Hyers-Ulam stability of second order difference equations

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Abstract. In this paper, we study the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the homogeneous and non-homogeneous linear difference equations of second order with constant co-efficients by applying Z-Transforms method.
Keywords: Hyers-Ulam stability, Hyers-Ulam-Rassias stability, homogeneous and non-homogeneous, linear difference equation, Z-Transforms method.

1. Introduction

The study of stability problem for various functional equations originated from a famous talk of S.M. Ulam [23]. In 1940, Ulam [23] raised the question concerning the stability of functional equations: "Give Conditions in order for a linear function near an approximately linear function to exist" (see [10, 15]). Since then, this question has attracted the attention of many researchers. The first partial solution to this problem was given by D.H. Hyers [10] in 1941. He gave an affirmative answer to the question of Ulam's for additive Cauchy equation in Banach Spaces. Thereafter, Aoki [2], Bourgin [4] and Rassias [20] have generalized and also improved the result reported in [10]. Since then, several mathematicians have been extensively established the stability results to the various functional equations in different directions (see [5, 8, 12, 25, 28, 29, 30]).

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A generalization of Ulam's problem was recently proposed by replacing functional equations with differential equations: The differential equation

$$\phi\left(f, x, x', x'', ...x^{(n)}\right) = 0$$

has the Hyers-Ulam stability if for a given $\epsilon > 0$ and a function x such that

$$\left|\phi\left(f, x, x', x'', ...x^{(n)}\right)\right| \leq \epsilon,$$

there exists a solution x_a of the differential equation

$$\phi\left(f, x, x', x'', ...x^{(n)}\right) = 0$$

such that $|x(t) - x_a(t)| \leq K(\epsilon)$ and $\lim_{\epsilon \to 0} K(\epsilon) = 0$. If the preceding statement is also true when we replace ϵ and $K(\epsilon)$ by $\phi(t)$ and $\varphi(t)$, where ϕ, φ are appropriate functions not depending on x and x_a explicitly, then we say that the corresponding differential equation has the generalized Hyers-Ulam stability.

Oblaza [16, 17] were among the first to contribute to deal with the Hyers-Ulam stability of differential equations. Thereafter Alsina and Ger [1] were the first authors, who investigated the Hyers-Ulam stability of differential equations of the form $x'(t) = x(t)$. They proved in [1] the following Theorem.

Theorem 1.1. *Assume that a differentiable function $f : I \to \mathbb{R}$ is a solution of the differential inequality $\|x'(t) - x(t)\| \leq \epsilon$, where I is an open sub interval of $\mathbb{R}$. Then there exists a solution $g : I \to \mathbb{R}$ of the differential equation $x'(t) = x(t)$ such that for any $t \in I$, we have $\|f(t) - g(t)\| \leq 3\epsilon$.*

This result of C. Alsina and R. Ger [1] has been generalized by Takahasi [22]. They proved in [22] that the Hyers-Ulam stability holds true for the Banach Space valued differential equation $y'(t) = \lambda y(t)$. These previous results are extended to the Hyers-Ulam stability of linear differential equation of first order, second order and higher order in [3, 9, 11, 12, 13, 15, 21, 24, 26] by different approaches.

Now a days, the investigation of Hyers-Ulam stability of difference equations has been given attention. Recently, some authors are studied the Hyers-Ulam stability of linear and non-linear recurrences, linear recurrence with constant coefficients, linear recurrence of higher order in [6, 7, 14, 18, 19]. In this paper, we are going to prove the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the homogeneous second order linear difference equation

$$x(n+2) + a\ x(n+1) + b\ x(n) = 0 \tag{1.1}$$

and the non-homogeneous linear difference equation

$$x(n+2) + a\ x(n+1) + b\ x(n) = p(n) \tag{1.2}$$

with initial conditions

$$x(0) = 0 \text{ and } x(1) = 1 \tag{1.3}$$

by using Z-Transforms for all $n \in \mathbb{N}$, where $a, b \neq 0$ are constants.

2. Preliminaries

Z-transforms plays an important role in discrete analysis as Laplace and Fourier transforms in continuous system. It has many properties similar to those of the Laplace transforms. The Z-transforms operate on sequences of the discrete integer valued arguments, not on functions of continuous arguments. For every operational rule or Laplace transforms, there is a corresponding operational rule of Z-transforms and for every application of the Laplace transforms, there is a corresponding application of Z-transforms.

