is a dense right hyperideal of R, we have $r_1r \neq 0$, $r_2r \in I$ for some $r \in R$. Since $I \subseteq K$, we obtain $r_1r \neq 0$, $r_2r \in K$. That is, $K \in D(R)$.

(v) We suppose that $Ia = 0$ for some $0 \neq a \in R$. Setting $r_1 = a = r_2$, we have there exists $r \in R$ such that $0 \neq ar \in I$. Hence $arRar \subseteq Iar = 0$. By semiprimeness of R, we have $ar = 0$. Then we have a contradiction and so $r(I) = 0$. Now we suppose $l(I) \neq 0$. Since R is semiprime hyperring, there exists $a, b \in l(I)$ such that $ab \neq 0$. Since I is a dense hyperideal, we can find $r \in R$ such that $abr \neq 0$ and $br \in I$. Then $abr \in aI = 0$ and again we have a contradiction.

(vi) Let $r_1 \neq 0$, $r_2 \in R$. Since $I \in D(R)$, there exists an element $r^{'} \in R$ such that $r_1r^{'} \neq 0$ and $r_2r^{'} \in I$. Hence by hypothesis, $(r_2r^{'} : K) \in D(R)$. From (v) , we have $l((r_2r^{'} : K)) = 0$ and so $r_1r^{'}r^{''} \neq 0$ and $r_2r^{'}r^{''} \in K$ for some $r^{''} \in (r_2r^{'} : K)$. Therefore $K \in D(R)$.

(vii) Let $x \in L$ and $r \in R$. By (ii), $(r : S)_R \in D(R)$ and so by (iii), $M = (r : S)_S \cap S \in D(R)$. For all $y \in M \subseteq S$ we get $ry \in S$ and

$$0 \in g(xry) - g(xry) = g(xr)y - g(x)ry = (g(xr) - g(x)r)y.$$

Hence $ay = 0$ for some $a \in g(xr) - g(x)r$. Since $y \in M$ and $M \in D(R)$, it follows from (v) that $0 \in g(xr) - g(x)r$, which implies that $-g(x)r$ is the inverse of $g(xr)$ in the canonical hypergroup $(R, +)$. Hence $-g(x)r = -g(xr)$ and so $g(xr) = g(x)r$.

(viii) Let $r_1 \neq 0$, $r_2 \in R$. By (ii) , $L = (r_2 : I) \in D(R)$ and by (v) there exists $r^{'} \in L$ such that $r_1r^{'} \neq 0$ and there exists $r^{''} \in J$ such that $r_1r^{'}r^{''} \neq 0$. Getting $r = r^{'}r^{''}$ we have $r_1r \neq 0$ and $r_2r = r_2(r^{'}r^{''}) = (r_2r^{'})r^{''} \in IJ$.

Now, we give an alternative definition of dense right hyperideals:

Corollary 3.1. *Let J be a right hyperideal of R. Then $J \in D(R)$ if and only if $l_R((a : J)) = 0$ for all $a \in R$.*

Proof. Let $J \in D(R)$. According to Proposition 3.1 (ii) and (v), $(a : J) \in D(R)$ and $l_R((a : J)) = 0$. Conversely, let $r_1 \neq 0$, $r_2 \in R$. By hypothesis, we have $r_1(r_2 : J) \neq 0$. Then we may choose $r \in (r_2 : J)$ such that $r_1r \neq 0$. Since $r \in (r_2 : J)$, we also have $r_2r \in J$. This completes the proof.

Definition 3.2. *A right hyperideal I of R is essential if for every nonzero right hyperideal K of R we have $I \cap K \neq 0$.*

Example 3.2. Define the hyperoperation $\oplus$ on the $R = [0,1]$ by

$$x \oplus y = \begin{cases} \{\max\{x,y\}\}, & \text{if } x \neq y \\ [0,x], & \text{if } x = y \end{cases}.$$

Then $([0,1], \oplus, \cdot)$ is a Krasner hyperring where $\cdot$ is the usual multiplication on real numbers. Let $I = [0, 0.3]$. Hence I is an essential hyperideal of R.

