Laplace transform of the product of two functions

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Abstract. In this paper, we prove many propositions of Laplace transform. In particular, we prove that the Laplace transform of product of two functions, under certain conditions, satisfies

$$\mathcal{L}_s\Big(f(t)h(t)\Big) = \int_s^\infty (\mathcal{L}^{-1}h)(\xi - s)(\mathcal{L}f)(\xi)d\xi = \int_0^\infty (\mathcal{L}^{-1}h)(\xi)(\mathcal{L}f)(\xi + s)d\xi.$$

Then we find interesting relation between the Laplace of the product of two functions and their convolution. Also, using these results, we prove a generalized result that the Laplace transform of the convolution of two function is the product of their Laplace transforms. Finally, we find particular solutions of integral, differential and difference equations as application for these results.

Keywords: Laplace transform, Laplace transform of the product of two functions, integro-differential equations, integro-difference equations.

1. Introduction

Suppose that f is a real-valued function of the (time) variable $t > 0$ and s is a real variable. We define the Laplace transform of $f(t)$ as

$$\mathcal{L}_s\left(f(t)\right) = (\mathcal{L}f)(s) = F(s) = \int_0^\infty e^{-st} f(t)dt. \tag{1.1}$$

One of the disappointments of the Laplace transform is that the Laplace transform of the product of two functions is not the product of their Laplace transforms. In fact, the Laplace transform of the convolution of two functions is the product of their Laplace transforms.

$$\mathcal{L}_s\left((f * g)(t)\right) = \mathcal{L}_s\left(\int_0^t f(u)g(t-u)du\right) = \mathcal{L}_s\left(f(t)\right)\mathcal{L}_s\left(g(t)\right).$$

In this paper we answer the following question:

In [6], it is proved that the Laplace transform of the product of $f(t)$ and $\frac{1}{t}$ is

$$\mathcal{L}_s\left(\frac{f(t)}{t}\right) = \int_s^\infty F(\xi)d\xi.$$

Now, what are the Laplace transforms of the product of $f(t)$ and $\frac{1}{t^r}$ $(r>0)$? of the product of $f(t)$ and $\frac{1}{t-a}$? of the product of $f(t)$ and $\frac{1}{t^2+a^2}$? or even the the product of $f(t)$ and $\frac{1}{t^2(t+3)(t^2+1)}$?

2. Main results

Theorem 2.1. *Assume that $\mathcal{L}(f(t)) = F(s)$ and $\mathcal{L}(g(t)) = G(s)$. If $\int_0^\infty\int_0^\infty e^{-(s+\xi)t}g(\xi)f(t)dtd\xi$ converges absolutely for $s>\alpha$, then $\mathcal{L}_s(f(t)G(t))$ is given as*

$$\begin{aligned}\mathcal{L}_s\Big(f(t)G(t)\Big) &= \mathcal{L}_s\Big(f(t)\mathcal{L}\left(g(t)\right)\Big)\\ &= \int_0^\infty g(\xi)F(\xi+s)d\xi = \int_s^\infty g(\xi-s)F(\xi)d\xi \text{ for } s>\alpha. \end{aligned} \tag{2.1}$$

Proof. By Fubini's Theorem (see [5]), since $\int_0^\infty\int_0^\infty e^{-(s+\xi)t}g(\xi)f(t)dtd\xi$ converges absolutely for $s>\alpha$ then

$$\int_0^\infty\int_0^\infty e^{-(s+\xi)t}g(\xi)f(t)dtd\xi = \int_0^\infty\int_0^\infty e^{-(s+\xi)t}g(\xi)f(t)d\xi dt.$$

Therefore, for $s>\alpha$

$$\begin{aligned}\mathcal{L}_s\Big(f(t)G(t)\Big) &= \int_0^\infty e^{-st}f(t)\int_0^\infty e^{-\xi t}g(\xi)d\xi dt\\ &= \int_0^\infty g(\xi)\int_0^\infty e^{-(s+\xi)t}f(t)dtd\xi\\ &= \int_0^\infty g(\xi)F(s+\xi)d\xi\\ &= \int_s^\infty g(\xi-s)F(\xi)d\xi.\end{aligned}$$

