Fuzzy bi-ideals in LA-rings

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Abstract. In this paper, we give the characterizations of different classes of LA-ring in terms of fuzzy left (resp. right, bi-, generalized bi-, $(1, 2)$-) ideals.

Keywords: Fuzzy left (resp. right, bi-, generalized bi-, $(1, 2)$-) ideals.

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1. Introduction

In 1972, a generalization of abelian semigroups initiated by Naseerdin et al [6]. In ternary commutative (abelian) law: $abc = cba$, they introduced the braces on the left side of this law and explored a new pseudo associative law, that is $(ab)c = (cb)a$. This law $(ab)c = (cb)a$ is called the left invertive law. A groupoid S is said to be a left almost semigroup (abbreviated as LA-semigroup) if it satisfies the left invertive law: $(ab)c = (cb)a$. An LA-semigroup is a midway structure between an abelian semigroup and a groupoid. Ideals in LA-semigroup have been investigated by [14].

In [4] (resp. [1]), a groupoid S is said to be medial (resp. paramedial) if $(ab)(cd) = (ac)(bd)$ (resp. $(ab)(cd) = (db)(ca)$). In [6], an LA-semigroup is medial, but in general an LA-semigroup needs not to be paramedial. Every LA-semigroup with left identity is paramedial by Stevanovic et al [14] and also satisfies $a(bc) = b(ac)$, $(ab)(cd) = (dc)(ba)$.

Kamran [5], extended the notion of LA-semigroup to the left almost group (LA-group). An LA-semigroup S is said to be a left almost group, if there exists left identity $e \in S$ such that $ea = a$ for all $a \in S$ and for every $a \in S$ there exists $b \in S$ such that $ba = e$.

Rehman et al [16], initiated the concept of left almost ring (abbreviated as LA-ring) of finitely nonzero functions which is a generalization of a commutative semigroup ring. By a left almost ring, we mean a non-empty set R with at least two elements such that $(R, +)$ is an LA-group, $(R, \cdot)$ is an LA-semigroup, both left and right distributive laws hold. For example, from a commutative ring $(R, +, \cdot)$, we can always obtain an LA-ring $(R, \oplus, \cdot)$ by defining for all $a, b \in R$, $a \oplus b = b - a$ and $a \cdot b$ is same as in the ring. Despite the fact that the structure is non-associative and non-commutative, however it possesses properties which usually come across in associative and commutative algebraic structures.

A non-empty subset A of an LA-ring R is called an LA-subring of R if $a - b$ and $ab \in A$ for all $a, b \in A$. A is called a left (resp. right) ideal of R if $(A, +)$ is an LA-group and $RA \subseteq A$ (resp. $AR \subseteq A$). A is called an ideal of R if it is both a left ideal and a right ideal of R.

An LA-subring A of R is called a bi-ideal of R if $(AR)A \subseteq A$. A non-empty subset A of R is called a generalized bi-ideal of R if $(A, +)$ is an LA-group and $(AR)A \subseteq A$. Every bi-ideal of R is a generalized bi-ideal of R. An LA-subring A of R is called a $(1, 2)$-ideal of R if $(AR)A^2 \subseteq A$.

We will initiate the concept of regular (resp. left regular, right regular, $(2, 2)$-regular, left weakly regular, right weakly regular, intra-regular) LA-rings. We will also define the concept of fuzzy left (resp. right, bi-,generalized bi-, $(1, 2)$-) ideals.

We will describe a study of regular (resp. left regular, right regular, $(2, 2)$-regular, left weakly regular, right weakly regular, intra-regular) LA-rings by the properties of fuzzy left (right, bi-, generalized bi-) ideals. In this regard, we will prove that in regular (resp. left weakly regular) LA-rings, the concept of fuzzy (right, two-sided) ideals coincides. We will also show that in right regular (resp. $(2,2)$-regular, right weakly regular, intra-regular) LA-rings, the concept of fuzzy (left, right, two-sided) ideals coincides. Also in left regular LA-rings with left identity, the concept of fuzzy (left, right, two-sided) ideals coincides. We will also characterize left weakly regular LA-rings in terms of fuzzy right (two-sided, bi-, generalize bi-) ideals.

2. Basic definitions and preliminary results

First time concept of fuzzy set introduced by Zadeh in his classical paper [19]. This concept has provided a useful mathematical tool for describing the behavior of systems that are too complex to admit precise mathematical analysis by classical methods and tools. Extensive applications of fuzzy set theory have been found in various fields such as artificial intelligence, computer science, management science, expert systems, finite state machines, Languages, robotics, coding theory and others.

It soon invoked a natural question concerning a possible connection between fuzzy sets and algebraic systems like (set, group, semigroup, ring, near-ring, semiring, measure) theory, groupoids, real analysis, topology, differential equations and so forth. Rosenfeld [**?**], was the first, who introduced the concept of fuzzy set in a group. The study of fuzzy set in semigroups was established by Kuroki [9]. He studied fuzzy ideals and fuzzy (interior, quasi-, bi-, generalized bi-, semiprime) ideals of semigroups. Liu [11], introduced the concept of fuzzy subrings and fuzzy ideals of a ring. Many authors have explored the theory of fuzzy rings (for example [3, 10, 12, 13, 18]). Gupta et al [3], gave the idea of intrinsic product of fuzzy subsets of a ring. Kuroki [10], characterized regular (intra-regular, both regular and intra-regular) rings in terms of fuzzy left (resp. right, quasi, bi-) ideals.

By a fuzzy subset μ of an LA-ring R, we mean a function $\mu : R \to [0,1]$ and the complement of μ is denoted by μ', is also a fuzzy subset of R defined by $\mu'(x) = 1 - \mu(x)$ for all $x \in R$.

A fuzzy subset μ of an LA-ring R is a fuzzy LA-subring of R, if $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$ and $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in R$.

Equivalent definition: A fuzzy subset μ of an LA-ring R is a fuzzy LA-subring of R, if $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$, $\mu(-x) \geq \mu(x)$ and $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in R$.

μ is a fuzzy left (resp. right) ideal of R, if $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$ and $\mu(xy) \geq \mu(y)$ (resp. $\mu(xy) \geq \mu(x)$) for all $x, y \in R$.

μ is a fuzzy ideal of R, if it is both a fuzzy left and a fuzzy right ideal of R. Every fuzzy ideal (whether left, right, two-sided) of R is a fuzzy LA-subring of R,here we are giving the example which show that the converse is not true.

