Numerical blow-up time and growth rate of a reaction-diffusion equation

Maan A. Rasheed*
Mustansiriyah University
Baghdad
Iraq
maan.rasheed.edbs@uomustansiriyah.edu.iq
maanabdulkadhim@gmail.com

Faez N. Ghaffoori
Mustansiriyah University
Baghdad
Iraq

Abstract. This paper is concerned with the numerical blow-up solutions of a homogeneous Dirichlet problem for a semilinear heat equation with a nonlinear reaction term, defined in one dimensional-space. Namely, we compute the approximate blow-up time and the blow-up growth rate constant for a numerical experiment of this problem by using a finite difference method. The numerical results confirm and support the known theoretical blow-up results.

Keywords: blow-up solution, semilinear heat equation, growth rate, finite difference.

1. Introduction

We consider the initial-boundary problem which takes the following form:

$$(1)\qquad \left\{ \begin{array}{ll} u_t = u_{xx} + \lambda u^p e^{qu}, & (x,t) \in (-1,1) \times (0,T)\,, \\ u\,(x,t) = 0, & x = \mp 1, \quad t \in (0,T)\,, \\ u\,(x,0) = u_0\,(x)\,, & x \in [-1,1] \end{array} \right\},$$

where $p \geq 1;\ q, \lambda > 0$, $u_0 \in C^2(R)$, nonzero , nonnegative, radially non increasing function, such that $u_0\,(x) = 0\ \ ,\ x = \mp 1$.

The blow-up phenomena in the homogeneous Dirichlet problem of semiliear heat equations, defined in bounded domains, has been studied, theoretically and numerically, with different types of nonlinear reaction term, by many authors, see for instance [1, 2, 3, 4, 5, 6]. The blow-up properties of problem (1) were studied in [4]. It has been shown that, with a large value initial function, the classical solution of problem (1) blows up in a finite time. Moreover, it has been shown that the blow-up can only occur at $x = 0$ and the upper point wise

$*$. Corresponding author

estimate takes the forms:

$$u(x,t) \leq \log A + \frac{2}{\alpha}\log\left(\frac{1}{|x|}\right), x \in (-1,1)/\{0\}, t \in (0,T) \tag{2}$$

where $A > 0, \alpha \in (0,1)$.

For blow-up rate estimate of this problem, it has been shown that there exist $C > 0$, such that:

$$u(x,t) \leq \frac{1}{q}(\log C - \log[q\lambda(T-t)]\ ), \qquad x \in (-1,1)\ ,\ \ t \in (0,T)\,. \tag{3}$$

Over the last decades, finite difference methods have been used to solve many parabolic problems, see for instance [7, 8, 9, 10]. In this paper, we aim to estimate blow-up time and blow-up growth rate constant, C, for some examples of problem (1), using explicit Euler finite difference method.

2. The discrete problem

In this section, we derive the explicit Euler finite difference formula for problem (1).

To derive the discrete problem of (1), we set $h = 2/J$, for J a positive integer, and define the grids $x_0 = -1$, $x_J = 1$, and $x_j = x_{j-1} + h$ for $j = 1, 2, \ldots, J-1$. Also, we introduce the time step k > 0 and the discrete time levels $t_0 = 0$, $t_{n+1} = t_n + k =$ nk, for $n = 0, 1, 2, \ldots$ We shall denote by U_j^n the approximate value of $u(x_j, t_n)$, for $1 \leq j \leq J-1$, and $n > 0$, obtained by numerical methods.

We approximate $u_{xx}(x_j, t_n)$ by the standard second order finite difference operator, while $u_t(x_j, t_n)$ is approximated by the forward finite difference operator. Thus, the discrete equation of the semilinear equation in (1) becomes:

$$\frac{U_j^{n+1} - U_j^n}{k} = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2} + f(U_j^n), \text{ for } 1 \leq j \leq J-1, n > 0,$$

where $f(U_j^n) = \lambda(U_j^n)^p e^{qU_j^n}$ i.e. the discrete problem of problem (1), becomes:

$$\left\{ \begin{array}{l} U_j^{n+1} = (1-2r)U_j^n + r(U_{j+1}^n + U_{j-1}^n) + kf(U_j^n), \\ \qquad\qquad\qquad \text{for } 1 \leq j \leq J-1, n > 0, \\ U_0^n = U_J^n = 0, \\ U_j^0 = u(x_j, 0) = u_0(x_j), \quad \text{for } 0 \leq\ j\ \leq\ J \end{array} \right\} \tag{4}$$

where $r = k/h^2$.