Firstly, we introduce some basic concepts of difference equations and Z-Transforms. Throughout this paper, $\mathbb{F}$ will denote the field of Complex numbers $\mathbb{C}$. If the function f_n is defined for discrete values $(n = 0, 1, 2, 3, ...)$ and $f_n = 0$ for $n < 0$, then the Z-Transform is defined to be

$$Z(f_n) = F(z) = \sum_{n=-\infty}^{\infty} \frac{f(n)}{z^n}$$

whenever the infinite series converges. Here z is a complex number. Z is an operator and $F(z)$ is the Z-Transform of $\{f(n)\}$. It is well known that Z is one-to-one and linear. If $Z[\{f(n)\}] = F(z)$, $n > 0$, then $f(0) = \lim_{z\to\infty} F(z)$ is known as the initial value problem of Z-Transform. If $Z[\{f(n)\}] = F(z)$, $n > 0$, then

$$\lim_{z\to\infty} F(n) = \lim_{z\to 1} (z-1)\ F(z)$$

is known as the Final value problem of Z-Transform. Let $\{f(n)\}$ and $\{g(n)\}$ are any two sequences and let the convolution of $\{f(n)\}$ and $\{g(n)\}$ be $\{h(n)\}$, where $\{h(n)\} = \{f(n)\} \ * \ \{g(n)\}$ then $\{h(n)\}$ is defined as

$$\{f(n)\} * \{g(n)\} = \sum_{n=-\infty}^{\infty} f(n)\ g(n-k) = \sum_{n=-\infty}^{\infty} g(n)\ f(n-k) = \{g(n)\} * \{f(n)\}.$$

Also, if $Z[\{f(n)\}] = F(z)$ and $Z[\{g(n)\}] = G(z)$, then

$$Z[\{h(n)\}] = Z[\{f(n)\}] * Z[\{g(n)\}] = F(z)\ G(z) = H(z),$$

where the region of convergence of $Z[\{h(n)\}]$ is the common region of convergence of $F(z)$ and $G(z)$.

If $F(z)$ is the Z-Transform of the sequence $\{f(k)\}$ then $\{f(k)\}$ is called the inverse Z-Transform of $F(z)$. The operator for inverse Z-Transform is denoted by Z^{-1}. Thus, if $Z[\{f(k)\}] = F(z)$, then $Z^{-1}[F(z)] = \{f(k)\}$.

Now, we will give the definition of Hyers-Ulam stability and Hyers-Ulam-Rassias stability for (1.1) and (1.2).

Definition 2.1. *We say that the homogeneous linear difference equation (1.1) has the Hyers-Ulam stability, if there is a positive constant K such that for every $\epsilon > 0$ and $x(n)$ satisfying the inequality*

$$|x(n+2) + a\ x(n+1) + b\ x(n)| \leq \epsilon$$

with (1.3) *then there exists* $y(n)$ *satisfying* (1.1) *with* $y(0) = 0$ *and* $y(1) = 1$ *such that* $|x(n) - y(n)| \leq K\ \epsilon$.

Definition 2.2. *We say that the non-homogeneous linear difference equation (1.2) has the Hyers-Ulam stability, if there is a positive constant* K *such that for every* $\epsilon > 0$ *and* $x(n)$ *satisfying the inequality*

$$|x(n+2) + a\ x(n+1) + b\ x(n) - p(n)| \leq \epsilon$$

with (1.3) *then there exists* $y(n)$ *satisfying (1.2) with* $y(0) = 0$ *and* $y(1) = 1$ *such that* $|x(n) - y(n)| \leq K\ \epsilon$.

Definition 2.3. *We say that the homogeneous linear difference equation* (1.1) *has the Hyers-Ulam-Rassias stability if there is a positive constant* K *such that for every* $\epsilon > 0$, $\phi(n)$ *and* $x(n)$ *be a positive sequences satisfying the inequality*

$$|x(n+2) + a\ x(n+1) + b\ x(n)| \leq \phi(n)\ \epsilon$$

with (1.3) *then there is a* $y(n)$ *satisfying (1.1) with* $y(0) = 0$ *and* $y(1) = 1$ *such that* $|x(n) - y(n)| \leq K(\epsilon)\ \phi(n)$.

Definition 2.4. *We say that the non-homogeneous linear difference equation (1.2) has the Hyers-Ulam-Rassias stability if there is a positive constant* K *such that for every* $\epsilon > 0$, $\phi(n)$ *and* $x(n)$ *be a positive sequences satisfying the inequality*

$$|x(n+2) + a\ x(n+1) + b\ x(n) - p(n)| \leq \phi(n)\ \epsilon$$

with (1.3) *then there exists* $y(n)$ *satisfying (1.2) with* $y(0) = 0$ *and* $y(1) = 1$ *such that* $|x(n) - y(n)| \leq K(\epsilon)\phi(n)$.