The following remark give the relationship between essential and dense right hyperideals:

Remark 3.2. Let J be a dense right hyperideal of R.Then J is an essential right hyperideal of R.

Proof. For $0 \neq a \in R$, pick $r \in R$ such that $0 \neq ar \in J$. Then $0 \neq ar \in J \cap aR$.

Remark 3.3. Let I be a 2-sided hyperideal of R. Then the following conditions are equivalent:

(i) $l(I) = 0$;

(ii) I is a dense right hyperideal;

(iii) I is an essential right hyperideal;

(iv) I is essential as a 2-sided hyperideal(i.e., for any hyperideal $J \neq 0$, $I \cap J \neq 0$).

Remark 3.4. Let I be a 2-sided hyperideal of R. Then:

(i) $l(I) = r(I)$;

(ii) $l(I) \cap I = 0$;

(iii) $I + l(I)$ is dense right hyperideal of R.

Remark 3.5. Let J be a right hyperideal of R and $f : J \to R$ be a good right R -hypermodule homomorphism. Then:

(i) If $a \in R$ and $r(a) \in D(R)$, then $a = 0$;

(ii) If $\ker f \in D(R)$, then $f = 0$.

Proof. (i) It is follows from Proposition 3.1 (v).

(ii) We assume that $\ker f \in D(R)$. Then we have $f(b)(b : \ker f) = 0$ for all $b \in J$. By Proposition 1 (ii), $(b : \ker f) \in D(R)$. According to the first statement we obtain $f(b) = 0$. Thus $f = 0$.

Let R is a semiprime hyperring $\Psi = \{f_U \mid f : U \to R$ is a good right R -homomorphism and $U \in D\}$.

Define a relation "$\approx$" on Ψ by $f_U \approx g_V :\Leftrightarrow$ there exists $K \in D$ and $K \subseteq U \cap V$ such that $f = g$ on K.

One readily checks that "$\approx$" is an equivalence relation. This gives a chance for us to get a partition of Ψ. We denote the equivalence class by $\bar{f}_U$, where

$\overline{f}_U := \{g : V \to R |\, f_U \approx g_V\}$ and denote by Q_{mr} set of all equivalence classes. We define a hyperaddition "+" on Q_{mr} as follows:

$$\overline{f}_U + \overline{g}_V := \overline{f+g}_{U\cap V}$$

where $f+g : U \cap V \to R$ is a good right R homomorphism. Assume that $f_{1U_1} \approx f_{2U_2}$ and $g_{1V_1} \approx g_{2V_2}$. Then $\exists K_1(\in D) \subseteq U_1 \cap U_2$ such that $f_1 = f_2$ on K_1 and $\exists K_2(\in D) \subseteq V_1 \cap V_2$ such that $g_1 = g_2$ on K_2. Taking $K = K_1 \cap K_2$ and so $K \in D$. For any $x \in K$, we have $(f_1 + g_1)(x) = f_1(x) + g_1(x) = \bigcup \{t(x) \mid t(x) \in f_1(x) + g_1(x)\} = \bigcup \{t(x) \in f_2(x) + g_2(x)\} = f_2(x) + g_2(x) = (f_2 + g_2)(x)$, and so $f_1+g_1 = f_2+g_2$ on K. Therefore $f_1+g_{1_{U_1\cap V_1}} \approx f_2+g_{2_{U_2\cap V_2}}$, which means that the addition in Q_{mr} is well-defined.