Example 1. Let $R=\{0,1,2,3,4,5,6,7\}$. Define $+$ and $\cdot$ in R as follows :

$+$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	2	0	3	1	6	4	7	5
2	1	3	0	2	5	7	4	6
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	6	4	7	5	2	0	3	1
6	5	7	4	6	1	3	0	2
7	7	6	5	4	3	2	1	0

and

$\cdot$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	4	4	0	0	4	4	0
2	0	4	4	0	0	4	4	0
3	0	0	0	0	0	0	0	0
4	0	3	3	0	0	3	3	0
5	0	7	7	0	0	7	7	0
6	0	7	7	0	0	7	7	0
7	0	3	3	0	0	3	3	0

Then R is an LA-ring and μ be a fuzzy subset of R. We define $\mu(0)=\mu(4)=0.7$, $\mu(1)=\mu(2)=\mu(3)=\mu(5)=\mu(6)=\mu(7)=0$. Then μ is a fuzzy LA-subring of R, but not fuzzy right ideal of R as

$$\begin{aligned} \mu(41) &= \mu_A(3)=0. \\ \mu(4) &= 0.7. \\ &\Rightarrow \mu(41)\ngeq \mu(4). \end{aligned}$$

A fuzzy LA-subring μ of an LA-ring R is a fuzzy bi-ideal of R if $\mu((xa)y)\geq \min\{\mu(x),\mu(y)\}$ for all $x,y,a\in R$.

A fuzzy subset μ of an LA-ring R is a fuzzy generalized bi-ideal of R if $\mu(x-y)\geq \min\{\mu(x),\mu(y)\}$ and $\mu((xa)y)\geq \min\{\mu(x),\mu(y)\}$ for all $x,y,a\in R$. Every fuzzy bi-ideal of R is a fuzzy generalized bi-ideal of R.

A fuzzy LA-subring μ of an LA-ring R is called a fuzzy $(1,2)$-ideal of R if $\mu((xa)(yz))\geq \min\{\mu(x),\mu(y),\mu(z)\}$ for all $x,y,z,a\in R$.

Let A be a non-empty subset of an LA-ring R. Then the characteristic function of A is denoted by χ_A and defined as

$$\chi_A(x)=\begin{cases}1, & \text{if } x\in A\\ 0, & \text{if } x\notin A\end{cases}$$

The product of two fuzzy subsets μ and ν is denoted by $\mu\circ\nu$ and defined as:

$$(\mu\circ\nu)(x)=\begin{cases}\bigvee\limits_{x=\sum\limits_{i=1}^{n}a_ib_i}\{\bigwedge\limits_{i=1}^{n}\{\mu(a_i)\wedge\nu(b_i)\}\}, & \text{if } x=\sum\limits_{i=1}^{n}a_ib_i,\ a_i,b_i\in R\\ 0, & \text{if } x\neq\sum\limits_{i=1}^{n}a_ib_i.\end{cases}$$

Now we are giving the some fundamental properties, which will be very helpful in the next section.

Theorem 1. *Let A and B be two non-empty subsets of an LA-ring R. Then the following conditions hold.*

(1) *If $A \subseteq B$ then $\chi_A \subseteq \chi_B$.*
(2) $\chi_A \circ \chi_B = \chi_{AB}$.
(3) $\chi_A \cup \chi_B = \chi_{A\cup B}$.
(4) $\chi_A \cap \chi_B = \chi_{A\cap B}$.

Proof. Straight forward. □

Example 2. Let $R = \{a, b, c, d\}$. Define $+$ and $\cdot$ in R as follows :

$+$	a	b	c	d
a	a	b	c	d
b	d	a	b	c
c	c	d	a	b
d	b	c	d	a

and

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	a	b
c	a	a	c	c
d	a	b	c	d

Then R is an LA-ring and μ be a fuzzy subset of R. We define $\mu(a) = \mu(c) = 0.7$, $\mu(b) = \mu(d) = 0$. Then μ is a fuzzy ideal of R.

Lemma 2.1. *Every fuzzy left (resp. right, two-sided) ideal of an LA-ring R is a fuzzy bi-ideal of R. But the converse is not true in general.*

Proof. Straight forward. □

Example 3. Above μ in example 1 is also a fuzzy bi-ideal of R,but not right ideal of R.

Lemma 2.2. *Every fuzzy bi-ideal of an LA-ring R is a fuzzy $(1, 2)$-ideal of R.*

Proof. Straight forward. □

Remark 1. Every fuzzy left (resp. right, two-sided) ideal of an LA-ring R is a fuzzy $(1, 2)$-ideal of R.

Proposition 2.1. *Let R be an LA-ring having the property $a = a^2$ for every $a \in R$. Then every fuzzy $(1, 2)$-ideal of R is a fuzzy bi-ideal of R.*

Proof. Suppose that μ is a fuzzy $(1, 2)$-ideal of R and $a, x, y \in R$. Thus

$$\mu((xa)y) = \mu((xa)(yy)) \geq \min\{\mu(x), \mu(y), \mu(y)\} = \min\{\mu(x), \mu(y)\}.$$

Therefore μ is a fuzzy bi-ideal of R. □

Lemma 2.3. *Let R be an LA-ring and $\emptyset \neq A \subseteq R$. Then A is an LA-subring of R if and only if the characteristic function χ_A of A is a fuzzy LA-subring of R.*

Proof. Straight forward. □

Lemma 2.4. *Let R be an LA-ring and $\emptyset \neq A \subseteq R$. Then A is a left (resp. right) ideal of R if and only if the characteristic function χ_A of A is a fuzzy left (resp. right) ideal of R.*

Proof. Straight forward. □

Proposition 2.2. *Let R be an LA-ring and $\emptyset \neq A \subseteq R$. Then A is a $(1,2)$-ideal of R if and only if the characteristic function χ_A of A is a fuzzy $(1,2)$-ideal of R.*

Proof. Let A be a $(1,2)$-ideal of R, this implies that A is an LA-subring of R. Then χ_A is a fuzzy LA-subring of R, by the Lemma 2.3. Let $a, x, y, z \in R$. If $x, y, z \in A$, then by definition $\chi_A(x) = 1 = \chi_A(y) = \chi_A(z)$. Since $(xa)(yz) \in A$, A being a $(1,2)$-ideal, so $\chi_A((xa)(yz)) = 1$. Thus $\chi_A((xa)(yz)) \geq \min\{\chi_A(x), \chi_A(y), \chi_A(z)\}$. Similarly, we have $\chi_A((xa)(yz)) \geq \min\{\chi_A(x), \chi_A(y), \chi_A(z)\}$, when $x, y, z \notin A$. Hence the characteristic function χ_A of A is a fuzzy $(1,2)$-ideal of R.

