Ideal theory on bounded semihoops

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Abstract. In this paper, we investigate some types of ideals on bounded semihoops. First, we introduce the notion of ideals on bounded semihoops and give some characterizations of ideals on bounded semihoops. Furthermore, we study primary ideals, prime ideals and maximal ideals. We discuss some relations between these ideals. In particular, every primary ideal is a prime ideal on bounded semihoops with $x^2 = x$. Also, we introduce the concept of perfect ideals and find that every perfect ideal is a primary ideal. In addition, we prove that I is a perfect ideal if and only if A/I is a perfect semihoop. Moreover, we introduce the notion of local semihoops. We characterize some equivalent conditions of local semihoops. In fact, A/I is local if and only if I is a primary ideal. Finally, we define the concept of locally finite, and show that I is a maximal ideal if and only if A/I is locally finite.
Keywords: semihoop, ideal, local semihoop.

1. Introduction

Various logical algebras have been proposed as the semantical systems of non-classical logic systems, for example, MV-algebras[3], BL-algebras [12], MTL-algebras [5], residuated lattices [10]. In fact, semihoops [6] are the fundamental residuated structures and contain all logic algebras based on residuated lattices. Semihoops are generalization of hoops. The algebraic hoop were investigated by Buchi and Owens in an unpublished manuscript in 1975. They have been studied by Blok and Ferreirim [15], Aglianò [11]et.al. In particular, from the structure theorem of finite basic hoops [11], one obtains an elegant short proof of

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the completeness theorem for propositional basic logic [11], introduced by Hájek [12]. It is proved that a hoop is a meet-semilattice ordered residuated, commutative monoid, integral, divisible. Juntao Wang [8] investigated monadic bounded hoops and proved the completeness of the monadic hoop logic. A. Namdar et.al [1] introduced some results in hoop algebras. R.T. Khorami et. al [14] introduced the notions of multiplier, $\odot$-closure modal operator on hoop-algebras. As a more general structure, a semihoop is a hoop without the divisibility. It follows that a semihoop does not satisfy the divisibility condition. Therefore, semihoops play an important role in fuzzy logics and the related algebraic structures.

From a logic point of view, various filters corresponding to various sets of provable formulae. Ideal theory is a very effective tool for investigating these various algebraic and logic systems. The notion of ideal has been introduced in many algebraic structure such as lattices, rings, MV-algebras and so on. For residuated lattices, Yi Liu and Ya Qin [17] introduced the notion of ideals in residuated lattices. Also, Lele and Nganou [4] introduced the notion of ideals in BL-algebras as a natural generalization of that of ideals in MV-algebras. Akbar Paad [2] introduced the notion of integral ideals and maximal ideals in BL-algebras. Furthermore, Wenjuan Chen [16] mainly investigated ideals and congruences in a quasi-pseudo-MV algebras. Forouzesh [7] presented fuzzy semi-maximal and fuzzy radical ideals in MV-algebras. However, the study of semihoops main focus on filters, the notion of ideal is missing.

In this paper, we will extend ideals from MV-algebras to semihoops for obtaining the most general results on logic algebraic ideals. For semihoops, there is no addition operation, so we introduce the definition of addition operation to define ideals on bound semihoops. One of our aims is to show the relationships among different ideals on bound semihoops, such as prime ideal, primary ideal and perfect ideal. On the other hand, we portray perfect semihoops, local semihoops and locally finite semihoops using perfect ideals, primary ideal and maximal ideals respectively. Above arguments motivate us to investigate ideals on bounded semihoops.

The papers organized as follows: In section 2, we give some basic results on bounded semihoops, which are needed in the rest papers. In section 3, we discuss characterizations of ideals and the relations between ideals and filters on a bounded semihoop. Also, we prove that every primary ideal with the condition of $x^2 = x$ is a prime ideal. In section 4, we study that every perfect ideal is a primary ideal and I is a perfect ideal if and only if A/I is a perfect semihoop. We prove that A is local if and only if it has unique maximal ideal, and A/I is local if and only if I is a primary ideal. Further, we obtain that $I \in \mathcal{M}(A)$ if and only if A/I is locally finite.

2. Preliminaries

In this section, we summarize some definitions and results on semihoops which will be used in the following sections.

Definition 2.1 ([6]). *An algebra* $(A, \odot, \to, \wedge, 1)$ *of type (2,2,2,0) is called a semihoop if it satisfies the following conditions:*

(1) $(A, \wedge, 1)$ *is a* $\wedge$*-semilattice with upper bound 1,*

(2) $(A, \odot, 1)$ *is a commutative monoid,*

(3) $(x \odot y) \to z = x \to (y \to z)$*, for all* $x, y, z \in A$.

On a semihoop $(A, \odot, \to, \wedge, 1)$, we define $x \leq y$ if and only if $x \to y = 1$ for any $x, y \in A$. It is easy to check that $\leq$ is a partial order relation on A and for all $x \in A, x \leq 1$.

For any $x \in A$, we define $x^0 = 1$ and $x^n = x^{n-1} \odot x$ for any natural number n.

Proposition 2.2 ([6]). *In any semihoop* $(A, \odot, \to, \wedge, 1)$*, the following properties hold for all* $x, y, z \in A$:

(1) $x \odot y \leq z$ *iff* $x \leq y \to z$,

(2) $x \odot y \leq x, y$,

(3) $1 \to x = x, x \to 1 = 1$,

(4) $x^n \leq x$,

(5) $y \leq x \to y$,

(6) *if* $x \leq y$*, then* $y \to z \leq x \to z, z \to x \leq z \to y$ *and* $x \odot z \leq y \odot z$,

(7) $x \leq (x \to y) \to y$,

(8) $x \to y \leq (z \to x) \to (z \to y), x \to y \leq (y \to z) \to (x \to z)$,

(9) $x \to (y \to z) = y \to (x \to z)$.

