Ordered LA-groups and ideals in ordered LA-semigroups

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Abstract. Kehayopulu et. al [7], have established a relation between the ideals in ordered semigroups and ordered groups. In this study, we extend these notions for a class of non-associative and non-commutative algebraic structures.

Keywords: Ordered (LA-semigroups, LA*-semigroups, LA-groups), left (right, interior, quasi-, bi-, generalized bi-) ideals.

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1. Introduction

A generalization of abelian semigroups has been established by Kazim et. al [5]. They introduced the braces on the left side of the ternary commutative law: $abc = cba$ and explored a new pseudo associative law, that is $(ab)c = (cb)a$. This they called the left invertive law. A groupoid S is a left almost semigroup (abbreviated as LA-semigroup) if it satisfies the left invertive law: $(ab)c = (cb)a$. An LA-semigroup is a midway structure between an abelian semigroup and a groupoid. Mushtaq et. al [8], investigated the concept of ideals of LA-semigroups.

A groupoid S is medial (resp. paramedial) if $(ab)(cd) = (ac)(bd)$ (resp. $(ab)(cd) = (db)(ca)$) in [2] (resp. [1]). An LA-semigroup is medial, but in general an LA-semigroup needs not to be paramedial in [5]. Every LA-semigroup with left identity is paramedial and also satisfies $a(bc) = b(ac)$, $(ab)(cd) = (dc)(ba)$ by Protic et. al [9].

The concept of ordered semigroups and ideals have been discussed in [6]. A non-empty subset A of an ordered semigroup S is an interior ideal of S if 1. $SAS \subseteq A$. 2. If $a \in A$ and $b \in S$ such that $b \leq a$ implies $b \in A$ (or $(A] \subseteq A$). A non-empty subset A of S is a quasi-ideal of S if 1. $(AS] \cap (SA] \subseteq A$. 2. If $a \in A$ and $b \in S$ such that $b \leq a$ implies $b \in A$ (or $(A] \subseteq A$). A subsemigroup A of S is a bi-ideal of S if 1. $ASA \subseteq A$. 2. If $a \in A$ and $b \in S$ such that $b \leq a$ implies $b \in A$ (or $(A] \subseteq A$). A non-empty subset A of S is a generalized bi-ideal of S if 1. $ASA \subseteq A$. 2. If $a \in A$ and $b \in S$ such that $b \leq a$ implies $b \in A$ (or $(A] \subseteq A$). Every bi-ideal of S is a generalized bi-ideal of S.

We give the concept of ordered (LA-semigroups, LA*-semigroups, LA-groups) and left (resp. right) simple ordered LA-semigroups. We investigate a relation between the ideals of ordered (LA-semigroups, LA*-semigroup) and (ordered LA-groups, left simple, right simple). Specifically, we show that: 1. If an ordered LA*-semigroup $(S, \cdot, \leq)$ with left identity e, is an ordered LA-group, then $Sa = aS = S$ for all $a \in S$. 2. An ordered LA*-semigroup $(S, \cdot, \leq)$ with left identity e is a left simple and a right simple if and only if it contains no proper quasi-ideal.

2. Ordered LA-groups and ideals in ordered LA-semigroups

An ordered LA-semigroup $(S, \cdot, \leq)$, is a poset (partially ordered set or simply ordered set), at the same time an LA-semigroup such that $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$ for all $a, b, c \in S$.

For $\emptyset \neq A \subseteq S$, we define $(A] = \{s \in S \mid s \leq a \text{ for some } a \in A\}$ and obviously $A \subseteq (A]$. If $A = \{a\}$, then we write $(a]$ instead of $(\{a\}]$. For $\emptyset \neq A, B \subseteq S$, then $AB = \{ab \mid a \in A, b \in B\}$, $((A]] = (A]$, $(A](B] \subseteq (AB]$, $((A](B]] = (AB]$, if $A \subseteq B$ then $(A] \subseteq (B]$, $(A \cap B] \neq (A] \cap (B]$ in general.

Now we define an ideal theory for a class of non-associative and non-commutative algebraic structures (ordered AG-groupoid), which are similar to the ideal theory of associative algebraic structure (ordered semigroup), but not identical.

A non-empty subset A of an ordered LA-semigroup S is an LA-subsemigroup of S if $A^2 \subseteq A$.

A non-empty subset A of an ordered LA-semigroup S is a left (resp. right) ideal of S if

1. $SA \subseteq A$ (resp. $AS \subseteq A$).
2. If $a \in A$ and $b \in S$ such that $b \leq a$ implies $b \in A$ (or $(A] \subseteq A$).

A is called an ideal of S if A is both a left and a right ideal of S. Every ideal (whether left, right, two-sided) is an LA-subsemigroup, but the converse is not in general.