3. Hyers-Ulam stability of (1.1) and (1.2)

In this section, we are going to prove the Hyers-Ulam stability of the second order homogeneous and non-homogeneous linear difference equation (1.1) and (1.2) by applying Z-Transforms. Now, we prove the Hyers-Ulam stability of the homogeneous linear difference equation (1.1) with initial conditions.

Theorem 3.1. *For any* $a, b \in \mathbb{F}$ *there is a* $\alpha, \beta \in \mathbb{F}$ *such that* $\alpha + \beta = -a$ *and* $\alpha\beta = b$ *and* $\alpha \neq \beta$. *For every* $\epsilon > 0$, *we can find a positive constant* K *such that a positive sequence* $\{x(n)\}$ *satisfies the inequality*

$$|x(n+2) + a\ x(n+1) + b\ x(n)| \leq \epsilon, \tag{3.1}$$

for each value of $n \in \mathbb{N}$ *with* (1.3), *then there exists a solution* $\{y(n)\}$ *of the second order linear difference equation satisfies (1.1) with* $y(0) = 0$ *and* $y(1) = 1$ *such that* $|x(n) - y(n)| \leq K\ \epsilon$.

Proof. Let us define a function $h : (0, \infty) \to \mathbb{F}$ such that

$$h(n) = x(n+2) + a\ x(n+1) + b\ x(n).$$

Also, in view of (3.1), we have $|h(n)| \leq \epsilon$. Now, taking Z-Transform to $h(n)$, then we have

$$Z[h(n)] = H(z) = (z^2 + az + b)\ X(z) - z\ (z+a)\ x(0) - z\ x(1). \tag{3.2}$$

In view of (3.2), a sequence $\{x_0(n)\}$ is a solution of (1.1) if and only if

$$(z^2 + az + b)\ X(z) - z\ (z+a)\ x_0(0) - z\ x_0(1) = 0. \tag{3.3}$$

Applying (1.3) in (3.2), we have

$$H(z) = (z^2 + az + b)\ X(z) - z. \tag{3.4}$$

Since, we have $z^2 + az + b = (z - \alpha)\ (z - \beta)$, then (3.4) becomes

$$X(z) - \frac{z}{(z-\alpha)\ (z-\beta)} = \frac{H(z)}{(z-\alpha)\ (z-\beta)}. \tag{3.5}$$

Now, let us choose the function $y(n)$ such that $y(n) = \frac{\alpha^n - \beta^n}{\alpha - \beta}\ x(1)$, then applying Z-Transform to $y(n)$, we get

$$Z[y(n)] = Y(z) = \frac{z\ x(1)}{(z-\alpha)\ (z-\beta)}.$$

Hence

$$(z-\alpha)\ (z-\beta)\ Y(z) - z(z+a)\ x(0) - z\ x(1) = 0. \tag{3.6}$$

Since, we have $x(0) = y(0) = 0$ and $x(1) = y(1) = 1$. Now,

$$\begin{aligned} Z[y(n+2) + a\ y(n+1) + b\ y(n)] &= Z[y(n+2)] + a\ Z[y(n+1)] + b\ Z[y(n)] \\ &= (z^2 + az + b)Y(z) - z(z+a)y(0) - zy(1) \\ &= 0. \qquad \text{[form (3.6)]} \end{aligned}$$

Since, Z is one-to-one operator, it holds that $y(n+2) + a\ y(n+1) + b\ y(n) = 0$, therefore $y(n)$ is a solution of (1.1). Then we have

$$Z[x(n)] - Z[y(n)] = X(z) - Y(z) = \frac{H(z)}{(z-\alpha)\ (z-\beta)} = Z\left[r(n) * h(n)\right],$$

where $r(n) = \frac{1}{z}\left\{\frac{\alpha^n - \beta^n}{\alpha - \beta}\right\}$. Since, Z-Transforms is linear and one-to-one, we have $x(n) - y(n) = (r(n) * h(n))$. Now, taking modulus on both sides we have,

$$\begin{aligned} |x(n) - y(n)| = |r(n) * h(n)| &= \left| \sum_{n=-\infty}^{\infty} r(n-k)\ h(n) \right| \\ &\leq \sum_{n=-\infty}^{\infty} |r(n-k)|\ |h(n)| \leq K\ \epsilon. \end{aligned}$$

for each n, where $K = \sum_{n=-\infty}^{\infty} |r(n-k)| = \left|\frac{1}{z(\alpha-\beta)}\right| \sum_{n=-\infty}^{\infty} \left|\{\alpha^{n-k} - \beta^{n-k}\}\right|$ occurs for each value of n. Then by the virtue of Definition 2.1, the linear difference equation (1.1) has the Hyers-Ulam stability. $\square$

Now, we illustrate the Theorem 3.1 by the following example.