A semihoop $(A, \odot, \to, \wedge, 1)$ is called a bounded semihoop if there exists an element $0 \in A$ such that $0 \leq x$ for any $x \in A$.

In a bounded semihoop $(A, \odot, \to, \wedge, 0, 1)$, we define $* : x^* = x \to 0$ for any $x \in A$. If $x^{**} = x$, for any $x \in A$, then the bounded semihoop $(A, \odot, \to, \wedge, 0, 1)$ is said to have a double negation property (DNP, for short).

We denote a bounded semihoop $(A, \odot, \to, \wedge, 0, 1)$ by A.

Proposition 2.3 ([13]). *In a bounded semihoop* A*, the following properties hold for all* $x, y \in A$:

(1) $1^* = 0, 0^* = 1$,

(2) $x \leq x^{**}$,

(3) $x \odot x^* = 0$,

(4) $y^* \leq y \to x$,

(5) $x \leq y$ *implies that* $y^* \leq x^*$,

(6) *if* A *has (DNP), then* $x \to y = y^* \to x^*$.

Proposition 2.4. *In a bounded semihoop* A*, the following properties hold for all* $x, y \in A$:

(1) $x \to y \leq y^* \to x^*$,

(2) *if* A *has (DNP), then* $x^* \to y = y^* \to x$.

Theorem 2.5 ([9]). *Let A be a hoop and X be a nonempty subset of A. Then* $(X] = \{a \in A : \exists n \in N : a \leq x_1 \ominus (x_2 \ominus \cdots \ominus (x_{n-1} \ominus x_n) \cdots), \text{ for } x_1, x_2 \cdots, x_n \in X\}$.

Proposition 2.6 ([9]). *Let A be a $\vee$-hoop with (DNP) and let P be a proper ideal of A. Then P is a prime ideal if and only if, for any $I, J \in \mathcal{ID}(A)$ such that $I \cap J \subseteq P$, we get $I \subseteq P$ or $J \subseteq P$.*

Proposition 2.7 ([13]). *Let A be a semihoop and for all x, y, we define $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$. Then the following conditions are equivalent:*

(1) $\vee$ *is an associative on A;*

(2) $x \leq y$ *implies* $\vee \sqcup z \leq y \vee z$, *for all* $x, y, z \in A$;

(3) $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$, *for all* $x, y, z \in A$;

(4) $\vee$ *is the join operation on A.*

Definition 2.8 ([13]). *A $\vee-$semihoop is a semihoop if it satisfies one of the equivalent conditions of Proposition 2.7.*

Remark: A semihoop with (DNP) may not be a hoop.

Example 2.9. Let $A = \{0, a, b, c, 1\}$ be a chain with $0 < a < b < c < 1$. Defined $\odot$ and $\to$ on A as follows,

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	0	a	b
c	0	0	a	a	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	c	1	1	1
c	a	c	c	1	1
1	0	a	b	c	1

We can check A is a bounded semihoop with (DNP). But it is not a hoop, because $b \odot (b \to c) = b \neq c \odot (c \to b) = a$.

3. Ideals on bounded semihoops

In this section, we introduce the notion of ideals and some important ideals such as primary ideals, prime ideals and maximal ideals on bounded semihoops. We discuss the relations between them on bounded semihoops. We will use these ideals in the next sections.

Definition 3.1. *In a bounded semihoop A, the binary operation $\oplus$ is defined by $x \oplus y = x^* \to y$ for any $x, y \in A$.*

Example 3.2. Let $A = \{0, a, b, 1\}$ be a chain with $0 < a < b < 1$. Defined $\odot$ and $\to$ on A as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	a	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	a	b	1	1
1	0	a	b	1

Then A is a bounded semihoop. Since $a \oplus 0 = 1$ and $0 \oplus a = a$, we get the operation $\oplus$ is not commutative in general.

In a bounded semihoop A with (DNP), the operation $\oplus$ is commutative. Since $x \oplus y = x^* \to y = y^* \to x = y \oplus x$, for any $x, y \in A$.

In a bounded semihoop A with (DNP), the operation $\oplus$ is associative. Since $(x \oplus y) \oplus z = (x^* \to y)^* \to z = (x^* \to y^{**})^* \to z = ((x^* \odot y^*) \to 0)^* \to z = (x^* \odot y^*)^{**} \to z = x^* \to (y^* \to z) = x \oplus (y \oplus z)$.

Proposition 3.3. *In a bounded semihoop A, the following properties hold for any $x, y, z \in A$:*

(1) *if $x \leq y$, then $x \oplus z \leq y \oplus z$,*
(2) $x \leq x \oplus y$,
(3) $x \oplus x^* = 1$,
(4) $0 \oplus x = x, x \oplus 0 = x^{**}$,
(5) $x \oplus y = 1$ *if and only if* $x^* \leq y$,
(6) $(x \oplus y) \oplus z \leq x \oplus (y \oplus z)$,
(7) *if A has (DNP), then $x^* \odot y^* = (x \oplus y)^*$,*
(8) *if A has (DNP), then $x^* \oplus y^* = (x \odot y)^*$.*

Proof. The proof of (1),(2),(3),(4),(5) are clear.