A non-empty subset A of S is an interior (resp. quasi-) ideal of S if $(SA)S \subseteq A$ (resp. $(AS] \cap (SA] \subseteq A$) and $(A] \subseteq A$.

An LA-subsemigroup A of S is a $(1,2)$-ideal of S if $(AS)A^2 \subseteq A$ and $(A] \subseteq A$. An LA-subsemigroup A of S is a bi-ideal of S if $(AS)A \subseteq A$ and $(A] \subseteq A$. A non-empty subset A of S is a generalized bi-ideal of S if $(AS)A \subseteq A$ and $(A] \subseteq A$. Every bi-ideal of S is a generalized bi-ideal of S.

In [3], a groupoid $(G, \cdot)$ is a left almost group (abbreviated LA-group) if

1. $(a \cdot b) \cdot c = (c \cdot b) \cdot a$ for all $a, b, c \in G$.
2. There exists $e \in G$ such that $e \cdot a = a$ for every $a \in G$.
3. For every $a \in G$ there exists $a' \in G$ such that $a' \cdot a = e$.

It is very easy to see that the left identity 'e' and the left inverse are unique. Left inverse is also right inverse as $aa' = (ea)a' = (a'a)e = ee = e$. This implies that a' is also a right inverse of a.

An ordered LA-group $(G, \cdot, \leq)$, is a poset, at the same time an LA-group such that $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$ for all $a, b, c \in G$.

An ordered LA-group is also called a po-LA-group for short. One of the immediate properties of a po-LA-group is this if $a \leq b$, then $b^{-1} \leq a^{-1}$. Now, let

$$\begin{aligned} a &\leq b \\ &\Rightarrow a^{-1}a \leq a^{-1}b \\ &\Rightarrow (a^{-1}a)b^{-1} \leq (a^{-1}b)b^{-1} \\ &\Rightarrow eb^{-1} \leq (b^{-1}b)a^{-1} \text{ by the left invertive law} \\ &\Rightarrow b^{-1} \leq ea^{-1} \\ &\Rightarrow b^{-1} \leq a^{-1} \end{aligned}$$

Example 1. Consider a set $S = \{e, a, b, c, d\}$ with the following multiplication "$\cdot$" and order relation "$\leq$"

$\cdot$	e	a	b	c	d
e	e	a	b	c	d
a	d	e	a	b	c
b	c	d	e	a	b
c	b	c	d	e	a
d	a	b	c	d	e

$\leq: = \{(e, e), (e, a), (e, b), (e, c), (e, d), (a, a), (b, b), (c, c), (d, d)\}$, then $(S, \cdot, \leq)$ is an ordered LA-group.

Lemma 1. *Every right ideal of an ordered LA-semigroup S with left identity e, is an ideal of S.*

Proof. Let I be a right ideal of S, this implies that $(I] \subseteq I$. Let $i \in I$ and $s \in S$. Now $si = (es)i = (is)e \in IS \in I$. Hence I is an ideal of S. □

Lemma 2. *Every two-sided ideal of an ordered LA-semigroup S is an interior ideal of S.*

Proof. Let I be a two sided ideal of S, this means that $(I] \subseteq I$. Now $(SI)S \subseteq IS \subseteq I$. Therefore I is an interior ideal of S. □

Proposition 1. *Let S be an ordered LA-semigroup with left identity e. Then any non-empty subset I of S is an interior ideal of S if and only if I is an ideal of S.*

Proof. Suppose that I is an interior ideal of S, this implies that $(I] \subseteq I$. Let $i \in I$ and $s \in S$. Now $is = (ei)s \in (SI)S \subseteq I$, thus I is a right ideal of S. Hence I is an ideal of S by Lemma 1. Converse is true by Lemma 2. □

Lemma 3. *Every right (two-sided) ideal of S is a bi-ideal of S.*

Proof. Let I be a right ideal of S, this means that $(I] \subseteq I$. Now $(IS)I \subseteq IS \subseteq I$. Hence I is a bi-ideal of S. □

Lemma 4. *Every left (right, two-sided) ideal of S is a quasi-ideal of S.*

Proof. Let I be a right ideal of S, this implies that $(I] \subseteq I$. Now $(IS] \cap (SI] \subseteq (IS] \subseteq (I] \subseteq I$. Therefore I is a quasi-ideal of S. □

Proposition 2. *Every quasi-ideal of S is an LA-subsemigroup of S.*

Proof. Let I be a quasi-ideal of S. Now $II \subseteq IS \subseteq (I](S] \subseteq (IS]$ and $II \subseteq SI \subseteq (S](I] \subseteq (SI]$. Thus $I^2 = II \subseteq (IS] \cap (SI] \subseteq I$. So I is an LA-subsemigroup of S. □

Proposition 3. *Let I and G be a right and a left ideal of an ordered LA-semigroup S, respectively. Then $I \cap G$ is a quasi-ideal of S.*