Example 3.2. For every $\epsilon > 0$, we can find a positive constant K such that a sequence $\{x(n)\}$ satisfies the inequality

$$|2\ x(n+2) + 5\ x(n+1) - 3\ x(n)| \leq \epsilon, \tag{3.7}$$

for each value of $t > 0$ with (1.3), then there exists a solution $y(n)$ with $y(0) = 0$ and $y(1) = 1$ such that $|x(n) - y(n)| \leq K\epsilon$.

Let $h(n) = 2\ x(n+2) + 5\ x(n+1) - 3\ x(n)$, for all n. Also, in view of (3.7), we have $|h(n)| \leq \epsilon$. Now, taking Z-Transform to $h(n)$, then we have

$$Z[h(n)] = H(z) = (2z^2 + 5z - 3)\ X(z) - z\ (2z+5)\ x(0) - 2z\ x(1).$$

Since, we have $2z^2 + 5z - 3 = 2(z - 1/2)\ (z+3)$, then we have

$$X(z) - \frac{2z}{2(z-1/2)\ (z+3)} = \frac{H(z)}{2(z-1/2)\ (z+3)}.$$

Thus

$$X(z) = \frac{H(z) + 2z}{2(z-1/2)\ (z+3)}.$$

Now, let us choose the function $y(n)$ such that $y(n) = \left(\frac{(1/2)^n - (-3)^n}{7/2}\right) x(1)$, then applying Z-Transform to $y(n)$, we get

$$Z[y(n)] = Y(z) = \frac{2z\ x(1)}{2(z-1/2)\ (z+3)}.$$

Hence $2(z-1/2)\ (z+3)\ Y(z) - z(2z+5)\ x(0) - 2z\ x(1) = 0$. Since, we have $x(0) = y(0) = 0$ and $x(1) = y(1) = 1$. Then $Z[2\,y(n+2) + 5\,y(n+1) - 3\,y(n)] = 0$. Since, Z is one-to-one operator, it holds that $2\ y(n+2) + 5\ y(n+1) - 3\ y(n) = 0$, therefore $y(n)$ is a solution of the difference equation. Then by using the Theorem 3.1, the difference equation $2\ x(n+2) + 5\ x(n+1) - 3\ x(n) = 0$ has the Hyers-Ulam stability.

Example 3.3. For every $\epsilon > 0$, we can find a positive constant K such that a sequence $\{x(n)\}$ satisfies the inequality

$$|x(n+2) + 3\ x(n+1) + 2\ x(n)| \leq \epsilon, \tag{3.8}$$

for each value of $n > 0$ with (1.3), then there exists a solution $y(n)$ with $y(0) = 0$ and $y(1) = 1$ such that $|x(n) - y(n)| \leq K\ \epsilon$. Let $h(n) = x(n+2) + 3\ x(n+1) +$

$2\ x(n)$. Also, in view of (3.8), we have $|h(n)| \leq \epsilon$. Now, taking Z-Transform to $h(n)$, then we have

$$Z[h(n)] = H(z) = (z^2 + 3z + 2)\ X(z) - z\ (z+3)\ x(0) - z\ x(1).$$

Since, we have $z^2 + 3z + 2 = (z+1)\ (z+2)$, then we have

$$X(z) - \frac{z}{(z+1)\ (z+2)} = \frac{H(z)}{(z+1)\ (z+2)}.$$

Now, let us choose the function $y(n)$ such that $y(n) = \{(-1)^n - (-2)^n\}\ x(1)$, then applying Z-Transform to $y(n)$, we get

$$Z[y(n)] = Y(z) = \frac{z\ x(1)}{(z+1)\ (z+2)}.$$

Hence $(z+1)\ (z+2)\ Y(z) - z(z+3)\ x(0) - z\ x(1) = 0$. Since, we have $x(0) = y(0) = 0$ and $x(1) = y(1) = 1$. Then $Z[y(n+2) + 3\ y(n+1) + 2\ y(n)] = 0$. Since, Z is one-to-one operator, it holds that $y(n+2)+3\ y(n+1)+2\ y(n) = 0$, therefore $y(n)$ is a solution. Then by using the Theorem 3.1, the difference equation $x(n+2) + 3\ x(n+1) + 2\ x(n) = 0$ has the Hyers-Ulam stability.