(6) $(x \oplus y) \oplus z \leq x \oplus (y \oplus z) \Leftrightarrow (x^* \to y)^* \to z \leq x^* \to (y^* \to z) \Leftrightarrow x^* \odot ((x^* \to y)^* \to z) \leq y^* \to z \Leftrightarrow x^* \leq ((x^* \to y)^* \to z) \to (y^* \to z) \Leftrightarrow x^* \to (((x^* \to y)^* \to z) \to (y^* \to z)) = 1$. Since $x^* \to (((x^* \to y)^* \to z) \to (y^* \to z)) \geq x^* \to (y^* \to (x^* \to y)^*) \geq x^* \to ((x^* \to y) \to y) = 1$, we have $x^* \to (((x^* \to y)^* \to z) \to (y^* \to z)) = 1$.

(7) Since $(x^* \odot y^*) \to (x \oplus y)^* = x^* \to (y^* \to (x \oplus y)^*) \geq x^* \to ((x \oplus y) \to y) = x^* \to ((x^* \to y) \to y) \geq x^* \to x^* = 1$, we have $(x^* \odot y^*) \to (x \oplus y)^* = 1$, and so $x^* \odot y^* \leq (x \oplus y)^*$.

On the other hand, $(x \oplus y)^* \leq x^* \odot y^* \Leftrightarrow (x^* \odot y^*)^* \leq (x^* \to y)^{**} \Leftrightarrow x^* \to y^{**} \leq x^* \to y \Leftrightarrow x^* \to y \leq x^* \to y$.

Therefore, $x^* \odot y^* = (x \oplus y)^*$.

(8) Since $(x^* \oplus y^*) \to (x \odot y)^* = (x^{**} \to y^*) \to (x \to y^*) \geq x \to x^{**} = 1$, we have $(x^* \oplus y^*) \to (x \odot y)^* = 1$, and so $x^* \oplus y^* \leq (x \odot y)^*$.

On the other hand, $(x \odot y)^* \leq x^* \oplus y^* \Leftrightarrow x \to y^* \leq x^{**} \to y^* \Leftrightarrow x \to y^* \leq x \to y^*$.

Therefore, $x^* \oplus y^* = (x \odot y)^*$. □

Definition 3.4. *Let A be a bounded semihoop and I be a non-empty subset of A. Then I is said to be an ideal of A, if it satisfies:*

($I1$) *$x \leq y$ and $y \in I$ imply $x \in I$, for any $x, y \in A$,*
($I2$) *$x \oplus y \in I$, for any $x, y \in I$.*

We can see that $\{0\}$ and A are ideals of A.

We denote the set of all ideals of A by $\mathcal{I}(A)$.

Definition 3.5. *Let A be a bounded semihoop. An ideal I of A is called a proper ideal if $1 \notin I$.*

Example 3.6. Let $A = \{0, a, b, c, d, 1\}$ with $0 < b, d < a < 1, 0 < d < c < 1$, where a and c are incomparable, b and d are incomparable. Defined $\odot$ and $\rightarrow$ on A as follows,

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	b	b	d	0	a
b	0	b	b	0	0	b
c	0	d	0	c	d	c
d	0	0	0	d	0	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	a	c	c	1
b	c	1	1	c	c	1
c	b	a	b	1	a	1
d	a	1	a	1	1	1
1	0	a	b	c	d	1

We can check A is a bounded semihoop, where $x \wedge y = x \odot (x \rightarrow y)$, for any $x, y \in A$ ([13]). It is routine to verify that $I_1 = \{0, b\}$ and $I_2 = \{0, c, d\}$ are ideals of A.

Theorem 3.7. *Let A be a bounded semihoop and I be a non-empty subset of A. Then the following conditions are equivalent:*

(1) *I is an ideal of A;*

(2) *$0 \in I$, and $x \oplus y \in I, x^* \odot y \in I$ and $x \in I$ imply $y \in I$, for any $x, y \in A$;*

(3) *$0 \in I$, and $x \oplus y \in I, (x^* \rightarrow y^*)^* \in I$ and $x \in I$ imply $y \in I$, for any $x, y \in A$;*

(4) *$x \oplus y \in I$, for any $x, y \in I$, and $x \in I$ implies $x \wedge y \in I$, for any $x, y \in A$.*

Proof. (1) $\Rightarrow$ (2) Suppose that I is an ideal of A. It follows from (I1) that $0 \in I$. Let $x, y \in A$ such that $x^* \odot y \in I$ and $x \in I$. Then $y \rightarrow (x \oplus (x^* \odot y)) = y \rightarrow (x^* \rightarrow (x^* \odot y)) = (y \odot x^*) \rightarrow (y \odot x^*) = 1$, and so $y \leq x \oplus (x^* \odot y)$. Since $x^* \odot y \in I$ and $x \in I$, it follows from (I2) that $x \oplus (x^* \odot y) \in I$. By (I1), we have $y \in I$. Therefore (2) holds.

(2) $\Rightarrow$ (3) Let $(x^* \rightarrow y^*)^* \in I$ and $x \in I$. Since $x^* \odot y^{**} \leq (x^* \odot y^{**})^{**} = (x^* \rightarrow y^{***})^* = (x^* \rightarrow y^*)^* \in I$ and $(x^* \rightarrow y^*)^* \in I$, we have $x^* \odot y^{**} \in I$. It follows from (2), we get $y^{**} \in I$. Since $y \leq y^{**}$, we have $y \in I$.

(3) $\Rightarrow$ (1) Suppose that $x, y \in A$, $x \leq y$ and $y \in I$, for any $x, y \in A$. Then $y^* \leq x^*$, and so $(y^* \rightarrow x^*)^* = 0 \in I$. Since $y \in I$, and it follows from (3), we have $x \in I$. Thus, I is an ideal of A.

(1) $\Rightarrow$ (4) Suppose that I is an ideal of A. It is clear that I satisfies (4).