Conversely, assume that the characteristic function χ_A of A is a fuzzy $(1,2)$-ideal of R, this implies that χ_A is a fuzzy LA-subring of R. Then A is an LA-subring of R by the Lemma 2.3. Let $t \in (AR)A^2$, so $t = (xa)(yz)$, where $x, y, z \in A$, $a \in R$. So by definition $\chi_A(x) = 1 = \chi_A(y) = \chi_A(z)$. Since $\chi_A((xa)(yz)) \geq \chi_A(x) \wedge \chi_A(y) \wedge \chi_A(z) = 1$, χ_A being a fuzzy $(1,2)$-ideal of R. Thus $\chi_A((xa)(yz)) = 1$, i.e., $(xa)(yz) \in A$. Hence A is a $(1,2)$-ideal of R. □

Remark 2. Let R be an LA-ring and $\emptyset \neq A \subseteq R$. Then A is a bi-ideal of R if and only if the characteristic function χ_A of A is a fuzzy bi-ideal of R.

Zadeh [19], introduced the concept of level set. Das [2], studied the fuzzy groups, level subgroups and gave the proper definition of a level set such that: let μ be a fuzzy subset of a non-empty set S, for $t \in [0,1]$, the set $\mu_t = \{x \in S \mid \mu(x) \geq t\}$, is called a level subset of the fuzzy subset μ.

Let μ be a fuzzy subset of an LA-ring R, then for all $t \in (0,1]$, we define a set $U(\mu; t) = \{x \in R \mid \mu(x) \geq t\}$, which is called an upper t-level of μ.

Lemma 2.5. *Let μ be a fuzzy subset of an LA-ring R. Then μ is a fuzzy LA-subring of R if and only if upper t-level $U(\mu; t)$ of μ is an LA-subring of R for all $t \in (0,1]$.*

Proof. Straight forward. □

Lemma 2.6. *Let μ be a fuzzy subset of an LA-ring R. Then μ is a fuzzy left (resp. right) ideal of R if and only if upper t-level $U(\mu; t)$ of μ is a left (resp. right) ideal of R for all $t \in (0,1]$.*

Proof. Straight forward. □

Proposition 2.3. *Let μ be a fuzzy subset of an LA-ring R. Then μ is a fuzzy $(1,2)$-ideal of R if and only if upper t-level $U(\mu;t)$ of μ is a $(1,2)$-ideal of R for all $t \in (0,1]$.*

Proof. Suppose that μ is a fuzzy $(1,2)$-ideal of R, this implies that μ is a fuzzy LA-subring of R. Then $U(\mu;t)$ is an LA-subring of R by the Lemma 2.5. Let $x, y, z \in U(\mu;t)$ and $a \in R$, then by definition $\mu(x), \mu(y), \mu(z) \geq t$. Now $\mu((xa)(yz)) \geq \mu(x) \wedge \mu(y) \wedge \mu(z) \geq t$, μ being a fuzzy $(1,2)$-ideal of R, i.e., $(xa)(yz) \in U(\mu;t)$. Therefore $U(\mu;t)$ is a $(1,2)$-ideal of R.

Conversely, assume that $U(\mu;t)$ is a $(1,2)$-ideal of R, this means that $U(\mu;t)$ is an LA-subring of R. Then μ is a fuzzy LA-subring of R by the Lemma 2.5. We have to show that $\mu((xa)(yz)) \geq \mu(x) \wedge \mu(y) \wedge \mu(z)$. We suppose a contradiction $\mu((xa)(yz)) < \mu(x) \vee \mu(y) \vee \mu(z)$. Let $\mu(x) = t = \mu(y) = \mu(z)$, so $\mu(x), \mu(y), \mu(z) \geq t$, i.e., $x, y, z \in U(\mu;t)$. But $\mu((xa)(yz)) < t$, i.e., $(xa)(yz) \notin U(\mu;t)$, which is a contradiction. Therefore $\mu((xa)(yz)) \geq \mu(x) \wedge \mu(y) \wedge \mu(z)$. □

Remark 3. Let μ be a fuzzy subbset of an LA-ring R. Then μ is a fuzzy bi-ideal of R if and only if upper t-level $U(\mu;t)$ of μ is a bi-ideal of R for all $t \in (0,1]$.

3. Characterizations of LA-rings

In this section, we characterize different classes of LA-ring in terms of fuzzy left (right, bi-, generalized bi-) ideals. An LA-ring R is a regular, if for every element $x \in R$, there exists an element $a \in R$ such that $x = (xa)x$. An LA-ring R is an intra-regular, if for every element $x \in R$, there exist elements $a_i, b_i \in R$ such that $x = \sum_{i=1}^{n} (a_i x^2) b_i$.

An LA-ring R is a left (resp. right) regular, if for every element $x \in R$, there exists an element $a \in R$ such that $x = ax^2$ (resp. $x^2 a$). An LA-ring R is completely regular if it is regular, left regular and right regular.

An LA-ring R is a $(2,2)$-regular if for every element $x \in R$, there exists an element $a \in R$ such that $x = (x^2 a)x^2$.

An LA-ring R is a locally associative LA-ring if $(a.a).a = a.(a.a)$ for all $a \in R$.

A ring R is a left (resp. right) weakly regular if $I^2 = I$ for every left (resp. right) ideal I of R, equivalently $x \in RxRx (x \in xRxR)$ for every $x \in R$. An LA-ring R is called a weakly regular if it is both a left weakly regular and a right weakly regular [15].

Now we define this notion in a class of non-associative and non-commutative rings (LA-ring).

An LA-ring R is called a left (resp. right) weakly regular if for every element $x \in R$, there exist elements $a, b \in R$ such that $x = (ax)(bx)$ (resp. $x = (xa)(xb)$). An LA-ring R is called a weakly regular if it is both a left weakly regular and a right weakly regular.

Lemma 3.1. *Every fuzzy right ideal of an LA-ring R with left identity e, is a fuzzy ideal of R.*

Proof. Let μ be a fuzzy right ideal of R and $x, y \in R$. Thus $\mu(xy) = \mu((ex)y) = \mu((yx)e) \geq \mu(yx) \geq \mu(y)$. Hence μ is a fuzzy ideal of R. □

Lemma 3.2. *Every fuzzy right ideal of a regular LA-ring R is a fuzzy ideal of R.*

Proof. Suppose that μ is a fuzzy right ideal of R. Let $x, y \in R$, this implies that there exists $a \in R$, such that $x = (xa)x$. Thus $\mu(xy) = \mu(((xa)x)y) = \mu((yx)(xa)) \geq \mu(yx) \geq \mu(y)$. Therefore μ is a fuzzy ideal of R. □

Proposition 3.1. *Let R be a regular LA-ring having the property $a = a^2$ for every $a \in R$, with left identity e. Then every fuzzy generalized bi-ideal of R is a fuzzy bi-ideal of R.*