Now, we would like to prove the Hyers-Ulam stability of the non-homogeneous linear difference equation (1.2) with initial conditions.

Theorem 3.4. *For any $a, b \in \mathbb{F}$ there is a $\alpha, \beta \in \mathbb{F}$ such that $\alpha + \beta = -a$ and $\alpha\beta = b$ and $\alpha \neq \beta$. For every $\epsilon > 0$, we can find a positive constant K such that a positive sequence $\{x(n)\}$ satisfies the inequality*

$$|x(n+2) + a\ x(n+1) + b\ x(n) - p(n)| \leq \epsilon, \tag{3.9}$$

for each value of n with (1.3)*, then there is a solution $\{y(n)\}$ of the second order linear difference equation satisfies (1.2) with $y(0) = 0$ and $y(1) = 1$ such that $|x(n) - y(n)| \leq K\ \epsilon$, with $p(0) = p(1) = 0$.*

Proof. If we define a function $h : (0, \infty) \to \mathbb{F}$ such that

$$h(n) = x(n+2) + a\ x(n+1) + b\ x(n) - p(n).$$

Also, in view of (3.9), we have $|h(n)| \leq \epsilon$. Now, taking Z-Transforms to $h(n)$, then we have

$$Z[h(n)] = H(z) = (z^2 + az + b)\ X(z) - z\ (z+a)\ x(0) - z\ x(1) - P(z). \tag{3.10}$$

In view of (3.10), a sequence $\{x_0(n)\}$ is a solution of (1.2) if and only if

$$(z^2 + az + b)\ X(z) - z\ (z+a)\ x_0(0) - z\ x_0(1) = P(z). \tag{3.11}$$

Applying (1.3) in (3.10), we have

$$H(z) = (z^2 + az + b)\ X(z) - z - P(z). \tag{3.12}$$

Since, we have $z^2 + az + b = (z - \alpha)\ (z - \beta)$, then (3.12) gives that

$$X(z) - \frac{z}{(z-\alpha)\ (z-\beta)} = \frac{H(z) + P(z)}{(z-\alpha)\ (z-\beta)}. \tag{3.13}$$

Now, we define a function $y(n)$ as

$$y(n) = \frac{\alpha^n - \beta^n}{\alpha - \beta}\ x(1) + [r(n) * p(n)],$$

where $r(n) = \frac{1}{z}\left\{\frac{\alpha^n-\beta^n}{\alpha-\beta}\right\}$. Now, applying Z-Transform to $y(n)$, we get

$$Z[y(n)] = Y(z) = \frac{z\ x(1)}{(z-\alpha)\ (z-\beta)} + \frac{P(z)}{(z-\alpha)\ (z-\beta)}.$$

Hence

$$(z-\alpha)\ (z-\beta)\ Y(z) - z(z+a)\ x(0) - z\ x(1) = P(z). \tag{3.14}$$

Since, we have $x(0) = y(0) = 0$ and $x(1) = y(1) = 1$. Now,

$$\begin{aligned} Z[y(n+2) + a\ y(n+1) + b\ y(n)] &= Z[y(n+2)] + a\ Z[y(n+1)] + b\ Z[y(n)] \\ &= (z^2 + az + b)Y(z) - z(z+a)y(0) - zy(1) \\ Z[y(n+2) + a\ y(n+1) + b\ y(n)] &= P(z) = Z[p(n)]. \qquad \text{[form (3.14)]} \end{aligned}$$

Since, Z is one-to-one operator, it holds that $y(n+2)+a\,y(n+1)+b\,y(n) = p(n)$, therefore $y(n)$ is a solution of (1.2). Then we have

$$Z[x(n)] - Z[y(n)] = X(z) - Y(z) = \frac{H(z)}{(z-\alpha)\ (z-\beta)} = Z\left[r(n) * h(n)\right],$$

Since, Z-Transforms is linear and one-to-one, so $x(n) - y(n) = (r(n) * h(n))$. Now, taking modulus on both sides we have,

$$\begin{aligned} |x(n) - y(n)| = |r(n) * h(n)| &= \left|\sum_{n=-\infty}^{\infty} r(n-k)\ h(n)\right| \\ &\le \sum_{n=-\infty}^{\infty} |r(n-k)|\ |h(n)| \\ &\le K\ \epsilon. \end{aligned}$$

for each n, where

$$K = \sum_{n=-\infty}^{\infty} |r(n-k)| = \left|\frac{1}{z(\alpha-\beta)}\right| \sum_{n=-\infty}^{\infty} \left|\left\{\alpha^{n-k} - \beta^{n-k}\right\}\right|$$

occurs for each value of n. Then by the virtue of Definition 2.2, the linear difference equation (1.2) has the Hyers-Ulam stability. □

4. Hyers-Ulam-Rassias stability of (1.1) and (1.2)

In this section, we prove the Hyers-Ulam-Rassias stability of the second order homogeneous and non-homogeneous linear difference equation (1.1) and (1.2) with initial conditions.