(4) $\Rightarrow$ (1) Suppose that I satisfies (4). Let $x \in I$, $y \in L$ such that $y \leq x$, then $0 = x \wedge 0 \in I$ and $y = x \wedge y \in I$. Therefore I is an ideal of A. □

Let A be a bounded semihoop and X be a non-empty subset of A. The ideal generated by X means the smallest ideal containing X, which is denoted by $\langle X \rangle$.

Corollary 3.8. *Let A be a bounded semihoop and I be an ideal of A. Then $x \in I$ if and only if $x^{**} \in I$.*

Proof. Suppose that $x \in I$. By Theorem 3.7, taking $y = x^{**}$ in (3), then $(x^* \to x^{***})^* = 0 \in I$. Thus $x^{**} \in I$. Conversely, suppose that $x^{**} \in I$. Since $x \leq x^{**}$ and I is an ideal, we have $x \in I$. □

Theorem 3.9. *Let A be a bounded semihoop and X be a non-empty subset of A. Then* $\langle X \rangle = \{a \in A |\ a \leq x_n \oplus (\cdots \oplus (x_3 \oplus (x_2 \oplus x_1)) \cdots), x_i \in X, i = 1, 2 \cdots n\}$.

Proof. The proof is similar to Theorem 2.5. □

Proposition 3.10. *Let A be a bounded semihoop and $x \in A$. Then* $\langle I \cup \{x\} \rangle = \{a \in A | a \leq y_1 \oplus (y_2 \oplus \cdots \oplus (y_n \oplus nx)), y_i \in I, i = 1, 2 \cdots n\} \bigcup \{a \in A | a \leq x \oplus (x \oplus \cdots \oplus (x \oplus y)), y \in I\}$.

Proof. Let $B = \{a \in A | a \leq y_1 \oplus (y_2 \oplus \cdots \oplus (y_n \oplus nx)), y_i \in I, i = 1, 2 \cdots n\} \bigcup \{a \in A | a \leq x \oplus (x \oplus \cdots \oplus (x \oplus y)), y \in I\}$. It is clear that $\langle I \cup \{x\} \rangle \subseteq B$ and $0 \in B$.

Suppose that $b, c \in A$ such that $b \leq c$ and $c \in B$. Then there exist $y_i \in I (i = 1, 2 \cdots n)$ and $y \in I$ such that $c \leq y_1 \oplus (y_2 \oplus \cdots \oplus (y_n \oplus nx)$ or $c \leq x \oplus (x \oplus \cdots \oplus (x \oplus y))$. Since $b \leq c$, we have $b \leq y_1 \oplus (y_2 \oplus \cdots \oplus (y_n \oplus nx)$ or $b \leq x \oplus (x \oplus \cdots \oplus (x \oplus y))$. Thus $b \in B$.

If for any $b, c \in B$, then there exist $y_i \in I, z_j \in I (i = 1, 2 \cdots n; j = 1, 2 \cdots m)$ and $y \in I, z \in I$ such that $b \leq y_1 \oplus (y_2 \oplus \cdots \oplus (y_n \oplus nx)$ or $b \leq x \oplus (x \oplus \cdots \oplus (x \oplus y))$ and $c \leq z_1 \oplus (z_2 \oplus \cdots \oplus (z_n \oplus nx)$ or $c \leq x \oplus (x \oplus \cdots \oplus (x \oplus z))$.

Case 1: if $b \leq y_1 \oplus (y_2 \oplus \cdots \oplus (y_n \oplus nx)$ and $c \leq z_1 \oplus (z_2 \oplus \cdots \oplus (z_n \oplus nx)$, By Proposition 3.3(6), we have $b \oplus c$

$\leq [y_1 \oplus (y_2 \oplus \cdots \oplus (y_n \oplus nx)] \oplus [z_1 \oplus (z_2 \oplus \cdots \oplus (z_n \oplus nx]$

$\leq y_1 \oplus \{[y_2 \oplus \cdots \oplus (y_n \oplus nx)] \oplus [z_1 \oplus (z_2 \oplus \cdots \oplus (z_n \oplus nx]\}$

$\leq y_1 \oplus (y_2 \oplus \{[y_3 \oplus (\cdots \oplus (y_n \oplus nx)] \oplus [z_1 \oplus (z_2 \oplus \cdots \oplus (z_n \oplus nx]\})$

$\leq \cdots \leq y_1 \oplus (y_2 \oplus (\cdots (y_n \oplus (nx \oplus (z_1 \oplus (z_2 \oplus (\cdots \oplus (z_n \oplus nx) \cdots))))) \cdots))$.

Hence, $b \oplus c \in B$.

The proof of other cases is similar to case 1.

Therefore, B is an ideal. Next, we prove that B is the smallest ideal that containing $\langle I \cup \{x\} \rangle$. Let C be an ideal such that $\langle I \cup \{x\} \rangle \subseteq C$. Then for any $b \in B, b \leq y_1 \oplus (y_2 \oplus \cdots \oplus (y_n \oplus nx)$ or $b \leq x \oplus (x \oplus \cdots \oplus (x \oplus y)), y_i \in I, y \in I$. Since C is an ideal and $\langle I \cup \{x\} \rangle \subseteq C$, we have $y_i \in C, y \in C$ and $y_1 \oplus (y_2 \oplus \cdots \oplus (y_n \oplus nx) \in C$, or $x \oplus (x \oplus \cdots \oplus (x \oplus y)) \in C$, and so $b \in C$. Hence $B \subseteq C$. Therefore $\langle I \cup \{x\} \rangle = \{a \in A | a \leq y_1 \oplus (y_2 \oplus \cdots \oplus (y_n \oplus nx)), y_i \in I, i = 1, 2 \cdots n\} \bigcup \{a \in A | a \leq x \oplus (x \oplus \cdots \oplus (x \oplus y)), y \in I\}$. □

We know if A has (DNP), then the operation $\oplus$ is commutative and associative. So we have the following corollary.