Now we will prove that Q_{mr} is a canonical hypergroup. Let $\overline{f}_U$, $\overline{g}_V$, $\overline{h}_H$ be elements of Q_{mr}. Since $U \cap (V \cap H) = (U \cap V) \cap H$, for all $x \in U \cap (V \cap H)$

$$\begin{aligned}
[(f+g)+h](x) = (f+g)(x) + h(x) &= \bigcup_{t(x)\in(f+g)(x)} t(x) + h(x) \\
&= \bigcup_{t(x)\in(f+g)(x)} \{k(x) \mid k(x) \in t(x) + h(x)\} \\
&= \bigcup \{k(x) \mid k(x) \in (f(x) + g(x)) + h(x)\} \\
&= \bigcup \{k(x) \mid k(x) \in f(x) + (g(x) + h(x))\} \\
&= \bigcup_{p(x)\in g(x)+h(x)} \{k(x) \mid k(x) \in f(x) + p(x)\} \\
&= \bigcup_{p(x)\in(g+h)(x)} f(x){+}p(x){=}f(x){+}(g+h)(x){=}[f+(g+h)](x)\ .
\end{aligned}$$

Hence $(f+g)+h = f+(g+h)$ on $U \cap (V \cap H)$. That is $(\overline{f}_U + \overline{g}_V) + \overline{h}_H = \overline{f}_U + (\overline{g}_V + \overline{h}_H)$. One can easily check that $\overline{f}_U + \overline{g}_V = \overline{g}_V + \overline{f}_U$. Taking $\overline{\theta}_R \in Q_{mr}$ where $\theta : R \to R$, $x \mapsto 0$ for all $x \in R$. Since $U \subseteq U \cap R$, $(\theta + f)(x) = \theta(x) + f(x) = 0 + f(x) = f(x)$ for all $x \in U$. Then we have $\overline{\theta}_R + \overline{f}_U = \overline{f}_U$ and similarly $\overline{f}_U + \overline{\theta}_R = \overline{f}_U$ for all $\overline{f}_U \in Q_{mr}$. Hence $\overline{\theta}_R$ is the additive identity in Q_{mr}. Let $-\overline{f}_U \in Q_{mr}$, where $-f : U \to R$, $x \mapsto -f(x) = (-f)(x)$ for all $x \in U$. Since $-f(x)$ is the unique inverse of $f(x)$ in R, we have $\theta(x) \in f(x) - f(x) = f(x) + (-f)(x)$ for all $x \in U$. So $\overline{\theta}_R \in \overline{f}_U + (-\overline{f}_U)$. Finally, let $\overline{f}_U$, $\overline{g}_V$, $\overline{h}_H$ be elements of Q_{mr} and $\overline{h}_H \in \overline{f}_U + \overline{g}_V$. So there exists a $f_1 \in \overline{f}_U$ and a $g_1 \in \overline{g}_V$ such that $h = f_1 + g_1$. For any $x \in K(\in D) \subseteq U \cap V$, we get $h(x) = (f_1 + g_1)(x) = f_1(x) + g_1(x) \subseteq f(x) + g(x)$. Since R is a hyperring, $h(x) \in f(x) + g(x)$ implies $g(x) \in -f(x) + h(x)$ and $f(x) \in h(x) - g(x)$. Thus we have $g(x) \in (-f+h)(x)$ and $f(x) \in (h-g)(x)$. That is, $\bar{g_V} \in -\bar{f_U} + \bar{h_H}$ and $\bar{f_U} \in \bar{h_H} - \bar{g_V}$. Therefore $(Q_{mr}, +)$ a canonical hypergroup.

Now we define a multiplication "." on Q_{mr} as follows: for all $\bar{f_U}, \bar{g_V} \in Q_{mr}$

$$\bar{f_U}\bar{g_V} := \overline{fg}_{g^{-1}(U)}$$

where $fg : g^{-1}(U) \to R$ is a good right R homomorphism. Assume that $f_{1U_1} \approx f_{2U_2}$ and $g_{1V_1} \approx g_{2V_2}$. Then $\exists K_1(\in D) \subseteq U_1 \cap U_2$ such that $f_1 = f_2$ on K_1 and $\exists K_2(\in D) \subseteq V_1 \cap V_2$ such that $g_1 = g_2$ on K_2. Taking $K := g_1^{-1}(K_1) \cap K_2$ and so $K \in D$. For any $x \in K$,