Proof. $((I \cap G)S] \cap (S(I \cap G)] \subseteq (IS] \cap (SG] \subseteq (I] \cap (G] \subseteq I \cap G$, since I is a right ideal and G is a left ideal of S. Also $(I \cap G] \subseteq I \cap G$. Consequently $I \cap G$ is a quasi-ideal of S. □

Lemma 5. *If an ordered LA-semigroup $(S, \cdot, \leq)$ with left identity e, is an ordered LA-group, then S does not contain any proper interior (resp. bi-, generalized bi-, quasi-) ideal.*

Proof. Let A be an interior ideal of S. Let $a \in S$ and $b \in A$, i.e., $b \in S$. This implies that there exists $b^{-1} \in S$ such that $bb^{-1} = b^{-1}b = e$ (e is the left identity of S). Now $a = ea = (b^{-1}b)a = (ab)b^{-1} \in (SA)S \subseteq A$. Thus $S \subseteq A$, i.e., $S = A$.

Suppose that A is a bi-ideal of S. Let $a \in S$ and $b \in A$, i.e., $b \in S$. This means that there exists $b^{-1} \in S$ such that $bb^{-1} = b^{-1}b = e$ (e is the left identity of S). Now

$$\begin{aligned} a &= (ee)a = (ae)e = (a(bb^{-1}))(bb^{-1}) \\ &= (b(ab^{-1}))(bb^{-1}) = ((bb^{-1})(ab^{-1}))b \\ &= ((ba)(b^{-1}b^{-1}))b = ((b^{-1}a)(b^{-1}b))b \\ &= ((b^{-1}a)(bb^{-1}))b \text{ since } b^{-1}b = bb^{-1} \\ &= (b((b^{-1}a)b^{-1}))b \in (AS)A \subseteq A. \end{aligned}$$

Thus $S \subseteq A$, i.e., $S = A$.

Assume that Q is a quasi-ideal of S and $0 \neq a \in Q$, then $e = aa^{-1} = a^{-1}a \in QS \cap SQ \subseteq (QS] \cap (SQ] \subseteq Q$, i.e., $e \in Q$. Let $x \in S$, and $x = ex = e(ex) = (xe)e \in QS \cap SQ \subseteq (QS] \cap (SQ] \subseteq Q$, i.e., $S \subseteq Q$. Hence $Q = S$. □

Proposition 4. *Let S be a right cancellative ordered LA-semigroup with left identity e. Then S is an ordered LA-group if and only if $Sa = S$ for all $a \in S$.*

Proof. Suppose that S is an ordered LA-group. Let $x \in S$ and $x = ex = (aa^{-1})x = (xa^{-1})a \in Sa$, i.e., $S \subseteq Sa$. Thus $Sa = S$ for all $a \in S$.

Conversely, assume that S is a right cancellative ordered LA-semigroup such that $Sa = S$ for all $a \in S$. From the given relation it follows that $ya = e$, for some $y \in S$. This implies that there exists left inverse of a in S, where a is an arbitrary element of S. For uniqueness, let $za = e$, $z \in S$. Now $ya = za$, i.e., $y = z$, by right cancellative. Hence S is an ordered LA-group. □

Definition 1. *An ordered LA-semigroup satisfies one the following conditions*

1. $(ab)c = b(ac)$,
2. $(ab)c = b(ca)$ *for all* $a, b, c \in S$.

then it is called weak associative ordered LA-semigroup and such that an ordered LA-semigroup is denoted by ordered LA*-semigroup. Note that condition 1. *and* 2. *are equivalent.*

Importance of definition 1, is that, when our need not fulfill by the left identity, then we involve the definition 1. That is why, we can prove this property, $Sa = S = aS$ for all $a \in S$, in the Proposition 5, but we cannot prove this property $Sa = S = aS$ for all $a \in S$, in the Proposition 4.

Proposition 5. *If an ordered LA***-semigroup* $(S, \cdot, \leq)$ *with left identity* e, *is an ordered LA***-group, then* $Sa = aS = S$ *for all* $a \in S$.

Proof. Assume that S is an ordered LA*-group and $Sa = S$. Since $aS \subseteq S$, let $x \in S$ and $x = ex = \left(a^{-1}a\right)x = a(a^{-1}x) \in aS$, i.e., $S \subseteq aS$. Thus $aS = S$. □

Converse of this statement is not true because associative law does not hold in an LA-group.

Suppose that S is an ordered LA*-semigroup such that $Sa = aS = S$ for all $a \in S$. From the given relation, it follows that $ax = e$ and $ya = e$, for some $x, y \in S$. This implies that x is the right inverse of a and y is the left inverse of a. Now we have to show that $x = y$.

$$\begin{aligned} y(ax) &= ye = e(ye) = (ee)y = ey = y \text{ and } (ya)x = ex = x \\ y(ax) &= (ay)x = ((ee)(ay))x = ((ya)(ee))x \\ &= ((ya)e)x = e((ya)x) = (ya)x. \\ &\Rightarrow x = y. \\ &\Rightarrow y(ax) = (ya)x. \end{aligned}$$

This means that associative law holds in S, which is a contradiction to fact that S is an LA-semigroup.