□

This theorem can be read as: if If $\int_0^\infty\int_0^\infty e^{-(s+\xi)t}(\mathcal{L}^{-1}h)(\xi)f(t)dtd\xi$ converges absolutely for $s>\alpha$. Then

$$\begin{aligned}\mathcal{L}_s\Big(f(t)h(t)\Big) &= \int_0^\infty (\mathcal{L}^{-1}h)(\xi)(\mathcal{L}f)(\xi+s)d\xi\\ &= \int_s^\infty (\mathcal{L}^{-1}h)(\xi-s)(\mathcal{L}f)(\xi)d\xi \text{ for } s>\alpha.\end{aligned} \tag{2.2}$$

Example 2.2. Using the fact $\mathcal{L}\left(e^{-at}\right) = \frac{1}{s+a}$ we get

$$\mathcal{L}_s\left(\frac{f(t)}{t+a}\right) = \mathcal{L}\left(f(t)\mathcal{L}(e^{-at})\right) = e^{as}\int_s^\infty e^{-a\xi}F(\xi)d\xi.$$

Example 2.3.

$$\begin{aligned}\mathcal{L}_s\Big(\frac{1-\cos(t)}{t^2}\Big) &= \int_s^\infty (u-s)(\frac{1}{u}-\frac{u}{u^2+1})du \\ &= \int_s^\infty \frac{du}{u^2+1} - s\int_s^\infty (\frac{1}{u}-\frac{u}{u^2+1})du \\ &= \arctan(\frac{1}{s}) + s\ln(\frac{s}{\sqrt{s^2+1}}).\end{aligned}$$

Example 2.4. For $s > 1$

$$\begin{aligned}\mathcal{L}_s\left(\frac{e^t-t-1}{t^2}\right) &= \int_s^\infty (\xi-s)\Big(\frac{1}{\xi-1}-\frac{1}{\xi^2}-\frac{1}{\xi}\Big)d\xi \\ &= (s-1)\ln(\frac{s-1}{s}) - 1.\end{aligned}$$

Using this result, we will solve equations contain mixture of differential, difference and integral operators

Example 2.5. To solve the integral equation that

$$\int_x^\infty y(u-x)\Big(\frac{2}{u^3}+\frac{3}{u^2}+\frac{2}{u}\Big)du = \frac{7}{x}-\frac{3}{x^2}, x > 0. \tag{2.3}$$

The left side can be written as $\mathcal{L}_x(Y(t)(t^2+3t+2))$, where $Y(t) = \mathcal{L}_t(y(x))$, and the right side is $\mathcal{L}_x(7-3t)$. Therefore, $Y(t) = \mathcal{L}_t(y(x)) = \frac{7-3t}{t^2+3t+2}$. This means that $y(x) = 10e^{-x} - 13e^{-2x}$ is a particular solution of the equation (2.3). To verify our answer, use interchanging the integration orders and the fact

$$Y(w) = \mathcal{L}_w\Big(10e^{-x}-13e^{-2x}\Big) = \frac{10}{w+1} - \frac{13}{w+2} = \frac{7-3w}{(w+1)(w+2)}$$

to get that

$$\begin{aligned}\int_x^\infty y(u-x)\Big(\frac{2}{u^3}+\frac{3}{u^2}+\frac{2}{u}\Big)du &= \int_x^\infty y(u-x)\int_0^\infty e^{-wu}\Big(w^2+3w+2\Big)dwdu \\ &= \int_0^\infty \Big(w^2+3w+2\Big)\int_x^\infty e^{-wu}y(u-x)dudw \\ &= \int_0^\infty e^{-wx}\Big(w^2+3w+2\Big)\int_0^\infty e^{-wv}y(v)dvdw \\ &= \int_0^\infty e^{-wx}\Big(w^2+3w+2\Big)Y(w)dw \\ &= \int_0^\infty e^{-wx}\Big(7-3w\Big)dw \\ &= \frac{7}{x}-\frac{3}{x^2}.\end{aligned}$$