$$(f_1 g_1)(x) = f_1(g_1(x)) = f_1(g_2(x)) = f_2(g_2(x)) = (f_2 g_2)(x).$$

Thus $f_1 g_1 = f_2 g_2$ on K. Hence "." is well-defined. Now we will prove that $(Q_{mr}, .)$ is a semigroup having zero as a bilaterally element . Let $\bar{f_U}, \bar{g_V}, \bar{h_H} \in Q_{mr}$. Since $h^{-1}(g^{-1}(U)) = (gh)^{-1}(U)$, we get for all $x \in h^{-1}(g^{-1}(U))$,

$$[(fg)h](x) = (fg)(h(x)) = f(g(h(x))) = f((gh)(x)) = (f(gh))(x).$$

Hence $(fg)h = f(gh)$ on $h^{-1}(g^{-1}(U))$. That is, $(\bar{f_U}\bar{g_V})\bar{h_H} = \bar{f_U}(\bar{g_V}\bar{h_H})$. Now we prove that $\bar{f_U}\bar{\theta_R} = \bar{\theta_R} = \bar{\theta_R}\bar{f_U}$ for all $\bar{f_U} \in Q_{mr}$. Since $\theta^{-1}(U) \subseteq \theta^{-1}(U) \cap R$ and $f\theta = \theta$ on $\theta^{-1}(U)$, we get $\bar{f_U}\bar{\theta_R} = \bar{\theta_R}$. Similarly $\bar{\theta_R}\bar{f_U} = \bar{\theta_R}$.

Let $\bar{f_U}$, $\bar{g_V}$, $\bar{h_H}$ be elements of Q_{mr}. Since $h^{-1}(U \cap V) = h^{-1}(U) \cap h^{-1}(V)$, we get for all $x \in h^{-1}(U \cap V)$,

$$\begin{aligned}[(f+g)h](x) &= (f+g)(h(x)) = f(h(x)) + g(h(x)) \\ &= (fh)(x) + (gh)(x) = (fh+gh)(x).\end{aligned}$$

Then $(f+g)h = fh+gh$ on $h^{-1}(U \cap V)$. That is, $(\bar{f_U} + \bar{g_V})\bar{h_H} = \bar{f_U}\bar{h_H} + \bar{g_V}\bar{h_H}$. Also since $(g+h)^{-1}(U) = g^{-1}(U) \cap h^{-1}(U)$, for all $x \in (g+h)^{-1}(U)$

$$\begin{aligned}[f(g+h)](x) &= f((g+h)(x)) = f(g(x) + h(x)) \\ &= f(g(x)) + f(h(x)) = (fg+fh)(x).\end{aligned}$$

Hence $f(g+h) = fg + fh$ on $(g+h)^{-1}(U)$. That is, $\bar{f_U}(\bar{g_V} + \bar{h_H}) = \bar{f_U}\bar{g_V} + \bar{f_U}\bar{h_H}$.

Therefore, $(Q_{mr}, +, .)$ is a hyperring.

Taking $\bar{1_R} \in Q_{mr}$ where $1 : R \to R$, $x \mapsto x$ for all $x \in R$. Let $\bar{f_U} \in Q_{mr}$. Since $1^{-1}(U) \subseteq U$, we get for all $x \in 1^{-1}(U)$, $(f1)(x) = f(1(x)) = f(x)$ and

$(1f)(x) = 1(f(x)) = f(x)$. Thus, $\overline{f}_U\overline{1}_R = \overline{1}_R\overline{f}_U = \overline{f}_U$. Hence $\overline{1}_R$ is the multiplicative identity in Q_{mr}.

We denote the hyperring constructed by $Q_{mr} = Q_{mr}(R)$ and we call it the maximal right quotient hyperring of R. One can of course, characterize Q_{ml}, the left quotient hyperring of R in similar manner. For purpose of convenience, we use q instead of $\overline{q}_V \in Q_{mr}$.