Theorem 4.1. *For any $a, b \in \mathbb{F}$ there is a $\alpha, \beta \in \mathbb{F}$ such that $\alpha + \beta = -a$ and $\alpha\beta = b$ and $\alpha \neq \beta$. If there exist a positive constant K, for every $\epsilon > 0$, $\phi(n)$ and $x(n)$ be a positive sequences satisfies the inequality*

$$|x(n+2) + a\ x(n+1) + b\ x(n)| \leq \epsilon\ \phi(n), \tag{4.1}$$

where $\phi(n)$ is bounded for each value of n with (1.3) *then there is a solution $\{y(n)\}$ of* (1.1) *with $y(0) = 0$ and $y(1) = 1$ such that $|x(n) - y(n)| \leq K\epsilon\ \phi(n)$.*

Proof. Let us define a function $h : (0, \infty) \to \mathbb{F}$ such that

$$h(n) = x(n+2) + a\ x(n+1) + b\ x(n),$$

for all n. Also, in view of (4.1), we have $|h(n)| \leq \epsilon\ \phi(n)$. Now, taking Z-Transform to $h(n)$, thus

$$Z[h(n)] = H(z) = (z^2 + az + b)\ X(z) - z\ (z + a)\ x(0) - z\ x(1). \tag{4.2}$$

In view of (4.2), a sequence $\{x_0(n)\}$ is a solution of (1.1) if and only if

$$(z^2 + az + b)\ X(z) - z\ (z + a)\ x_0(0) - z\ x_0(1) = 0. \tag{4.3}$$

Applying (1.3) in (4.2), we obtain that

$$H(z) = (z^2 + az + b)\ X(z) - z. \tag{4.4}$$

Since, we have $z^2 + az + b = (z - \alpha)\ (z - \beta)$, then (4.4) gives that

$$X(z) - \frac{z}{(z-\alpha)\ (z-\beta)} = \frac{H(z)}{(z-\alpha)\ (z-\beta)}. \tag{4.5}$$

Now, let us choose $y(n)$ as $y(n) = \frac{\alpha^n - \beta^n}{\alpha - \beta}\ x(1)$, then applying Z-Transform to $y(n)$,

$$Z[y(n)] = Y(z) = \frac{z\ x(1)}{(z-\alpha)\ (z-\beta)}.$$

$$(z-\alpha)\ (z-\beta)\ Y(z) - z(z+a)\ x(0) - z\ x(1) = 0. \tag{4.6}$$

Since, we have $x(0) = y(0) = 0$ and $x(1) = y(1) = 1$. Now,

$$\begin{aligned} Z[y(n+2) + a\ y(n+1) + b\ y(n)] &= (z^2 + az + b)\ Y(z) - z\ (z+a)\ y(0) - z\ y(1) \\ &= 0. \qquad \text{[form (4.6)]} \end{aligned}$$

Since, Z is one-to-one operator, it holds that $y(n+2)+a\ y(n+1)+b\ y(n)=0$, therefore $y(n)$ is a solution of (1.1). Then we have

$$Z[x(n)]-Z[y(n)]=X(z)-Y(z)=\frac{H(z)}{(z-\alpha)\ (z-\beta)}=Z\left[r(n)*h(n)\right]$$

where, $r(n)=\frac{1}{z}\left\{\frac{\alpha^n-\beta^n}{\alpha-\beta}\right\}$. Since, Z-Transforms is linear and one-to-one, so $x(n)-y(n)=(r(n)*h(n))$. Now,

$$|x(n)-y(n)|=|r(n)*h(n)|=\left|\sum_{n=-\infty}^{\infty} r(n-k)\ h(n)\right|$$
$$\leq \sum_{n=-\infty}^{\infty} |r(n-k)|\ \ |h(n)| \leq K\phi(n)\epsilon.$$

for each n, where $K=\sum_{n=-\infty}^{\infty}|r(n-k)|=\left|\frac{1}{z(\alpha-\beta)}\right|\sum_{n=-\infty}^{\infty}\left|\left\{\alpha^{n-k}-\beta^{n-k}\right\}\right|$ occurs for each value of n. Then by the virtue of Definition 2.3, the linear difference equation (1.1) has the Hyers-Ulam-Rassias stability. □

Now, we illustrate the Theorem 4.1 by the following examples.