Proof. Let μ be a fuzzy generalized bi-ideal of R and $x, y \in R$, this implies that there exists $a \in R$ such that $x = (xa)x$. We have to show that μ is a fuzzy LA-subring of R. Thus

$$\begin{aligned} \mu(xy) &= \mu(((xa)x)y) = \mu(((xa)x^2)y) = \mu(((xa)(xx))y) \\ &= \mu((x((xa)x))y) \geq \min\{\mu(x), \mu(y)\}. \end{aligned}$$

Hence μ is a fuzzy LA-subring of R. □

Lemma 3.3 ([7, Lemma 8])**.** *Let R be an LA-ring with left identity e. Then Ra is the smallest left ideal of R containing a.*

Lemma 3.4 ([7, Lemma 9])**.** *Let R be an LA-ring with left identity e. Then aR is a left ideal of R.*

Proposition 3.2 ([7, Proposition 5])**.** *Let R be an LA-ring with left identity e. Then $aR \cup Ra$ is the smallest right ideal of R containing a.*

Lemma 3.5. *Let R be an LA-ring. Then $\mu \circ \nu \subseteq \mu \cap \nu$ for every fuzzy right ideal μ and for every fuzzy left ideal ν of R.*

Proof. Let μ be a fuzzy right and ν be a fuzzy left ideal of R and $x \in R$. If x cannot be expressible as $x = \sum_{i=1}^{n} a_i b_i$, where $a_i, b_i \in R$ and n is any positive integer, then obviously $\mu \circ \nu \subseteq \mu \cap \nu$, otherwise we have

$$\begin{aligned} (\mu \circ \nu)(x) &= \bigvee_{x=\sum_{i=1}^{n} a_i b_i} \{\bigwedge_{i=1}^{n} \{\mu(a_i) \wedge \nu(b_i)\}\} \\ &\leq \bigvee_{x=\sum_{i=1}^{n} a_i b_i} \{\bigwedge_{i=1}^{n} \{\mu(a_i b_i) \wedge \nu(a_i b_i)\}\} \\ &= \bigvee_{x=\sum_{i=1}^{n} a_i b_i} \{\bigwedge_{i=1}^{n} (\mu \cap \nu)(a_i b_i)\} = (\mu \cap \nu)(x). \\ &\Rightarrow \mu \circ \nu \subseteq \mu \cap \nu. \end{aligned}$$

□

Theorem 2. *Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is a regular.*

(2) *$\mu \cap \nu = \mu \circ \nu$ for every fuzzy right ideal μ and for every fuzzy left ideal ν of R.*

Proof. Suppose that (1) holds. Since $\mu \circ \nu \subseteq \mu \cap \nu$, for every fuzzy right ideal μ and every fuzzy left ideal ν of R by the Lemma 3.5. Let $x \in R$, this implies that there exists an element $a \in R$ such that $x = (xa)x$. Thus

$$\begin{aligned}(\mu \circ \nu)(x) &= \bigvee_{x=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \{\mu(a_i) \wedge \nu(b_i)\}\} \\ &\geq \min\{\mu(xa), \nu(x)\} \geq \min\{\mu(x), \nu(x)\} \\ &= (\mu \wedge \nu)(x) = (\mu \cap \nu)(x). \\ &\Rightarrow \mu \cap \nu \subseteq \mu \circ \nu.\end{aligned}$$

Hence $\mu \cap \nu = \mu \circ \nu$, i.e., (1) $\Rightarrow$ (2). Assume that (2) is true and $a \in R$. Then Ra is a left ideal of R containing a by the Lemma 3.3 and $aR \cup Ra$ is a right ideal of R containing a by the Proposition 3.2. So χ_{Ra} is a fuzzy left ideal and $\chi_{aR\cup Ra}$ is a fuzzy right ideal of R, by the Lemma 2.4. By our assumption $\chi_{aR\cup Ra} \cap \chi_{Ra} = \chi_{aR\cup Ra} \circ \chi_{Ra}$, i.e., $\chi_{(aR\cup Ra)\cap Ra} = \chi_{(aR\cup Ra)Ra}$. Thus $(aR\cup Ra)\cap Ra = (aR\cup Ra)Ra$. Since $a \in (aR\cup Ra)\cap Ra$, i.e., $a \in (aR\cup Ra)Ra$, so $a \in (aR)(Ra) \cup (Ra)(Ra)$. Now $(Ra)(Ra) = ((Re)a)(Ra) = ((ae)R)(Ra) = (aR)(Ra)$. This implies that

$$(aR)(Ra) \cup (Ra)(Ra) = (aR)(Ra) \cup (aR)(Ra) = (aR)(Ra).$$

Thus $a \in (aR)(Ra)$. Then

$$\begin{aligned}a &= (ax)(ya) = ((ya)x)a = (((ey)a)x)a = (((ay)e)x)a \\ &= ((xe)(ay))a = (a((xe)y))a \in (aR)a, \text{ for any } x, y \in R.\end{aligned}$$

This means that $a \in (aR)a$, i.e., a is regular. Hence R is a regular, i.e., (2) $\Rightarrow$ (1). □

Theorem 3. *Let R be a regular locally associative LA-ring having the property $a = a^2$ for every $a \in R$. Then for every fuzzy bi-ideal μ of R, $\mu(a^n) = \mu(a^{2n})$ for all $a \in R$, where n is any positive integer.*

Proof. For $n = 1$. Let $a \in R$, this implies that there exists an element $x \in R$ such that $a = (ax)a$. Now $a = (ax)a = (a^2x)a^2$, because $a = a^2$. Thus

$$\begin{aligned}\mu(a) &= \mu\left((a^2x)a^2\right) \geq \min\{\mu\left(a^2\right), \mu\left(a^2\right)\} = \mu\left(a^2\right) \\ &= \mu(aa) \geq \min\{\mu(a), \mu(a)\} = \mu(a). \\ &\Rightarrow \mu(a) = \mu\left(a^2\right).\end{aligned}$$

Now $a^2 = aa = ((a^2x)a^2)((a^2x)a^2) = (a^4x^2)a^4$, then the result is true for $n = 2$. Suppose that the result is true for $n = k$, i.e., $\mu(a^k) = \mu\left(a^{2k}\right)$. Now $a^{k+1} = a^k a = ((a^{2k}x^k)a^{2k})((a^2x)a^2) = (a^{2(k+1)}x^{k+1})a^{2(k+1)}$. Thus

$$\begin{aligned}\mu\left(a^{k+1}\right) &= \mu\left((a^{2(k+1)}x^{k+1})a^{2(k+1)}\right) \geq \min\{\mu(a^{2(k+1)}), \mu\left(a^{2(k+1)}\right)\} \\ &= \mu\left(a^{2(k+1)}\right) = \mu\left(a^{k+1}a^{k+1}\right) \\ &\geq \min\{\mu\left(a^{k+1}\right), \mu\left(a^{k+1}\right)\} = \mu(a^{k+1}). \\ &\Rightarrow \mu(a^{k+1}) = \mu(a^{2(k+1)}).\end{aligned}$$

Hence by induction method, the result is true for all positive integers. □

Lemma 3.6. *Let R be a $(2,2)$-regular LA-ring. Then every fuzzy left (resp. right) ideal of R is a fuzzy ideal of R.*

Proof. Let μ be a fuzzy right ideal of R and $x, y \in R$, this implies that there exists an element $a \in R$ such that $x = (x^2a)x^2$. Thus $\mu(xy) = \mu(((x^2a)x^2)y) = \mu((yx^2)(x^2a)) \geq \mu(yx^2) \geq \mu(y)$. Hence μ is a fuzzy ideal of R. Similarly, for left ideal. □

Remark 4. The concept of fuzzy (left, right, two-sided) ideals coincides in $(2,2)$-regular LA-rings.