Proposition 3.2. *The maximal right quotient hyperring Q_{mr} of R satisfies:*

(i) *R is a subhyperring of Q_{mr};*

(ii) *For all $q \in Q_{mr}$ there exists $U \in D$ such that $qU \subseteq R$;*

(iii) *If $q \in Q_{mr}$ and $U \in D$, then $qU = 0$ if and only if $q = \theta$;*

(iv) *If $U \in D$ and $f : U \to R$ is a good right R -hypermodule homomorphism, then there exists $q \in Q_{mr}$ such that $f(x) = qx$ for all $x \in U$.*

Furthermore, if Q' is any hyperring satisfying $(i) - (iv)$, then Q' is isomorphic to Q_{mr}.

Proof. (i) For a fixed element a in R, consider a mapping $\lambda_a : R \to R$ by $\lambda_a(r) = ar$ for all $r \in R$. It is easy to prove that the mapping λ_a is a good right R homomorphism. Define a mapping $\varphi : R \to Q_{mr}$ by $\varphi(a) = \lambda_a$ for $a \in R$. Clearly the mapping φ is injective homomorphism and so R is a subring of Q_{mr}.

(ii) Let $q = f_U \in Q_{mr}$. One sees that $f_U\lambda_{a_R} = \lambda_{f(a)_R}$ for all $a \in U$, i.e., $q\varphi(U) \subseteq \varphi(R)$. Since $R \cong \varphi(R)$, we can write $qU \subseteq R$.

(iii) Let $q = f_U \in Q_{mr}$ and $U \in D$ such that $q\varphi(U) = 0$. Then $q = \theta$. Indeed, we have $\theta = f_U\lambda_{a_R} = \lambda_{f(a)_R}$ for all $a \in U$. Then $f(a) = 0$ and so $f(U) = 0$. By Remark 3.5, $q = \theta$.

(iv) Let $U \in D$ and $f : U \to R$ is a good right R -hypermodule homomorphism. Hence $f_U\lambda_{a_R} = \lambda_{f(a)_R}$ for all $a \in U$. That is, $q\varphi(a) = \varphi(f(a))$ for all $a \in U$, where $q = f_U$. Thus we write $qx = f(x)$ for all $x \in U$.

Now suppose $Q' \supseteq R$ is a hyperring having properties $(i) - (iv)$. Define the mapping $\alpha : Q' \to Q_{mr}$, $q \longmapsto \lambda_{q_{(q:R)_R}}$. One can easily check that α is an isomorphism of hyperrings.

Lemma 3.1. *Let $q_1, q_2, ..., q_n \in Q_{mr}$ and $I, J \in D(R)$. Then there exists $L \in D(R)$ such that $L \subseteq J$ and $q_iL \subseteq I$ for all $i = 1, 2, ..., n$.*

Proof. Let $J_i = (q_i : R)_R$. Then $J_i \in D$ for all $i = 1, 2, ..., n$. Consider the mapping $f_i : J_i \to R$, $f_i = \lambda_{q_i}$. By Proposition 3.1, $K_i = f_i^{-1}(I) = \{x \in J_i \mid q_ix \in I\} \in D$. Then $L = (\bigcap_{i=1}^{n} K_i) \cap J$ we have the desired dense right hyperideal.

Lemma 3.2. *Let K be a dense right hyperideal of a semiprime hyperring R and S a subhyperring of $Q_{mr}(R)$ such that $K \subseteq S$. Then:*

(i) *S is a semiprime hyperring;*

(ii) *A right hyperideal J of S is dense if and only if $(J \cap R)K \in D(R)$ (in particular, if $I \in D(R)$ then $IS \in D(S)$);*

(iii) A right hyperideal J of S is essential if and only if $(J \cap R)K$ is an essential right hyperideal of R.

Since the proof is similar to the proof of the corresponding lemma in ring theory, we omitted it.

Proposition 3.3. *Let K be a dense right hyperideal of R and S a subhyperring of $Q_{mr}(R)$ such that $K \subseteq S$. Then $Q_{mr}(S) = Q_{mr}(R)$.*

The following result is an immediate corollary of Proposition 3.3.