Corollary 3.11. *Let A be a bounded semihoop with (DNP). Then* $\langle I \cup \{x\} \rangle = \{a \in A | a \leq y \oplus nx, y \in I\}$.

Let A be a bounded semihoop. We define a binary relation $\sim_I$ on A as follows:

$x \sim_I y$ if and only if $(x \to y)^* \in I$ and $(y \to x)^* \in I$.

Proposition 3.12. *Let A be a bounded semihoop and I be an ideal of A. Then $\sim_I$ is a congruence relation on a bounded semihoop A.*

Proof. It is easy to check that $\sim_I$ is an equivalence relation on A.

Suppose that $x \sim_I y$. Then $(x \to y)^* \in I$ and $(y \to x)^* \in I$. Since $(x \wedge z) \to (y \wedge z) \geq x \to y$ for any $z \in I$, we have $((x \wedge z) \to (y \wedge z))^* \leq (x \to y)^*$. Since I is an ideal, we have $((x \wedge z) \to (y \wedge z))^* \in I$. By the similar way, we obtain $((y \wedge z) \to (x \wedge z))^* \in I$. Therefore, $x \wedge z \sim_I y \wedge z$.

By the similar way, we obtain $x \odot z \sim_I y \odot z$ and $x \to z \sim_I y \to z$.

Therefore, $\sim_I$ is a congruence relation on A. □

Let $A/I = \{[x] | x \in A\}$, where $[x] = \{y \in A | x \sim_I y\}$. Then the binary relation $\leq$ on A/I which is defined by, $[x] \leq [y]$ if and only if $(x \to y)^* \in I$, is an order relation on A/I. Now, $(A/I, \wedge_{A/I}, \otimes, \rightsquigarrow, 0_{A/I})$ is a bounded semihoop, where for any $x, y \in A$:

$$0_{A/I} = [0], [x] \wedge_{A/I} [y] = [x \wedge_A y], [x] \otimes [y] = [x \odot y], [x] \rightsquigarrow [y] = [x \to y].$$

Definition 3.13. *Let A be a bounded semihoop and P be a proper ideal of A. P is called a primary ideal, if for all $x, y \in A, x \odot y \in P$ implies $x^n \in P$ or $y^n \in P$, for some $n \in N$.*

Example 3.14. (1) Let $A = \{0, a, b, 1\}$ be a chain. Defined $\odot$ and $\to$ on A as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	a	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

We can check A is a bounded semihoop ([13]). It is routine to verify that $I = \{0\}$ is a primary ideal of A.

(2) In the example 3.6, $I_1 = \{0, b\}$ is a primary ideal.

Proposition 3.15. *Let A be a bounded semihoop and P be a proper ideal of A. Then P is a primary ideal if and only if $[x] \otimes [y] = [0]$ implies that there exists $n \in N$ such that $[x]^n = [0]$ or $[y]^n = [0]$, for any $[x], [y] \in A/P$.*

Proof. ($\Rightarrow$) Let P be a primary ideal of A and $[x] \otimes [y] = [0]$, for some $[x], [y] \in A/P$. Since $[x] \otimes [y] = [x \odot y] = [0]$, we have $((x \odot y) \to 0)^* \in P$ and so $(x \odot y)^{**} \in P$. By Corollary 3.8, $x \odot y \in P$. Since P is a primary ideal, there exists $n \in N$ such that $x^n \in P$ or $y^n \in P$. If $x^n \in P$, then $(x^n \to 0)^* = (x^n)^{**}$. By Corollary 3.8, $(x^n)^{**} \in P$. Since $(0 \to x^n)^* = 0 \in P$, by definition of quotient, $[x]^n = [0]$. By the similar way, we have $[y]^n = [0]$.

($\Leftarrow$) Let $x \odot y \in P$, for some $x, y \in A$. Then $((x \odot y) \to 0)^* = (x \odot y)^{**}$. By Corollary 3.8, we have $(x \odot y)^{**} \in P$. Since $0 \to (x \odot y)^* = 0 \in P$, we get $x \odot y \equiv_P 0$. Hence $[x] \otimes [y] = [x \odot y] = [0]$. So, there exists $n \in N$ such that $[x]^n = [0]$ or $[y]^n = [0]$. If $[x]^n = [0]$, then $(x^n \to 0)^* = (x^n)^{**} \in P$ and so $x^n \in P$. By the similar way, we can prove that $y^n \in P$. Therefore P is a primary ideal. □

Definition 3.16. *A proper ideal I of a bounded semihoop A is called a prime ideal of A, if $H \cap G \subseteq I$ implies $H \subseteq I$ or $G \subseteq I$, for any $H, G \in \mathcal{I}(A)$.*

Proposition 3.17. *Let A be a bounded $\vee$-semihoop with (DNP)and I be a proper ideal of A. Then the following conditions are equivalent:*

(1) *I is a prime ideal;*

(2) *if $x \wedge y \in I$, for any $x, y \in A$, then $x \in I$ or $y \in I$.*

Proof. The proof is similar to Theorem 2.6. □

Proposition 3.18. *Let A be a bounded semihoop and $x^2 = x$, for any $x \in A$. Then every primary ideal of A is a prime ideal.*

Proof. Let P be a primary ideal of A and $x \wedge y \in P$, for any $x, y \in P$. Since $x \odot y \leq x \wedge y$, then $x \odot y \in P$. Since P is a primary ideal, there exists $n \in N$ such that $x^n \in P$ or $y^n \in P$. Since $x^2 = x$, if $x^n \in P$, we get $x^n = x \in P$. If $y^n \in P$, then by the similar way, we have $y \in P$. Therefore P is a prime ideal. □

Open problem: Is there a prime ideal, which is not primary on a bounded semihoop?