Proposition 3.3. *Every fuzzy generalized bi-ideal of $(2,2)$-regular LA-ring R with left identity e, is a fuzzy bi-ideal of R.*

Proof. Suppose that μ is a fuzzy generalized bi-ideal of R and $x, y \in R$, this means that there exists an element $a \in R$ such that $x = (x^2a)x^2$. We have to show that μ is a fuzzy LA-subring of R. Thus

$$\begin{aligned}\mu(xy) &= \mu(((x^2a)x^2)y) = \mu(((x^2a)(xx))y) \\ &= \mu((x((x^2a)x))y) \geq \min\{\mu(x), \mu(y)\}.\end{aligned}$$

Therefore μ is a fuzzy LA-subring of R. □

Theorem 4. *Let R be a $(2,2)$-regular locally associative LA-ring. Then for every fuzzy bi-ideal μ of R, $\mu(a^n) = \mu(a^{2n})$ for all $a \in R$, where n is any positive integer.*

Proof. Same as Theorem 3. □

Lemma 3.7. *Let R be a right regular LA-ring. Then every fuzzy left (resp. right) ideal of R is a fuzzy ideal of R.*

Proof. Let μ be a fuzzy right ideal of R and $x, y \in R$, this implies that there exists $a \in R$ such that $x = x^2a$. Thus

$$\begin{aligned}\mu(xy) &= \mu((x^2a)y) = \mu(((xx)a)y) = \mu(((ax)x)y) \\ &= \mu((yx)(ax)) \geq \mu(yx) \geq \mu(y).\end{aligned}$$

Hence μ is a fuzzy ideal of R. Similarly, for left ideal. □

Remark 5. The concept of fuzzy (left, right, two-sided) ideals coincides in right regular LA-rings.

Proposition 3.4. *Every fuzzy generalized bi-ideal of a right regular LA-ring R with left identity e, is a fuzzy bi-ideal of R.*

Proof. Suppose that μ is a fuzzy generalized bi-ideal of R and $x, y \in R$, this means that there exists $a \in R$ such that $x = x^2a$. We have to show that μ is a fuzzy LA-subring of R. Thus

$$\begin{aligned}\mu(xy) &= \mu((x^2a)y) = \mu(((xx)(ea))y) = \mu(((ae)(xx))y) \\ &= \mu((x((ae)x))y) \geq \min\{\mu(x), \mu(y)\}.\end{aligned}$$

Therefore μ is a fuzzy LA-subring of R. □

Lemma 3.8. *Let R be a left regular LA-ring with left identity e. Then every fuzzy left (resp. right) ideal of R is a fuzzy ideal of R.*

Proof. Assume that μ is a fuzzy right ideal of R and $x, y \in R$, then there exists an element $a \in R$ such that $x = ax^2$. Thus

$$\begin{aligned}\mu(xy) &= \mu((ax^2)y) = \mu((a(xx))y) = \mu((x(ax))y) \\ &= \mu(y(ax))x) \geq \mu(y(ax)) \geq \mu(y).\end{aligned}$$

So μ is a fuzzy ideal of R. Similarly, for left ideal. □

Remark 6. The concept of fuzzy (left, right, two-sided) ideals coincides in left regular LA-rings with left identity.

Proposition 3.5. *Every fuzzy generalized bi-ideal of a left regular LA-ring R with left identity e, is a fuzzy bi-ideal of R.*

Proof. Let μ be a fuzzy generalized bi-ideal of R and $x, y \in R$, this implies that there exists $a \in R$ such that $x = ax^2$. We have to show that μ is a fuzzy LA-subring of R. Thus $\mu(xy) = \mu((ax^2)y) = \mu((a(xx))y) = \mu((x(ax))y) \geq \min\{\mu(x), \mu(y)\}$. Hence μ is a fuzzy LA-subring of R. □

Theorem 5. *Let R be a regular and right regular locally associative LA-ring. Then for every fuzzy right ideal μ of R, $\mu(a^n) = \mu(a^{3n})$ for all $a \in R$, where n is any positive integer.*

Proof. For $n = 1$. Let $a \in R$, this means that there exists an element $x \in R$ such that $a = (ax)a$ and $a = a^2x$. Now $a = (ax)a = (ax)(a^2x) = a^3x^2$. Thus

$$\begin{aligned} \mu(a) &= \mu(a^3x^2) \geq \mu(a^3) = \mu(aa^2) \geq \min\{\mu(a), \mu(a^2)\} \\ &\geq \min\{\mu(a), \mu(a), \mu(a)\} = \mu(a). \\ &\Rightarrow \mu(a) = \mu(a^3). \end{aligned}$$

Here $a^2 = aa = (a^3x^2)(a^3x^2) = a^6x^4$, then the result is true for $n = 2$. Assume that the result is true for $n = k$, i.e., $\mu(a^k) = \mu(a^{3k})$. Now $a^{k+1} = a^ka = (a^{3k}x^{2k})(a^3x^2) = a^{3(k+1)}x^{2(k+1)}$. Thus

$$\begin{aligned} \mu(a^{k+1}) &= \mu(a^{3(k+1)}x^{2(k+1)}) \geq \mu(a^{3(k+1)}) = \mu(a^{3k+3}) \\ &= \mu(a^{k+1}a^{2k+2}) \geq \min\{\mu\left(a^{k+1}\right), \mu\left(a^{2k+2}\right)\} \\ &\geq \min\{\mu\left(a^{k+1}\right), \mu\left(a^{k+1}\right), \mu\left(a^{k+1}\right)\} = \mu\left(a^{k+1}\right). \\ &\Rightarrow \mu(a^{k+1}) = \mu(a^{3(k+1)}). \end{aligned}$$