Theorem 3.1. *Let R be a semiprime hyperring and $Q = Q_{mr}(R)$. Then $Q_{mr}(Q) = Q$.*

Corollary 3.2. *Let R be a semiprime hyperring, I be an hyperideal of R and $J = l_R(I)$. Then $Q_{mr}(R) = Q_{mr}(I) \oplus Q_{mr}(J)$.*

Proof. According to Remark 3.4, $I \oplus J \in D(R)$ and by Proposition 3.3, $Q_{mr}(R) = Q_{mr}(I \oplus J)$. From equality $Q_{mr}(I \oplus J) = Q_{mr}(I) \oplus Q_{mr}(J)$, the proof is obtained.

Definition 3.3. *Let R be a hyperring. The set*

$$Z_r(R) = \{x \in R \mid r_R(x) \text{ is an essential right hyperideal}\}$$

is called the right singular hyperideal of R.

Example 3.3. Let R be a hyperring and $H(R) = \left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \mid a, b \in R \right\}$ be a collection of 2×2 matrices over R. A hyperoperation "+" and a multiplication "·" are defined on $H(R)$ as follows:

$$\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} + \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix} = \left\{ \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \mid x \in a + c \ , \ y \in b + d \right\}$$

$$\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \cdot \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix} = \begin{pmatrix} 0 & ad \\ 0 & bd \end{pmatrix} \text{ for all } \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}, \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix} \in H(R).$$

Clearly, $(H(R), +, \cdot)$ is a Krasner hyperring. We will show that $Z_r(H(R)) = 0$. Indeed, for each $0 \neq A \in H(R)$, $r_{H(R)}(A) = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in H(R) \mid AB = 0 \right\}$ but $I = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix} \in H(R) \mid y \in R \right\}$ is a nonzero right hyperideal of $H(R)$ and $r_{H(R)}(A) \cap I = 0$. Thus $r_{H(R)}(A)$ is not an essential right hyperideal of $H(R)$.

Lemma 3.3. *Let R be a semiprime hyperring, K an essential right hyperideal of R and $r \in R$. Then:*

(i) $(r : K)_R$ is an essential right hyperideal of R;

(ii) $Z_r(R)$ is a hyperideal of R;

(iii) $Z_r(R) = 0$ if and only if every essential right hyperideal is dense;

(iv) For any subhyperring $R \subseteq S \subseteq Q_{mr}(R)$, $Z_r(R) = R \cap Z_r(S)$.

Proof. (i) Clearly, $(r : K)_R = \{x \in R \mid rx \in K\}$ is a right hyperideal of R. Let $L \neq 0$ be a right hyperideal of R. If $rL = 0$, then $L \subseteq (r : K)_R$. Hence $0 \neq L = L \cap (r : K)_R$. Now, suppose that $rL \neq 0$. Since rL is a right hyperideal of R and K an essential right hyperideal of R , $rL \cap K \neq 0$. But $rL \cap K = r[L \cap (r : K)_R]$. Thus $L \cap (r : K)_R \neq 0$ and so $(r : K)_R$ is essential.

(ii) Let $x, y \in Z_r(R)$ and $r \in R$. We first verify that $x - y \subseteq Z_r(R)$. Let $a \in x-y$. We shall show that $r_R(a)$ is essential. Since $r_R(x) \cap r_R(y) \subseteq r_R(x-y)$ and $r_R(x) \cap r_R(y)$ is an essential right hyperideal, $r_R(x-y)$ is essential as well. Further $r_R(x-y) \subseteq r_R(a)$. Indeed, for any $b \in r_R(x-y)$, $(x-y)b = 0$. Since $a \in x-y$, $ab \in (x-y)b = 0$. Thus $ab = 0$ and so $b \in r_R(a)$. Hence $r_R(a)$ is an essential right hyperideal of R. As $r_R(x) \subseteq r_R(rx)$ and $r_R(x)$ is essential, $rx \in Z_r(R)$. By the above result, the right hyperideal $(r : r_R(x))$ is essential. From $(r : r_R(x)) \subseteq r_R(xr)$ it follows that $xr \in Z_r(R)$. Therefore $Z_r(R)$ is a hyperideal of R.