Definition 3.19. *Let A be a bounded semihoop and I be a proper ideal of A. I is called maximal ideal of A, if it is not properly contained in the any other proper ideal of A. We denote the set of all maximal ideal of bounded semihoops A by $\mathcal{M}(A)$.*

Proposition 3.20. *Let A be a bounded semihoop and I be a proper ideal of A. Then the following conditions are equivalent:*

(1) *$I \in \mathcal{M}(A)$;*

(2) *if $x \notin I$, then $\langle I \cup \{x\} \rangle = A$, for any $x \in A$.*

Proof. (1) $\Rightarrow$ (2) If $x \notin I$, then $I \subsetneqq \langle I \cup \{x\} \rangle$. Since $I \in \mathcal{M}(A)$, we have $\langle I \cup \{x\} \rangle = A$.

(2) $\Rightarrow$ (1) Let G be an ideal of A such that $I \subseteq G$ and $I \neq G$, then there exists $x \in G$ such that $x \notin I$. It follows from(2), $\langle I \cup \{x\} \rangle = A$. Since $A = \langle I \cup \{x\} \rangle \subseteq G$, then we get $A = G$, which is a contradiction. Therefore, $I \in \mathcal{M}(A)$. □

Proposition 3.21. *Let A be a bounded semihoop and M be a proper ideal of A. If M is a maximal ideal of A, then for all $x \notin M$ if and only if there exists $n \in N$ such that $(nx)^* \in M$.*

Proof. ($\Rightarrow$) Suppose that M is a maximal ideal of A. If $x \notin M$, by Proposition 3.20, we get $\langle M \cup \{x\} \rangle = A$, Since $1 \in A$, we have $1 \in \langle M \cup \{x\} \rangle$. By Corollary 3.11, there exist $n \in N, y \in M$ such that $1 \leq y \oplus (nx)$. Since $1 \leq y^* \to (nx)$, we have $1 \odot y^* \leq nx$, and so $(nx)^* \leq y^{**}$. Since M is an ideal and $y \in M$, we have $y^{**} \in M$. Hence $(nx)^* \in M$.

($\Leftarrow$) Suppose that $x \in M$. Then we have $nx \in M$. Since $(nx)^* \in M$, we have $(nx) \oplus (nx)^* = 1 \in M$, which is contradiction. Therefore, $x \notin M$. □

4. Perfect ideals of bounded semihoops

In this section, we introduce the notion of perfect ideals and discuss the relation of perfect ideals on perfect semihoops. In addition, we study the relation ideals on local semihoops and give some characterizations of local semihoops.

Definition 4.1. *The order of $x \in A$, in symbols ord(x), is the smallest $n \in N$ such that $nx = 1$. If no such n exists, then ord$(x) = \infty$.*

Proposition 4.2. *The following conditions hold, for any $x, y \in A$:*

(1) *$\langle x \rangle$ is a proper ideal of A if and only if ord$(x) = \infty$,*
(2) *if $x \leq y$ and ord$(y) = \infty$, then ord$(x) = \infty$,*
(3) *if $x \leq y$ and ord$(x) < \infty$, then ord$(y) < \infty$,*
(4) *for every $M \in \mathcal{M}(A), M \subseteq \{x \in A | ord(x) = \infty\}$,*
(5) *ord$(1) < \infty$ and ord$(0) = \infty$.*

Proof. (1) Let $\langle x \rangle$ be a proper ideal of A. If there exists $n \in N$ such that $nx = 1$, then $1 \in \langle x \rangle$, which is a contradiction. Conversely, let $\langle x \rangle = A$. Since $1 \in A$, we have $1 \in \langle x \rangle$ and so there exists $n \in N$ such that $nx = 1$. Thus ord$(x) < \infty$, which is a contradiction.

(2) Let $x \leq y$, for some $x, y \in A$. Then $nx \leq ny$, for all $n \in N$. Since ord$(y) = \infty$, we have $nx \neq 1$, for all $n \in N$. Therefore ord$(x) = \infty$.

(3) Let $x \leq y$ and ord$(x) < \infty$, for some $x, y \in A$. Then there exists $n \in N$ such that $nx = 1$ and so $ny = 1$. Hence ord$(y) < \infty$.

(4) Let $x \in M$ and ord$(x) < \infty$. Then exists $n \in N$ such that $nx = 1$. Thus $1 \in M$, which is a contradiction.

(5) The proof is clear. □

Definition 4.3. *A is called a perfect semihoop, if ord$(x) < \infty$, then ord$(x^*) = \infty$, and if ord$(x) = \infty$, then ord$(x^*) < \infty$, for any $x \in A$.*

Definition 4.4. *Let A be a bounded semihoop and P be a proper ideal of A. P is called perfect ideal of A, if P satisfies the condition, for any $x \in A, x^n \in P$, for some $n \in N$ if and only if $(x^*)^m \notin P$ for all $m \in N$.*

Example 4.5. Let $A = \{0, a, b, c, 1\}$ be a chain with $0 < a < b < c < 1$. Define $\odot$ and $\to$ on A as follows,

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

We can check A is a bounded semihoop. It is routine to verify that $I = \{0, a\}$ is a perfect ideal.