Example 4.2. For every $\epsilon>0$, $\phi(n)$ and $x(n)$ be a positive sequences, if there exists a constant $K>0$ such that

$$|x(n+2)+3\ x(n+1)+2\ x(n)| \leq \epsilon\ \phi(n), \tag{4.7}$$

with (1.3) then there is a solution $y(n)$ with $y(0)=0$ and $y(1)=1$ such that $|x(n)-y(n)|\leq K\epsilon\ \phi(n)$. Let $h(n)=x(n+2)+3\ x(n+1)+2\ x(n)$, for all n. Also, in view of (4.7), we have $|h(n)|\leq \epsilon\ \phi(n)$. Now, taking Z-Transform to $h(n)$, then we have

$$Z[h(n)]=H(z)=(z^2+3z+2)\ X(z)-z\ (z+3)\ x(0)-z\ x(1).$$

Since, we have $z^2+3z+3=(z+1)\ (z+2)$, then we have

$$X(z)-\frac{z}{(z+1)\ (z+2)}=\frac{H(z)}{(z+1)\ (z+2)}.$$

Now, let us choose $y(n)=\{(-1)^n-(-2)^n\}\ x(1)$, then applying Z-Transform, we get

$$Z[y(n)]=Y(z)=\frac{z\ x(1)}{(z+1)\ (z+2)}.$$

Hence

$$(z+1)\ (z+2)\ Y(z)-z(z+3)\ x(0)-z\ x(1)=0.$$

Since, we have $x(0) = y(0) = 0$ and $x(1) = y(1) = 1$. Then

$$Z[y(n+2) + 3\ y(n+1) + 2\ y(n)] = 0.$$

Since, Z is one-to-one operator, it holds that $y(n+2)+3\ y(n+1)+2\ y(n) = 0$, therefore $y(n)$ is a solution. Then by using the Theorem 4.1, the difference equation $x(n+2)+3\ x(n+1)+2\ x(n) = 0$ has the Hyers-Ulam-Rassias stability.

Example 4.3. For every $\epsilon > 0$, $\phi(n)$ and $x(n)$ be a positive sequences, if there exists a constant $K > 0$ such that

$$|2\ x(n+2) + 5\ x(n+1) - 3\ x(n)| \leq \phi(n)\epsilon, \tag{4.8}$$

for each value of $t > 0$ with (1.3), then there exists a solution $y(n)$ with $y(0) = 0$ and $y(1) = 1$ such that $|x(n) - y(n)| \leq K\phi(n)\epsilon$.

Let $h(n) = 2\ x(n+2) + 5\ x(n+1) - 3\ x(n)$, for all n. Also, in view of (4.8), we have $|h(n)| \leq \phi(n)\epsilon$. Now, taking Z-Transform to $h(n)$, then we have

$$Z[h(n)] = H(z) = (2z^2 + 5z - 3)\ X(z) - z\ (2z+5)\ x(0) - 2z\ x(1).$$

Since, we have $2z^2 + 5z - 3 = 2(z - 1/2)\ (z+3)$, then we have

$$X(z) - \frac{2z}{2(z-1/2)\ (z+3)} = \frac{H(z)}{2(z-1/2)\ (z+3)}.$$

Thus,

$$X(z) = \frac{H(z) + 2z}{2(z-1/2)\ (z+3)}.$$

Now, let us choose the function $y(n)$ such that $y(n) = \left(\frac{(1/2)^n - (-3)^n}{7/2}\right) x(1)$, then applying Z-Transform to $y(n)$, we get

$$Z[y(n)] = Y(z) = \frac{2z\ x(1)}{2(z-1/2)\ (z+3)}.$$

Hence $2(z-1/2)\ (z+3)\ Y(z) - z(2z+5)\ x(0) - 2z\ x(1) = 0$. Since, we have $x(0) = y(0) = 0$ and $x(1) = y(1) = 1$. Then $Z[2\,y(n+2)+5\,y(n+1)-3\,y(n)] = 0$. Since, Z is one-to-one operator, it holds that $2\ y(n+2) + 5\ y(n+1) - 3\ y(n) = 0$, therefore $y(n)$ is a solution of the difference equation. Then by using the Theorem 4.1, the difference equation $2\ x(n+2) + 5\ x(n+1) - 3\ x(n) = 0$ has the Hyers-Ulam-Rassias stability.