Hence by induction method, the result is true for all positive integers. □

Theorem 6. *Let R be a right regular locally associative LA-ring. Then for every fuzzy right ideal μ of R, $\mu(a^n) = \mu(a^{2n})$ for all $a \in R$, where n is any positive integer.*

Proof. For $n = 1$. Let $a \in R$, then there exists an element $x \in R$ such that $a = a^2x$. Thus

$$\begin{aligned} \mu_A(a) &= \mu_A(a^2x) \geq \mu_A(a^2) = \mu_A(aa) \\ &\geq \min\{\mu_A(a), \mu_A(a)\} = \mu_A(a) \Rightarrow \mu(a) = \mu(a^2). \end{aligned}$$

Now $a^2 = aa = (a^2x)(a^2x) = a^4x^2$, then the result is true for $n = 2$. Suppose that the result is true for $n = k$, i.e., $\mu(a^k) = \mu(a^{2k})$. Now $a^{k+1} = a^ka = (a^{2k}x^k)(a^2x) = a^{2(k+1)}x^{(k+1)}$. Thus

$$\begin{aligned} \mu(a^{k+1}) &= \mu(a^{2(k+1)}x^{(k+1)}) \geq \mu(a^{2(k+1)}) \\ &= \mu(a^{2k+2}) = \mu(a^{k+1}a^{k+1}) \\ &\geq \min\{\mu\left(a^{k+1}\right), \mu\left(a^{k+1}\right)\} = \mu\left(a^{k+1}\right) \Rightarrow \mu(a^{k+1}) = \mu(a^{2(k+1)}). \end{aligned}$$

Hence by induction method, the result is true for all positive integers. □

Lemma 3.9. *Let R be a right regular locally associative LA-ring with left identity e. Then for every fuzzy right ideal μ of R, $\mu(ab) = \mu(ba)$ for all $a, b \in R$.*

Proof. Let $a, b \in R$. By using Theorem 6 (for $n = 1$). Now

$$\begin{aligned} \mu(ab) &= \mu((ab)^2) = \mu((ab)(ab)) \\ &= \mu((ba)(ba)) = \mu((ba)^2) = \mu(ba). \end{aligned}$$

□

Remark 7. It is easy to see that, if R is a left regular locally associative LA-ring with left identity e. Then for every fuzzy left ideal μ of R, $\mu(a^n) = \mu(a^{2n})$ for all $a \in R$, where n is any positive integer. And also for every fuzzy left ideal μ of R, $\mu(ab) = \mu(ba)$ for all $a, b \in R$.

Lemma 3.10. *Let R be a right weakly regular LA-ring. Then every fuzzy left (resp. right) ideal of R is a fuzzy ideal of R.*

Proof. Suppose that μ is a fuzzy right ideal of R and $x, y \in R$, this means that there exist $a, b \in R$ such that $x = (xa)(xb)$. Thus

$$\begin{aligned} \mu(xy) &= \mu(((xa)(xb))y) = \mu((((xb)a)x)y) \\ &= \mu((((ab)x)x)y) = \mu((yx)((ab)x)) \\ &= \mu((yx)(nx)) \text{ say } ab = n \\ &\geq \mu(yx) \geq \mu(y). \end{aligned}$$

Therefore μ is a fuzzy ideal of R. Similarly, for left ideal. □

Remark 8. The concept of fuzzy (left, right, two-sided) ideals coincides in right weakly regular LA-rings.

Proposition 3.6. *Every fuzzy generalized bi-ideal of a right weakly regular LA-ring R with left identity e, is a fuzzy bi-ideal of R.*

Proof. Assume that μ is a fuzzy generalized bi-ideal of R and $x, y \in R$, then there exist elements $a, b \in R$ such that $x = (xa)(xb)$. We have to show that μ is a fuzzy LA-subring of R. Thus $\mu(xy) = \mu(((xa)(xb))y) = \mu((x((xa)b))y) \geq \min\{\mu(x), \mu(y)\}$. So μ is a fuzzy LA-subring of R. □

Lemma 3.11. *Let R be a left weakly regular LA-ring. Then every fuzzy right ideal of R is a fuzzy ideal of R.*

Proof. Let μ be a fuzzy right ideal of R and $x, y \in R$, this implies that there exist $a, b \in R$ such that $x = (ax)(bx)$. Thus $\mu(xy) = \mu(((ax)(bx))y) = \mu(y(bx))(ax) \geq \mu(y(bx)) \geq \mu(y)$. Hence μ is a fuzzy ideal of R. □

Lemma 3.12. *Let R be a left weakly regular LA-ring with left identity e. Then every fuzzy left ideal of R is a fuzzy ideal of R.*

Proof. Suppose that μ is a fuzzy left ideal of R and $x, y \in R$, this means that there exist $a, b \in R$ such that $x = (ax)(bx)$. Thus

$$\begin{aligned} \mu(xy) &= \mu(((ax)(bx))y) = \mu(((ab)(xx))y) \\ &= \mu((x((ab)x))y) = \mu((y((ab)x))x) \geq \mu(x). \end{aligned}$$

Therefore μ is a fuzzy ideal of R. □

Proposition 3.7. *Every fuzzy generalized bi-ideal of a left weakly regular LA-ring R with left identity e, is a fuzzy bi-ideal of R.*

Proof. Assume that μ is a fuzzy generalized bi-ideal of R and $x, y \in R$, then there exist elements $a, b \in R$ such that $x = (ax)(bx)$. We have to show that μ is a fuzzy LA-subring of R. Thus

$$\begin{aligned} \mu(xy) &= \mu(((ax)(bx))y) = \mu(((ab)(xx))y) \\ &= \mu((x((ab)x))y) \geq \min\{\mu(x), \mu(y)\}. \end{aligned}$$

So μ is a fuzzy LA-subring of R. □

Remark 9. It is easy to see that, if R is a left (resp. right) weakly regular locally associative LA-ring. Then for every fuzzy left (resp. right) ideal μ of R, $\mu(a^n) = \mu(a^{2n})$ for all $a \in R$, where n is any positive integer.

Theorem 7. *Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) R is a left weakly regular.

(2) $\mu \cap \nu = \mu \circ \nu$ for every fuzzy right ideal μ and for every fuzzy left ideal ν of R.