Proposition 4.6. *Let A be a bounded semihoop. Then every perfect ideal of A is a primary ideal.*

Proof. Let A be a bounded semihoop and P be a perfect ideal of A. Suppose that $x \odot y \in P$, for some $x, y \in A$ such that $x^m \notin P$, for all $m \in N$. Since $x \odot y \in P$ and P is an ideal, we have $(x \odot y)^{**} \in P$. Since $(x \odot y)^{**} = (y \odot x)^{**} = (y \to x^*)^* \in P$, by definition of quotient, $y/P \leq x^*/P$. Thus $y^n/P \leq (x^*)^n/P$, for all $n \in N$. Since $x^m \notin P$ and P is a perfect ideal, we have for some $n \in N$ such that $(x^*)^n \in P$ and so $(x^*)^n/P = 0/P$. Thus $y^n/P = 0/P$. Hence $y^n \in P$. Therefore, P is a primary ideal. □

The following example shows that the converse of the proposition 4.6 may not be true.

Example 4.7. In the example 3.14(1), $I = \{0\}$ is a primary ideal, but it is not a perfect ideal, because $a^2 = 0 \in I$, but $(a^*)^3 = 0 \in I$.

Theorem 4.8. *Let A be a bounded semihoop with (DNP) and I be a proper ideal of A. Then I is a perfect ideal if and only if A/I is a perfect semihoop.*

Proof. Let I be a proper ideal. Suppose that $x/I \in A/I$ such that $\text{ord}(x/I) < \infty$. Then there exists $n \in N$ such that $n(x/I) = 1/I$, and so $(n(x/I))^* = 0/I$. By Proposition 3.3(7), $((x/I)^*)^n = (n(x/I))^*$, then $((x/I)^*)^n = 0/I$. Hence $(x^*)^n \in I$. Since I is a perfect ideal, we have $x^m \notin I$, for any $x \in A$ and for any $m \in N$. Thus $x^m/I \neq 0/I$, hence $(x^m/I)^* \neq 1/I$. By Proposition 3.3(8), $m(x^*/I) = (x^m/I)^*$, then $m(x^*/I) \neq 1/I$. Thus $\text{ord}(x^*/I) = \infty$. If there exists $n \in N$ such that $n(x^*/I) = 1/I$, then $(n(x^*/I))^* = 0/I$. Since $(x^{**}/I)^n = (n(x^*/I))^*$, we have $(x^{**})^n/I = 0/I$, then $(x^{**})^n \in I$ and so $x^n \in I$, which is a contradiction with $x^m \notin I$, for any $m \in N$. Therefore, A/I is a perfect. Let $\text{ord}(x/I) = \infty$. Then $n(x/I) \neq 1/I$, for all $n \in N$, and so $(n(x/I))^* \neq 0/I$. By Proposition 3.3(7), $((x/I)^*)^n = (n(x/I))^*$, then $((x/I)^*)^n \neq 0/I$. Thus $(x^*)^n \notin I$. Since I is a perfect ideal, we have there exists $m \in N, x^m \in I$, then $x^m/I = 0/I$. Thus $(x^m/I)^* = 1/I$. By Proposition 3.3(8), $m(x^*/I) = (x^m/I)^*$, $m(x^*/I) = 1/I$, we have $\text{ord}(x^*/I) < \infty$. Therefore, A/I is a perfect.

Conversely, let I be a proper ideal of A such that $x^n \in I$, for some $n \in N$ and for any $x \in A$. Then $x^n/I = 0/I$ and so $(x^n/I)^* = 1/I$. By Proposition 3.3(8), $n(x^*/I) = (x^n/I)^*$, then $n(x^*/I) = 1/I$. Hence $\mathrm{ord}(x^*/I) < \infty$. Since A/I is a perfect, we get $\mathrm{ord}(x/I) = \infty$, that is $m(x/I) \neq 1/I$, for all $m \in N$. Hence $(m(x/I))^* \neq 0/I$. By Proposition 3.3(7), $(x^*/I)^m = (m(x/I))^*$, then $(x^*)^m/I \neq 0/I$, and so $(x^*)^m \notin I$, for all $m \in N$. The proof of the other implication is similar. Therefore, I is a perfect ideal. □

Definition 4.9. *Let A be a bounded semihoop. A is called a local semihoop if $ord(x) < \infty$ or $ord(x^*) < \infty$, for any $x \in A$.*

Definition 4.10. *The radical of A, denoted by $Rad(A)$, is defined by $Rad(A) = \bigcap_{M \in \mathcal{M}(A)} M$.*

Theorem 4.11. *Let A be a bounded semihoop with(DNP). Then the following conditions are equivalent:*

(1) *A is local;*
(2) *A has unique maximal ideal;*
(3) *$Rad(A) = \{x \in A | ord(x) = \infty\}$.*

Proof. (1) ⇒ (2) Let $M \in \mathcal{M}(A)$. Then we prove $M = \{x \in A | ord(x) = \infty\}$. By Proposition 4.2(4), $M \subseteq \{x \in A | ord(x) = \infty\}$. Next, we prove that $\{x \in A | ord(x) = \infty\} \subseteq M$. Suppose that there exists $x \in \{x \in A | ord(x) = \infty\}$ such that $x \notin M$, then $\mathrm{ord}(x) = \infty$. By Proposition 3.21, there exists $n \in N$ such that $(nx)^* \in M$, then $\mathrm{ord}(nx)^* = \infty$. Since A is local, we have $\mathrm{ord}(nx) < \infty$. Hence $\mathrm{ord}(x) < \infty$, which is a contradiction, then $x \in M$. Thus $M = \{x \in A | ord(x) = \infty\}$. Therefore, A has unique maximal ideal.

(2) ⇒ (1) Suppose that A has unique maximal ideal M. Let $x \in A$ such that $\mathrm{ord}(x) = ord(x^*) = \infty$. By Proposition 4.2(1), $\langle x \rangle$ and $\langle x^* \rangle$ are proper ideal of A. By Zorn′s Lemma, they are contained in a maximal ideal. Since A has unique maximal ideal M, we get $\langle x \rangle \subseteq M$ and $\langle x^* \rangle \subseteq M$, then $x, x^* \in M$. By Proposition 3.3(3), $x \oplus x^* = 1 \in M$, which is a contradiction. Hence $\mathrm{ord}(x) < \infty$ or $\mathrm{ord}(x^*) < \infty$. Therefore, A is local.