Theorem 4.4. *For every $a, b \in \mathbb{F}$ there is a $\alpha, \beta \in \mathbb{F}$ such that $\alpha + \beta = -a$ and $\alpha\beta = b$ and $\alpha \neq \beta$. If there exist a positive constant K, for every $\epsilon > 0$, $\phi(n)$ and $x(n)$ be a positive sequences satisfies the inequality*

$$|x(n+2) + a\ x(n+1) + b\ x(n) - p(n)| \leq \epsilon\ \phi(n), \tag{4.9}$$

where $\phi(n)$ is bounded for each value of n with (1.3) *then there exists a solution $y(n)$ of* (1.2) *with $y(0) = 0$ and $y(1) = 1$ such that $|x(n) - y(n)| \leq K\epsilon\phi(n)$ with $p(0) = p(1) = 0$.*

Proof. If we define a function $h : (0, \infty) \to \mathbb{F}$ such that

$$h(n) = x(n+2) + a\ x(n+1) + b\ x(n) - p(n).$$

Also, in view of (4.9), we have $|h(n)| \leq \epsilon\ \phi(n)$. Now, taking Z-Transform to $h(n)$, then we have

$$Z[h(n)] = H(z) = (z^2 + az + b)\ X(z) - z\ (z+a)\ x(0) - z\ x(1) - P(z). \tag{4.10}$$

In view of (4.10), a sequence $\{x_0(n)\}$ is a solution of (1.2) if and only if

$$(z^2 + az + b)\ X(z) - z\ (z+a)\ x_0(0) - z\ x_0(1) = P(z). \tag{4.11}$$

Applying (1.3) in (4.10), we obtain that

$$H(z) = (z^2 + az + b)\ X(z) - z - P(z). \tag{4.12}$$

Since, we have $z^2 + az + b = (z - \alpha)\ (z - \beta)$, then (4.12) gives that

$$X(z) - \frac{z}{(z-\alpha)\ (z-\beta)} = \frac{H(z) + P(z)}{(z-\alpha)\ (z-\beta)}. \tag{4.13}$$

Now, we define a function $y(n)$ such that

$$y(n) = \frac{\alpha^n - \beta^n}{\alpha - \beta}\ x(1) + [r(n) * p(n)],$$

where $r(n) = \frac{1}{z}\left\{\frac{\alpha^n - \beta^n}{\alpha - \beta}\right\}$. Now, applying Z-Transform to $y(n)$, we get

$$Z[y(n)] = Y(z) = \frac{z\ x(1)}{(z-\alpha)\ (z-\beta)} + \frac{P(z)}{(z-\alpha)\ (z-\beta)}.$$

Hence, we have,

$$(z-\alpha)\ (z-\beta)\ Y(z) - z(z+a)\ x(0) - z\ x(1) = P(z) \tag{4.14}$$

Since, we have $x(0) = y(0) = 0$ and $x(1) = y(1) = 1$. Now,

$$\begin{aligned} Z[y(n+2) + a\ y(n+1) + b\ y(n)] &= Z[y(n+2)] + a\ Z[y(n+1)] + b\ Z[y(n)] \\ &= (z^2 + az + b)Y(z) - z(z+a)y(0) - zy(1) \\ Z[y(n+2) + a\ y(n+1) + b\ y(n)] &= P(z) = Z[p(n)]. \qquad \text{[form (4.14)]} \end{aligned}$$

Since, Z is one-to-one operator, it holds that $y(n+2) + a\ y(n+1) + b\ y(n) = p(n)$, therefore $y(n)$ is a solution of (1.2). Then we have

$$Z[x(n)] - Z[y(n)] = X(z) - Y(z) = \frac{H(z)}{(z-\alpha)\ (z-\beta)} = Z\left[r(n) * h(n)\right],$$

Since, Z-Transforms is linear and one-to-one, thus $x(n) - y(n) = (r(n) * h(n))$. Now,

$$|x(n) - y(n)| = |r(n) * h(n)| = \left| \sum_{n=-\infty}^{\infty} r(n-k)\ h(n) \right|$$
$$\leq \sum_{n=-\infty}^{\infty} |r(n-k)|\ |h(n)| \leq K\epsilon\ \phi(n).$$

for each n, where $K = \sum_{n=-\infty}^{\infty} |r(n-k)| = \left|\frac{1}{z(\alpha-\beta)}\right| \sum_{n=-\infty}^{\infty} \left|\left\{\alpha^{n-k} - \beta^{n-k}\right\}\right|$ occurs for each value of n. Then by the virtue of Definition 2.4, the linear difference equation (1.2) has the Hyers-Ulam-Rassias stability. □

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