(iii) Suppose that $Z_r(R) = 0$. Let J be an essential right hyperideal of R. Taking into account (i), we have $l_R((a : J)) = 0$ for all $a \in R$. By Corollary 3.1, we obtain $J \in D(R)$. The converse statement follows from Proposition 3.1 (v).

(iv) Note that $r_R(x) = r_S(x) \cap R$ for all $x \in R$. Thus, by Lemma 3.2, $r_R(x)$ is an essential right hyperideal of S. Hence $Z_r(R) = R \cap Z_r(S)$.

Lemma 3.4. *Let R be a semiprime hyperring, $Q = Q_{mr}(R)$ and K a subhypermodule of the right R -hypermodule Q. Suppose that $\alpha : K \to Q$ is a good homomorphism of right R -hypermodules. Then:*

(i) The rule $\widehat{\alpha} : KQ \to Q$, $\widehat{\alpha}\left(\sum_{i=1}^n k_i q_i\right) = \sum_{i=1}^n \alpha(k_i) q_i$, where $k_i \in K$, $q_i \in Q$ defines a good homomorphism of right Q -hypermodules.

(ii) If K is a right hyperideal of the hyperring Q, then α is a good homomorphism of right Q -hypermodules.

Proof. (i) It is enough to check that $\widehat{\alpha}$ is well-defined. Indeed, let $\sum_{i=1}^n k_i q_i = 0$, where $k_i \in K$, $q_i \in Q$. By Lemma 3.1, there exists a dense right hyperideal L of R such that $q_i L \subseteq R$ for all i. For any $x \in L$ we have

$$\left(\sum_{i=1}^n \alpha(k_i) q_i\right) x = \sum_{i=1}^n \alpha(k_i)(q_i x) = \alpha\left(\sum_{i=1}^n k_i q_i x\right) = 0.$$

That is, $\sum_{i=1}^n \alpha(k_i) q_i = 0$ and $\widehat{\alpha}$ is well-defined.

(ii) If K is a right hyperideal of the hyperring Q, then $\alpha = \widehat{\alpha}$ which means that α is a good homomorphism of right Q -hypermodules.

Definition 3.4. *An element $a \in R$ is said to be regular if $a \in aRa$. That is, there exists an element $b \in R$ such that $a = aba$. A hyperring R is said to be regular (in the sense of von Neumann) if every element of R is regular.*

Theorem 3.2. *Let R be a semiprime hyperring and $Q = Q_{mr}(R)$. Then the following conditions are equivalent:*

(i) Q is a von Neumann regular hyperring;

(ii) $Z_r(R) = 0$.

Proof. (i)$\Rightarrow$ (ii). We suppose that Q is a von Neumann regular hyperring. Let $0 \neq q \in Q$. Hence $qpq = q$ for some $p \in Q$. Clearly, $r_Q(pq) = r_Q(q)$ and $(pq)^2 = pq$. Then $r_Q(pq) = (1 - pq)Q$. Indeed, for any $x \in r_Q(pq)$, we write $pqx = 0$. Since Q is a hyperring, $0 = pqx \in pqq^{'} - pqq^{'} = pqq^{'} - pqpqq^{'} = pq(1 - pq)q^{'}$ where $q^{'} \in Q$. Thus, $x \in (1 - pq)Q$, i.e.; $r_Q(pq) \subseteq (1 - pq)Q$. Similarly, $(1 - pq)Q \subseteq r_Q(pq)$. Since $(1 - pq)Q \cap pqQ = 0$, $(1 - pq)Q$ is not essential. Therefore $Z_r(Q) = 0$.

By Lemma 3.3, $Z_r(R) = 0$.