Proof. Suppose that (1) holds. Since $\mu \circ \nu \subseteq \mu \cap \nu$, for every fuzzy right ideal μ and every fuzzy left ideal ν of R by the Lemma 3.5. Let $x \in R$, this implies that there exist $a, b \in R$ such that $x = (ax)(bx) = (ab)(xx) = x((ab)x)$. Now

$$\begin{aligned} (\mu \circ \nu)(x) &= \bigvee_{x=\sum_{i=1}^{n} a_i b_i} \{\bigwedge_{i=1}^{n} \{\mu(a_i) \wedge \nu(b_i)\}\} \\ &\geq \mu(x) \wedge \nu((ab)x) \geq \mu(x) \wedge \nu(x) = (\mu \cap \nu)(x). \\ &\Rightarrow \mu \cap \nu \subseteq \mu \circ \nu. \end{aligned}$$

Hence $\mu \cap \nu = \mu \circ \nu$, i.e., $(1) \Rightarrow (2)$. Assume that (2) is true and $a \in R$. Then Ra is a left ideal of R containing a by the Lemma 3.3 and $aR \cup Ra$ is a right ideal of R containing a by the Proposition 3.2. So χ_{Ra} is a fuzzy left ideal and $\chi_{aR\cup Ra}$ is a fuzzy right ideal of R, by the Lemma 2.4. Then by our assumption $\chi_{aR\cup Ra} \cap \chi_{Ra} = \chi_{aR\cup Ra} \circ \chi_{Ra}$, i.e., $\chi_{(aR\cup Ra)\cap Ra} = \chi_{(aR\cup Ra)Ra}$ by the Theorem 1. Thus $(aR \cup Ra) \cap Ra = (aR \cup Ra)Ra$. Since $a \in (aR \cup Ra) \cap Ra$, i.e., $a \in (aR \cup Ra)Ra$, so $a \in (aR)(Ra) \cup (Ra)(Ra)$. This implies that $a \in (aR)(Ra)$ or $a \in (Ra)(Ra)$. If $a \in (Ra)(Ra)$, then R is a left weakly regular. If $a \in (aR)(Ra)$, then

$$\begin{aligned} (aR)(Ra) &= ((ea)(RR))(Ra) = ((RR)(ae))(Ra) \\ &= (((ae)R)R)(Ra) = ((aR)R)(Ra) \\ &= ((RR)a)(Ra) = (Ra)(Ra). \end{aligned}$$

Hence R is a left weakly regular, i.e., $(2) \Rightarrow (1)$. □

Theorem 8. *Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is a left weakly regular.*

(2) *$\mu \cap \gamma \subseteq \mu \circ \gamma$ for every fuzzy bi-ideal μ and for every fuzzy ideal γ of R.*

(3) *$\nu \cap \gamma \subseteq \nu \circ \gamma$ for every fuzzy generalized bi-ideal ν and for every fuzzy ideal γ of R.*

Proof. Assume that (1) holds. Let ν be a fuzzy generalized bi-ideal and γ be a fuzzy ideal of R. Let $x \in R$, this means that there exist $a, b \in R$ such that $x = (ax)(bx) = (ab)(xx) = x((ab)x)$. Thus

$$\begin{aligned}
(\mu \circ \gamma)(x) &= \bigvee_{x=\sum_{i=1}^{n} a_i b_i} \{\bigwedge_{i=1}^{n} \{\mu(a_i) \wedge \gamma(b_i)\}\} \\
&\geq \mu(x) \wedge \gamma((ab)x) \\
&\geq \mu(x) \wedge \gamma(x) = (\mu \cap \gamma)(x). \\
&\Rightarrow \nu \cap \gamma \subseteq \nu \circ \gamma.
\end{aligned}$$

Therefore (1) $\Rightarrow$ (3). It is clear that (3) $\Rightarrow$ (2). Suppose that (2) holds. Then $\mu \cap \gamma \subseteq \mu \circ \gamma$, where μ is a fuzzy right ideal of R. Since $\mu \circ \gamma \subseteq \mu \cap \gamma$, so $\mu \circ \gamma = \mu \cap \gamma$. Therefore R is a left weakly regular by the Theorem 7, i.e., (2) $\Rightarrow$ (1). □

Theorem 9. *Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is a left weakly regular.*

(2) *$\mu \cap \gamma \cap \delta \subseteq (\mu \circ \gamma) \circ \delta$ for every fuzzy bi-ideal μ, every fuzzy ideal γ and for every fuzzy right ideal δ of R.*

(3) *$\nu \cap \gamma \cap \delta \subseteq (\nu \circ \gamma) \circ \delta$ for every fuzzy generalized bi-ideal ν, every fuzzy ideal γ and for every fuzzy right ideal δ of R.*

Proof. Suppose that (1) holds. Let ν be a fuzzy generalized bi-ideal, γ be a fuzzy ideal and δ be a fuzzy right ideal of R. Let $x \in R$, then there exist elements $a, b \in R$ such that $x = (ax)(bx)$. Now

$$\begin{aligned}
x &= (ax)(bx) = (xb)(xa) \\
xb &= ((ax)(bx))b = ((xx)(ba))b \\
&= (b(ba))(xx) = c(xx) = x(cx) \quad \text{say } c = b(ba)
\end{aligned}$$

Thus

$$\begin{aligned}
((\nu \circ \gamma) \circ \delta)(x) &= \bigvee_{x=\sum_{i=1}^{n} a_i b_i} \{\bigwedge_{i=1}^{n} \{(\nu \circ \gamma)(a_i) \wedge \delta(b_i)\}\} \\
&\geq (\nu \circ \gamma)(xb) \wedge \delta(xa)
\end{aligned}$$

$$\begin{aligned}
&\geq (\nu \circ \gamma)(xb) \wedge \delta(x) \\
&= \bigvee_{xb=\sum_{i=1}^{n} p_i q_i} \{\bigwedge_{i=1}^{n} \{\nu(p_i) \wedge \gamma(q_i)\}\} \wedge \delta(x) \\
&\geq \nu(x) \wedge \gamma(cx) \wedge \delta(x) \\
&\geq \nu(x) \wedge \gamma(x) \wedge \delta(x) = (\mu \cap \gamma \cap \delta)(x). \\
&\Rightarrow \mu \cap \gamma \cap \delta \subseteq (\nu \circ \gamma) \circ \delta.
\end{aligned}$$

Hence (1) $\Rightarrow$ (3). Since (3) $\Rightarrow$ (2), every fuzzy bi-ideal of R is a fuzzy generalized bi-ideal of R. Assume that (2) is true. Then $\mu \cap \gamma \cap R \subseteq (\mu \circ \gamma) \circ R$, where μ is a fuzzy right ideal of R, i.e., $\mu \cap \gamma \subseteq \mu \circ \gamma$. Since $\mu \circ \gamma \subseteq \mu \cap \gamma$, so $\mu \circ \gamma = \mu \cap \gamma$. Hence R is a left weakly regular by the Theorem 7, i.e., (2) $\Rightarrow$ (1). □

Lemma 3.13. *Let R be an intra-regular LA-ring. Then every fuzzy left (resp. right) ideal of R is a fuzzy ideal of R.*