Example 2.6. In this example, we find a solution for the difference-integral equation

$$y(x+1) + \int_x^\infty (\xi - x)^2 y(\xi) d\xi = \frac{6}{(x+1)^4} + \frac{2}{x}, x > -1.$$

The left side can be written as $\mathcal{L}_x(e^{-t}f(t) + \frac{2f(t)}{t^3})$, where $y(x) = \mathcal{L}_x(f(t))$, and the right side is $\mathcal{L}_x(t^3e^{-t} + 2)$. Therefore, for $t > 0$, we have $f(t) = t^3$. This means $y(x) = \mathcal{L}_x(f(t)) = \frac{6}{x^4}$ is a solution of this equation.

Example 2.7. We wish to find a solution for the integro-differential equation

$$Y''(x) + Y(x+1) + \int_x^\infty \sin(u - x)Y(u)du = e^{-3x}. \tag{2.4}$$

This equation is resulting by taking the Laplace transform of

$$t^2 y(t) + e^{-t}y(t) + \frac{y(t)}{t^2+1} = \delta(t-3),$$

where $Y(x) = \mathcal{L}_x(y(t))$. Hence,

$$y(t) = \frac{(t^2+1)\delta(t-3)}{(t^2+1)(t^2+e^{-t})+1}.$$

Therefore,

$$Y(x) = \mathcal{L}_x(y(t)) = \frac{10e^{-3x}}{10(9+e^{-3})+1}$$

is a particular solution for the equation (2.4).

The following result gives the Laplace transform for "the weighted convolution"

Corollary 2.8. *The Laplace transform of the product of $h(t)$ and the convolution of $f * g$ is given as*

$$\mathcal{L}_s(h(t)(f*g)(t)) = \int_s^\infty (\mathcal{L}^{-1}h)(\xi - s)F(\xi)G(\xi)d\xi.$$

In particular, if $h(t) = 1$ then we get

$$\mathcal{L}_s((f*g)(t)) = \int_s^\infty \delta(\xi - s)F(\xi)G(\xi)d\xi = F(s)G(s),$$

which is the well-known result that the Laplace transform of the convolution of two function is the product of their Laplace transforms.

Example 2.9. Use $f(t) = g(t) = \sin t$ and $h(t) = \frac{2}{t^3}$ to get that

$$\begin{aligned}\mathcal{L}_s\Big(\frac{\sin t - t\cos t}{t^3}\Big) &= \mathcal{L}_s\Big(\frac{2(f*g)(t)}{t^3}\Big)\\ &= \int_s^\infty \frac{(\xi - s)^2}{(\xi^2+1)^2}d\xi = \frac{1}{4}(\pi s^2 - 2(s^2+1)\arctan(s) - 2s + \pi).\end{aligned}$$

One of the application of this result is the following interesting relations between convolution and the Laplace transform of the product of two functions

Corollary 2.10.

$$\int_0^\infty h(u)\mathcal{L}_u(f(t)(\mathcal{L}g)(t))du = \int_0^\infty (\mathcal{L}f)(\xi)(g*h)(\xi)d\xi.$$

In particular,

$$\mathcal{L}_u(f(t)(\mathcal{L}g)(t))(a) = \int_0^\infty (\mathcal{L}f)(\xi)(e^{-at}*h(t))(\xi)d\xi.$$

Proof.

$$\begin{aligned}\int_0^\infty h(u)\mathcal{L}_u(f(t)(\mathcal{L}g)(t))du &= \int_0^\infty h(u)\Big(\int_u^\infty g(\xi-u)F(\xi)d\xi\Big)du\\ &= \int_0^\infty h(u)\Big(\int_u^\infty g(\xi-u)F(\xi)d\xi\Big)du\\ &= \int_0^\infty F(\xi)\Big(\int_0^\xi g(\xi-u)h(u)du\Big)d\xi\\ &= \int_0^\infty F(\xi)(g*h)(\xi)d\xi.\end{aligned}$$

□

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