(ii)$\Rightarrow$ (i). Suppose that $Z_r(R) = 0$. Then, according to Lemma 3.3 (*iii*), the set $D(R)$ coincides with the set of all essential right hyperideals of R. Let $q = f_U \in Q = Q_{mr}$ and $K = \ker f$. Choosing L to be a right hyperideal of R maximal with the properties $L \subseteq U$ and $L \cap K = 0$, we note that $L \cong qL$. Then $K + L$ is an essential right hyperideal of R and so $K + L \in D(R)$. We choose M to be a right hyperideal of R maximal with the property $M \cap qL = 0$. Obviously, $M \oplus qL$ is an essential right hyperideal of R. Thus $M \oplus qL \in D(R)$. The mapping $g : M \oplus qL \to L$, $g(x) = l$ where $x \in m + ql$, $m \in M$ and $l \in L$. Hence $p = g_{M \oplus qL} \in Q$ and $(fgf)(k + l) = f(k + l)$ for all $k \in K$, $l \in L$. Thus $pqp = p$ and so Q is von Neumann regular hyperring.

References

[1] K.I. Beidar, W.S. Martindale III, and A.V. Mikhalev, *Rings with generalized identities*, Marcel Dekker, Inc. 1996.

[2] P. Corsini and V. Leoreanu, *Application of hyperstructure theory*, Kluwer Academic Pub, 2003.

[3] B. Davvaz, *Isomorphism theorems of hyperrings*, Indian J. Pure Appl. Math., 35 (2004), 321-331.

[4] B. Davvaz and V. Leoreanu-Fotea, *Hyperring theory and applications*, International Academic Press, 2007.

[5] B. Davvaz and A. Salasi, *A realization of hyperrings*, Comm. Algebra, 34 (2006), 4389-4400.

[6] K. Hila and B. Davvaz, *An introduction to the theory of algebraic multi-hyperring spaces*, An. Stiint. Univ. "Ovidius" Constanţa Ser. Mat., 22 (2014), 59-72.

[7] M. Krasner, *A class of hyperrings and hyperfields*, Int. J. Math. Math. Sci., (1983), 307-312.

[8] F. Marty, *Sur uni generalization de la notion de group*, 8th Congress Math. Scandenaves, Stockholm, (1934), 45-49.

[9] S. Mirvakili and B. Davvaz, *Applications of the α^*-relation to Krasner hyperrings*, Journal of Algebra, 362 (2012), 145-156.

[10] S. Mirvakili and B. Davvaz, *Strongly transitive geometric spaces: applications to hyperrings*, Revista de la Union Matematica Argentina, 53 (2012) 43-53.

[11] S. Mirvakili and B. Davvaz, *Relationship between rings and hyperrings by using the notion of fundamental relations*, Communications in Algebra, 41 (2013), 70-82.

[12] A. Mortazavi and B. Davvaz, *Krasner hypermodules of generalized fractions*, Sigma Journal of Engineering and Natural Sciences, 9 (2018), 39-62.

[13] A. Nakassis, *Recent results in hyperring and hyperfield theory*, Internat. J. Math. Math. Sci., 11 (1988), 209-220.

[14] S. Omidi and B. Davvaz, *On ordered Krasner hyperrings*, Iranian Journal of Mathematical Sciences and Informatics, 12 (2017), 35-49.

[15] S. Omidi, B. Davvaz and P. Corsini, *Operations on hyperideals in ordered Krasner hyperrings*, An. Stiint. Univ. "Ovidius" Constanta Ser. Mat., 24 (2016), 275-293.

[16] Y. Utumi, *On quotient rings*, Osaka J. Math., 8 (1956), 1-18.

[17] T. Vougiouklis, *The fundamental relation in hyperrings. The general hyperfield*, Algebraic hyperstructures and applications (Xánthi, 1990), 203-211, World Sci. Publ., Teaneck, NJ, 1991.

[18] E.R. Yanehsari, and M. Asghari-Larimi, *Some properties of hyperrings*, Int. Res. J. Appl. Basic. Sci., 4 (2013), 2375-2380.

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