It is well known that, $2r \leq\ 1$, is the stability condition of the explicit Euler method for the heat equation. Therefore, h and k must be chosen such that this condition is satisfied. Moreover, the rate of convergence for this method is $\mathrm{O}\left(k\ +\ h^2\right)$.

The discrete formula (4) has been used by some authors to compute the numerical solution of the semilinear heat equation in one dimensional space, where the nonlinear term is of exponential or power type functions, for instance, see [9, 10].

3. Convergence of numerical blow-up solution and time

The next theorem summaries some results, those have been proved in [10], for more general nonlinear terms f (including power and exponential types), which guarantees that for small h, the solution of the discrete problem (4) convergences to the exact solution of problem (1)

Theorem 3.1. *Let $U^n = [U_0^n,\ U_1^n, \ldots\ldots, U_J^n]$ is the solution of* (4).
If $0 < k \le \frac{h^2}{2}$ then
1- The solution of (4) *is nonnegative, and*

$$U_j^{n+1} \ge U_j^n, \quad U_{J-j}^n = U_j^n \quad for \quad 0 \le\ j\ \le\ J\ , n \ge 0$$

$$2 - ||U^n - u^n|| \le Ch^2, \quad n \ge 0,$$

where $u^n = [u_0(x_j),\ u(x_1, t_n), \ldots\ldots, u(x_J, t_n)]$. i.e. $U^n \to\ u^n$, where $h \to 0$ which means, at each point (x_j, t_n), the numerical solution given by (4) *convergences to the exact solution of problem* (1).

We will see, the solutions of (4) do not exist for all $n \in N$, because they become unbounded for some n.

Definition 3.2 ([10])**.** Let $\{U^n\} = (U_0^n\ ,\ U_1^n\ , \ldots .U_J^n)\ n \ge 0$ is the numerical solution of problem (4), we say that $\{U^n\}$ blows up in the finite time, if there exists $m \in N$ such that $||U^m||_\infty$ is unbounded, where $||U^n||_\infty = \max_{j=0,\ldots,J} |U_j^n|$.
Moreover, $T_J^m = mk$ is called the numerical blow-up time of problem (1).

In fact, the size of spatial grid h and the choice of time steps k play an important role in determining the numerical blow-up time.

The next theorem, which has been proved in [10], studies the convergence of the blow-up time of the discrete problem (4) to the blow-up time of the of problem (1), for more general types of f (including power and exponential types).

Theorem 3.3. *Let $\{U^n\} = (U_0^n\ ,\ U_1^n\ , \ldots .U_J^n)\ n \ge 0$ be the solution of* (4), *such that $0 < k \le \frac{h^2}{2}$.*

1. *There is $m \in N$ such that $\{U^n\}$ blows up in the finite time T_J^m*
2. *Let T be the blow-up time of* (1), *then $T_J^m \to T$, where $J \to \infty$, which means, the numerical blow-up time approaches to the theoretical blow-up time, for h sufficiently small.*

4. Numerical experiment

In this section, we use the Euler discrete formula, to study the numerical solutions of problem (1) with $u_0(x) = 5\left(1 - x^2\right)$.

It is clear that u_0 satisfies all our assumptions in (1), and takes its maximum value at the point $x = 0$, therefore according to the blow-up results of problem (1), which we have proved in [4], the blow-up in equation (1) should only occur at $x = 0$.

We aim to confirm these theoretical results and find the approximate values to the blow-up time. Moreover, we estimate the blow-up growth rate constant depending on the numerical results.

Our aim is to use the discrete equation (4) to compute numerically, the blow-up time and solutions of problem (1), and the computational codes are written in Matlab software.

In fact, the blow-up time will be taken experimentally, at the first time that $||U^n||_\infty \geq 10^6$. For converging, we will choose, $k = h^2/2$, and we will get a symmetric numerical solution which takes its maximum at $x_{J/2} = 0$, with respect to the meshes *J=150, 200, 250, 300* and *350*.

Moreover, we estimate the constant of the blow-up growth rate in the blow-up estimates (3), at each fixed value of J, as follows $C_J = \max_{0 \leq n \leq m-1} C(n)$, where $C(n) = q\lambda\,(T_J^m - t_n)\, exp(qU^n_{J/2})$.

The numerical solutions will be carried out for three cases:

- Case1: $p = q = \lambda = 1$,
- Case 2: $p = 1,\ q = \lambda = 1/2$
- Case 3: $p = 2, q = 1, \lambda = 1/2$.