An ordered LA-semigroup S is said to be left (resp. right) simple if it does not contain any proper left (resp. right) ideal of S or equivalently, if for every left (resp. right) ideal A of S, we have $A = S$.

Equivalent definition: An ordered LA-semigroup S is a left (resp., right) simple if and only if $(Sa] = S$ (resp. $(aS] = S$) for every $a \in S$.

Theorem 1. *If an ordered LA-semigroup* $(S, \cdot, \leq)$ *with left identity* e *is a left* (*resp. right*) *simple. Then* S *does not contain any proper interior ideal.*

Proof. Suppose that S is a right simple, this implies that $(aS] = S$ for every $a \in S$. Let A be a right ideal of S, then $(bS] = S$ for every $b \in A \subseteq S$. As $a \in S$ and $b \in A$. Now

$$\begin{aligned} a &\in (bS] = (b(bS]] \subseteq (b(bS)] = (b((eb)S)] \\ &= (b((Sb)e)] = ((Sb)(be)] \subseteq ((SA)S] \subseteq (A] = A, \end{aligned}$$

since every right ideal of S is an interior ideal of S, (because every right ideal of S is an ideal of S by the Lemma 1). Thus $S \subseteq A$, so $A = S$. Similarly, for left simple. □

Theorem 2. *If an ordered LA-semigroup $(S, \cdot, \leq)$ is a left and a right simple. Then S does not contain any proper bi-ideal.*

Proof. Assume that S is a left and right simple, then $(Sa] = S$ and $(aS] = S$ for every $a \in S$. Let A be an ideal of S, this implies that A is a left and a right ideal of S. Thus $(Sb] = S$ and $(bS] = S$ for every $b \in A \subseteq S$. As $a \in S$ and $b \in A$. Now $a \in (Sb] = ((bS]b] \subseteq ((bS)b] \subseteq ((AS)A] \subseteq (A] = A$, since every ideal of S is a bi-ideal of S by Lemma 3. Thus $S \subseteq A$, so $A = S$. □

Remark 1. If an ordered LA-semigroup $(S, \cdot, \leq)$ is a left and a right simple. Then S does not contain any proper generalized bi-ideal.

Theorem 3. *If an ordered LA*-semigroup $(S, \cdot, \leq)$ with left identity e is a left and a right simple if and only if it contains no proper quasi-ideal.*

Proof. Let S be a left and a right simple, then $(Sa] = S$ and $(aS] = S$ for every $a \in S$. Suppose that A is an ideal of S, this implies that A is a left and a right ideal of S. Thus $(Sb] = S$ and $(bS] = S$ for every $b \in A \subseteq S$. As $a \in S$ and $b \in A$. Now $a \in (bS] \cap (Sb] \subseteq (AS] \cap (SA] \subseteq A$, since A is quasi-ideal of S (because every ideal of S is a quasi-ideal of S by Lemma 4). Hence $S = A$. Conversely, assume that S contains no proper quasi-ideal. So S contains no proper left ideal and no proper right ideal, because every left and right ideal of S is a quasi ideal of S by Lemma 4. Let $a \in S$. First we have to show that $(Sa]$ is a left ideal of S and $(aS]$ is a right ideal of S. Now

$$\begin{aligned} S(Sa] &\subseteq (S](Sa] \subseteq (S(Sa)] \\ &= ((Se)(Sa)] = ((SS)(ea)] \subseteq (Sa]. \end{aligned}$$

and $((Sa]] \subseteq (Sa]$. Now

$$\begin{aligned} (aS]S &\subseteq (aS](S] \subseteq ((aS)S] = ((aS)(eS)] = ((ae)(SS)] \\ &\subseteq ((ae)S] = (e(aS)] = (aS], \text{ by LA}^*\text{-semigroup.} \end{aligned}$$

and $((aS]] \subseteq (aS]$. Therefore $(Sa]$ is a left ideal of S and $(aS]$ is a right ideal of S. Now we have to show that $(aS] = S = (Sa]$. Obviously, $(Sa] \subseteq S$ and $(aS] \subseteq S$. Now $a = ea \in Sa \subseteq (Sa]$ and $a = ea = (ee)a = e(ae) = ae \in aS \subseteq (aS]$, by LA*-semigroup, i.e., $S \subseteq (Sa]$ and $S \subseteq (aS]$. Thus $(aS] = (Sa] = S$ for all $a \in S$. Hence S is a left and a right simple. □

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