(2) ⇒ (3) Let A has unique maximal ideal M such that $M = \{x \in A | ord(x) = \infty\}$. Since $Rad(A) = \bigcap_{M_i \in \mathcal{M}(A)} M_i$ and A has one maximal ideal M, we have $Rad(A) = M = \{x \in A | ord(x) = \infty\}$.

(3) ⇒ (2) Let $Rad(A) = \{x \in A | ord(x) = \infty\}$. Since $Rad(A) = \bigcap_{M \in \mathcal{M}(A)} M$, for all $M \in \mathcal{M}(A)$, we have $Rad(A) \subseteq M$, so $\{x \in A | ord(x) = \infty\} \subseteq M$. On the other side, by Proposition4.2(4), $M \subseteq \{x \in A | ord(x) = \infty\}$, for all $M \in \mathcal{M}(A)$. Thus, $M = \{x \in A | ord(x) = \infty\}$, for all $M \in \mathcal{M}(A)$. Therefore, A has unique maximal ideal. □

Theorem 4.12. *Let A be a bounded semihoop with (DNP) and I be a proper ideal of A. Then A/I is local if and only if I is a primary ideal.*

Proof. ($\Rightarrow$) Let $x \odot y \in I$, for some $x, y \in A$. Then $x \odot y/I = 0/I$, and so $x/I \leq y^*/I$. Suppose that $x^n \notin I$, for any $n \in N$. Then $x^n/I \neq 0/I$. Hence $(x^n/I)^* \neq 1/I$. By Proposition 3.3(8), $n(x^*/I) = (x^n/I)^*$. Then $n(x^*/I) \neq 1/I$, for any $n \in N$. Since A/I is local, there exists $m \in N$ such that $m(x/I) = 1/I$. Since $x/I \leq y^*/I$, we have $m(y^*/I) = 1/I$ and so $(m(y^*/I))^* = 0/I$. By Proposition 3.3(7), $(y^{**}/I)^m = (m(y^*/I))^*$, then $(y^{**}/I)^m = 0/I$, that is $(y^{**})^m \in I$ and so $y^m \in I$. Therefore, I is a primary ideal.

($\Leftarrow$) Let I be a primary ideal. Suppose that $x/I \in A/I$. Since $x \odot x^* = 0 \in I$, we have $x^n \in I$ or $(x^*)^n \in I$, for some $n \in N$. Hence $x^n/I = 0/I$ or $(x^*)^n/I = 0/I$. If $x^n/I = 0/I$, then $(x^n/I)^* = 1/I$. By Proposition 3.3(8), $n(x^*/I) = ((x/I)^n)^*$. Then $n(x^*/I) = 1/I$ and so $\mathrm{ord}(x^*/I) < \infty$. If $(x^*)^n/I = 0/I$, then by the similar way, we have $\mathrm{ord}(x/I) < \infty$. Therefore, A/I is local. □

Corollary 4.13. *If any proper ideals of A are primary, then A is local.*

Proof. Suppose that every proper ideal of A is primary. Since $\{0\}$ is a proper ideal of A, we have $\{0\}$ is primary. Thus, by Theorem 4.12, $A/\{0\}$ is local. Moreover, since $A/\{0\} \cong A$, we get A is local. □

Definition 4.14. *A is said to be local finite if for any $0 \neq x \in A$, ord$(x) < \infty$.*

Theorem 4.15. *Let A be a bounded semihoop with(DNP) and I be a proper ideal of A. Then $I \in \mathcal{M}(A)$ if and only if A/I is locally finite.*

Proof. ($\Rightarrow$) Let $I \in \mathcal{M}(A)$. Since I is a proper ideal of A, we have $A/I \neq \emptyset$. Let for any $x/I \in A/I$, but $x/I \neq 0/I$. Since $x \notin I$, by Proposition 3.21, there exists $n \in N$ such that $(nx)^* \in I$. Since $(nx \to 1)^* = 0 \in I$ and $(1 \to nx)^* = (nx)^* \in I$, we have $[nx] = [1]$. Therefore, A/I is locally finite.

($\Leftarrow$) Let I be a proper ideal of A and A/I be a locally finite. Suppose that $x \notin I$. Then $x/I \in A/I$. Since A/I is a locally finite, we have $\mathrm{ord}(x/I) < \infty$, that is there exists $n \in N$ such that $n(x/I) = 1/I$. Hence $(n(x/I))^* = 0/I$ and so $(nx)^* \in I$. By Proposition 3.21, $I \in \mathcal{M}(A)$. □

5. Conclusions

In this paper, motivated by the research of previous ideals on MV-algebras, BL-algebras and residuated lattices, we extend the concept of ideals to more general fuzzy structures, namely bounded semihoops. We introduce the notion of perfect semihoops and portray perfect semihoops by applying perfect ideals. Furthermore, we use primary ideals to characterize local semihoops. Finally, we give the concept of locally finite bounded semihoops and characterize them with maximal ideals. In the future, we will continue our discussion of the relationship between primary ideals and prime ideals. Also, we try to use the prime ideals and maximal ideals to research topological space on bounded semihoops.

Acknowledgement

This research was supported by a grant of National Natural Science Foundation of China (11571281). The Natural Science Basic Research Plan in Shaanxi Province of China (2020JQ-762) and Natural Science Foundation of Education Committee of Shannxi Province (19JK0653).

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Accepted 25.04.2019