Proof. Suppose that μ is a fuzzy right ideal of R. Let $x, y \in R$, this implies that there exist $a_i, b_i \in R$, such that $x = \sum_{i=1}^{n}(a_i x^2)b_i$. Thus $\mu(xy) = \mu(((a_i x^2)b_i)y) = \mu((yb_i)(a_i x^2)) \geq \mu(yb_i) \geq \mu(y)$. Hence μ is a fuzzy ideal of R. Similarly, for left ideal. □

Proposition 3.8. *Every fuzzy generalized bi-ideal of an intra-regular LA-ring R with left identity e, is a fuzzy bi-ideal of R.*

Proof. Let μ be a fuzzy generalized bi-ideal of R and $x, y \in R$, this implies that there exist $a_i, b_i \in R$ such that $x = \sum_{i=1}^{n}(a_i x^2)b_i$. We have to show that μ is a fuzzy LA-subring of R. Now

$$\begin{aligned}
x &= (a_i x^2)b_i = (a_i x^2)(eb_i) = (a_i e)(x^2 b_i) \\
&= (a_i e)((xx)b_i) = (a_i e)((b_i x)x) = (x(b_i x))(ea_i) \\
&= (x(b_i x))a_i = (a_i(b_i x))x = (a_i(b_i x))(ex) \\
&= (xe)((b_i x)a_i) = (b_i x)((xe)a_i) = (b_i x)((a_i e)x) \\
&= (x(a_i e))(xb_i) = x((x(a_i e))b_i) = xn, \quad \text{say } n = (x(a_i e))b_i
\end{aligned}$$

Thus $\mu(xy) = \mu((xn)y) \geq \min\{\mu(x), \mu(y)\}$. Hence μ is a fuzzy LA-subring of R. □

Theorem 10. *Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is an intra-regular.*

(2) *$\mu \cap \nu \subseteq \mu \circ \nu$ for every fuzzy right ideal ν and for every fuzzy left ideal μ of R.*

Proof. Assume that (1) holds. Let $x \in R$, then there exist elements $a_i, b_i \in R$ such that $x = \sum_{i=1}^{n}(a_i x^2)b_i$. Now

$$\begin{aligned} x &= (a_i x^2)b_i = (a_i(xx))b_i = (x(a_i x))(eb_i) \\ &= (xe)((a_i x)b_i) = (a_i x)((xe)b_i). \end{aligned}$$

Thus

$$\begin{aligned} (\mu \circ \nu)(x) &= \bigvee_{x=\sum_{i=1}^{n} a_i b_i} \{\bigwedge_{i=1}^{n} \mu(a_i) \wedge \nu(b_i)\} \\ &\geq \min\{\mu(a_i x), \nu((xe)b_i)\} \geq \min\{\mu(x), \nu(x)\} \\ &= (\mu \wedge \nu)(x) = (\mu \cap \nu)(x). \\ &\Rightarrow \mu \cap \nu \subseteq \mu \circ \nu. \end{aligned}$$

Hence (1) $\Rightarrow$ (2). Suppose that (2) is true and $a \in R$. Then Ra is a left ideal of R containing a by the Lemma 3.3 and $aR \cup Ra$ is a right ideal of R containing a by the Proposition 3.2. This means that χ_{Ra} is a fuzzy left ideal and $\chi_{aR\cup Ra}$ is a fuzzy right ideal of R, by the Lemma 2.4. By our supposition $\chi_{aR\cup Ra} \cap \chi_{Ra} \subseteq \chi_{Ra} \circ \chi_{aR\cup Ra}$, i.e., $\chi_{(aR\cup Ra)\cap Ra} \subseteq \chi_{(Ra)(aR\cup Ra)}$. Thus $(aR \cup Ra) \cap Ra \subseteq Ra(aR \cup Ra)$. Since $a \in (aR \cup Ra) \cap Ra$, i.e., $a \in Ra(aR \cup Ra) = (Ra)(aR) \cup (Ra)(Ra)$. Now

$$\begin{aligned} (Ra)(aR) &= (Ra)((ea)(RR)) = (Ra)((RR)(ae)) \\ &= (Ra)(((ae)R)R) = (Ra)((aR)R) \\ &= (Ra)((RR)a) = (Ra)(Ra). \end{aligned}$$

This implies that

$$\begin{aligned} (Ra)(aR) \cup (Ra)(Ra) &= (Ra)(Ra) \cup (Ra)(Ra) \\ &= (Ra)(Ra) = ((Ra)a)R \\ &= ((Ra)(ea))R = ((Re)(aa))R \\ &= (Ra^2)R. \end{aligned}$$

Thus $a \in (Ra^2)R$, i.e., a is an intra regular. Therefore R is an intra-regular, i.e., (2) $\Rightarrow$ (1). □

Theorem 11. *Let R be an intra-regular locally associative LA-ring. Then for every fuzzy right ideal μ of R, $\mu(a^n) = \mu(a^{2n})$ for all $a \in R$, where n is any positive integer.*

Proof. For $n = 1$. Let $a \in R$, this implies that there exist elements $x_i, y_i \in R$ such that $a = \sum_{i=1}^{n}(x_i a^2)y_i$. Thus

$$\begin{aligned} \mu(a) &= \mu((x_i a^2)y_i) \geq \mu(x_i a^2) \geq \mu(a^2) \\ &= \mu(aa) \geq \min\{\mu(a), \mu(a)\} = \mu(a). \\ &\Rightarrow \mu(a) = \mu(a^2). \end{aligned}$$

Now $a^2 = aa = ((x_i a^2) y_i)((x_i a^2) y_i) = (x_i^2 a^4) y_i^2$, then the result is true for $n = 2$. Assume that the result is true for $n = k$, i.e., $\mu(a^k) = \mu(a^{2k})$. Now $a^{k+1} = a^k a = ((x_i^k a^{2k}) y_i^k)((x_i a^2) y_i) = (x_i^{k+1} a^{2(k+1)}) y_i^{k+1}$. Thus

$$\begin{aligned} \mu(a^{k+1}) &= \mu((x_i^{k+1} a^{2(k+1)}) y_i^{k+1}) \geq \mu(x_i^{k+1} a^{2(k+1)}) \\ &\geq \mu(a^{2(k+1)}) = \mu(a^{k+1} a^{k+1}) \\ &\geq \min\{\mu\left(a^{(k+1)}\right), \mu\left(a^{(k+1)}\right)\} = \mu\left(a^{(k+1)}\right). \\ &\Rightarrow \mu(a^{k+1}) = \mu(a^{2(k+1)}). \end{aligned}$$

Hence by induction method, the result is true for all positive integers. □

Proposition 3.9. *Let R be an intra-regular locally associative LA-ring with left identity e. Then for every fuzzy right ideal μ of R, $\mu(ab) = \mu(ba)$ for all $a, b \in R$.*

Proof. Same as Lemma 3.9. □

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