In Tables 4, 4 and 4, we present the numerical blow-up times and the numerical values of the constant of blow-up growth rate C_J, for Cases 1, 2 and 3 respectively, with respect to the meshes J=150, 200, 250 ,300 and *350*, while in Table 4, we present the iterative errors obtained by using the error form $E_J = |T_J^m - T_{J-50}^m|$, for Cases 1, 2and 3, where $m = m(J)$ refers to the number of iteration, when numerical blow-up occurs.

J	k	m	$T = mk$	C
150	8.8889e-05	18	0.00151	1.4539e+05
200	5.0000e-05	29	0.00140	5.6757e+97
250	3.2000e-05	42	0.00131	3.8499e+12
300	2.2222e-05	58	0.00126	7.6391e+05
350	1.6327e-05	78	0.00125	9.0910e+04

Table 1: Case 1.

J	k	m	$T = mk$	C
150	8.8889e-05	837	0.07431	3.1238e+02
200	5.0000e-05	1482	0.07405	1.1749e+17
250	3.2000e-05	2312	0.07395	4.1404e+124
300	2.2222e-05	3326	0.07388	5.9799e+104
350	1.6327e-05	4524	0.07384	1.7548e+12

Table 2: Case 2.

J	k	m	$T = mk$	C
150	8.8889e-05	9	7.11111e-04	1.7108e+05
200	5.0000e-05	13	6.00000e-04	2.1918e+15
250	3.2000e-05	18	5.44000e-04	4.7385e+115
300	2.2222e-05	24	5.11111e-04	2.7181e+02
350	1.6327e-05	30	4.93469e-04	7.7754e+09

Table 3: Case 3.

J	E_J, Case 1	E_J, Case 2	E_J, Case 3
200	0.00011	0.00026	1.11111e-04
250	0.00009	0.00010	5.59999e-05
300	0.00005	0.00007	e-053.28890
350	0.00001	0.00004	e-051.76420

Table 4: Errors bounds.

The next figures show the evolutions in time, of the numerical blow-up solutions of problem (1), for cases 1, 2 and 3, with respect to $J = 350$ and $0 \leq n \leq m$.

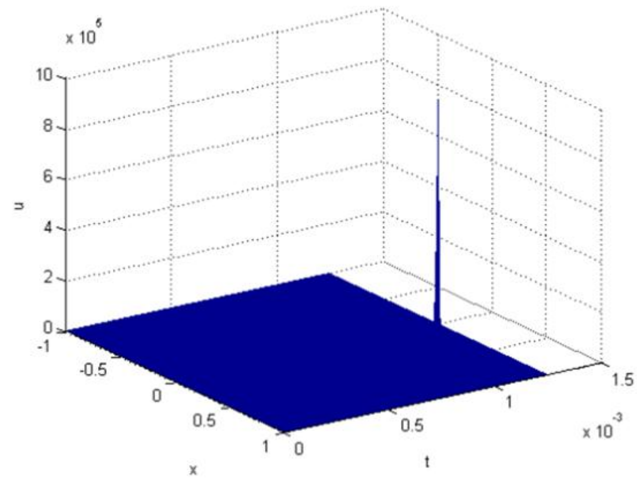

Figure 1: Case 1

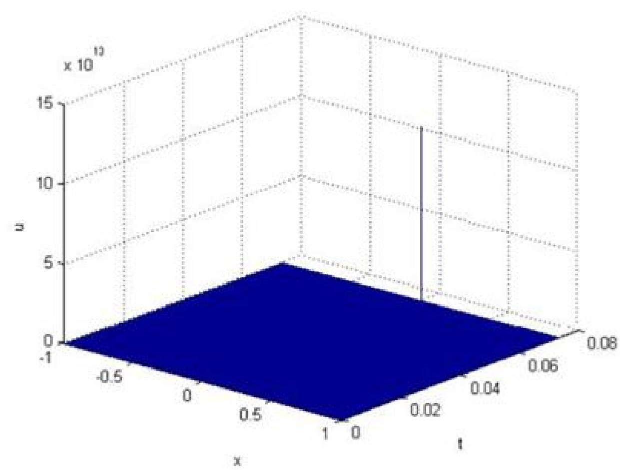

Figure 2: Case 2

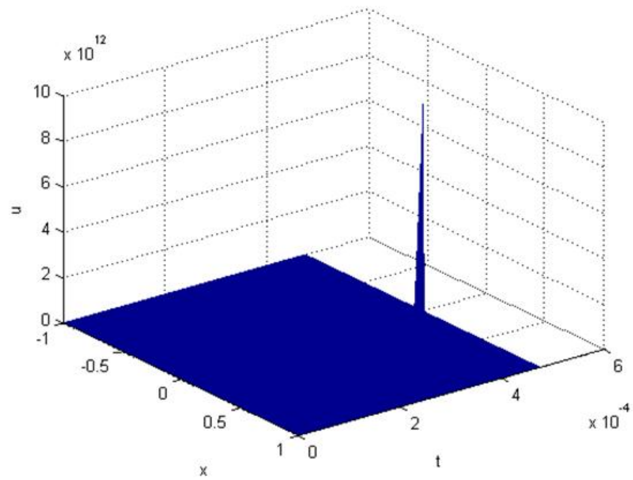

Figure 3: Case 3

From our numerical results Tables 4–4 and Figures 1-3, we can point out the following observations:

1. The numerical solution blows up only at the centre, $x = 0$, and that confirms the theoretical results proved in [4].

2. Decreasing the space-step and time-step, leads to decreasing the numerical blow-up time, which means the numerical blow-up occurs faster, as we refine the grid points.

3. The error bound, E_J , decreases as J increases, which indicates that: the numerical solutions converge as the space-step and time-step tend to zero.

4. With large values to the one or more of p, λ, or q, the solutions divergences radially, while decreasing at least one of them, leads to increasing the numerical blow-up time. And that support the know blow-up properties for problem (1).

5. The numerical constant C_J plays an important role in estimating the growth rate of the corresponding numerical blow-up solution and deriving a discrete formula to the blow-up rate estimate (3) as follows:

$$U_j^n \leq \frac{1}{q} \left(\log C_J \ - \log \left[q\lambda \left(T_J^m - t_n \right) \right] \ \right),$$

$$\text{for} \ \ 0 \leq n \leq m-1, i = 1, 2, \dots J-1$$

5. Conclusion

In this paper, the numerical blow-up solutions of problem (1) are studied. Namely, we compute the approximate blow-up time and the blow-up growth rate constant for a numerical experiment of this problem by using explicit Euler difference method. The numerical results obtained by the proposed methods are analyzed, simulated and presented in the form of tables and figures. We have observed that the numerical result confirms and supports the known theoretical blow-up results. Moreover, the proposed method is successfully implemented with good efficiency and high order of convergence.

Acknowledgments

The authors would like to thank Mustansiriyah University (www.uomustansiriyah.edu.iq) Baghdad-Iraq for its support in the present work.

References

[1] S. Kaplan, *On the growth of solutions of quasilinear parabolic equations,* Comm. Pure Appl. Math., 16 (1963), 327-330

[2] A. Friedman, B. McLeod, *Blow-up of positive solutions of semilinear heat equations, Indiana Univ. Math. J.* , 34 (1985), 425-447.

[3] P. Quittner, Ph. Souplet, *Superlinear parabolic problems. Blow-up, global existence and steady states*, Birkhuser Advanced Texts, Birkhuser, Basel, 2007.

[4] M.A. Rasheed, *On blow-up solutions of parabolic problems*, Ph.D. thesis, University of Sussex, UK, 2012.

[5] M.A. Rasheed, L.J. Barghooth, *Blow-up set and upper rate estimate for a semilinear heat equation,* Journal of Physics: Conf. Series 1294 (2019) 032013.

[6] M.A. Rasheed, H.A.S. Al-Dujaly, T.J. Aldhlki, *Blow-up rate estimates for a system of reaction-diffusion equations with gradient terms,* International Journal of Mathematics and Mathematical Sciences, 2019, 1-7.

[7] M.A. Rasheed, A.T. Balasim, A.F. Jameel, *Some results for the vorticity transport equation by using A.D.I scheme,* AIP Conference Proceedings, 2138, 030031 (2019).

[8] M.A. Rasheed, S. Laverty, B. Bannish, *Numerical solutions of a linear age-structured population model*, AIP Conference Proceedings, 2096, 020002 (2019).

[9] L.M. Abia, J.C. Lopez-Marcos and J. Martinez, *Blow-up for semidiscretizations of reaction-diffusion equations*, Appl. Numer. Math., 20 (1996), 145-156.

[10] L.M. Abia, J.C. Lopez-Marcos, J. Martinez, *The Euler method in the numerical integration of reaction-diffusion problems with blow-up*, Applied Numerical Maths., 38 (2001), 287-313.

Accepted: 18.12.2019