Ordered hypervector spaces

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Abstract. In this paper, we first introduce the concept of ordered hypervector spaces, then we peresent several examples to better explain the introduced concepts. we finally prove that although ordered hypervector spaces do not necessarily satisfy the antisymmetric condition, the positive weak linear functionals on cofinal weak subhypervector spaces can be extended to the whole weak hypervector space.

Keywords: ordered hypervector spaces, positive weak linear functional, order unit element, cofinal set.

1. Introduction

The concept of hyperstructure was first introduced by Marty [3] in 1934. During the recent decades, many researchers have focused on this field and applied it as a corner stone to construct some other structures such as hypergroups, hypermodules, hyperfields, and hypervector spacses. These constructions have been

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applied to many disciplines such as geometry, hypergraphs, binary relations, combinatories, codes, cryptography, probability, etc. A wealth of applications of these concepts are given in [1, 2, 4] and [18].

In 1988, Scafati-Tallini first introduced the concept of hypervector spaces. She later considered more properties of such spaces. In [7, 13], authors considered hypervector spaces in the viewpoint of analysis and introduced some concepts such as normed hypervector spaces and the dimension of hypervector spaces. They also introduced operators on these space as well as some other important concepts.

In this paper we introduce the concept of ordered hypervector spaces and prove some Han-Banach results about it. we prove that each positive weak linear functional on cofinal weak subhypervector spaces can be extended to a positive weak linear functional on the whole weak hypervector space.

2. Preliminaries

Definition 2.1 ([16]). A weak or weakly distributive hypervector space over a field F ($\mathbb{C}$ *or* $\mathbb{R}$) is a quadruple $(X, +, \circ, F)$ such that $(X, +)$ is an abelian group and $o : F \times X \to P_*(X)$ is a multivalued product such that $(P_*(X) = \{A \subseteq X|\, A \neq \phi\})$

1- $\forall a \in F,\ \forall x, y \in X,\ [ao(x + y)] \cap [aox + aoy] \neq \emptyset;$

2- $\forall a, b \in F,\ \forall x \in X,\ [(a + b)ox] \cap [aox + box] \neq \emptyset;$

3- $\forall a, b \in F,\ \forall x \in X,\ ao(box) = (ab)ox;$

4- $\forall a \in F,\ \forall x \in X,\ ao(-x) = (-a)ox = -(aox);$

5- $\forall x \in X,\ x \in 1ox.$

The properties 1 and 2 are called weak right and left distributive laws respectively. Note that the set $ao(box)$ in 3 is of the form $\bigcup_{y \in box} aoy$.

Throughout this article, by "weak hypervector space" we mean "weak distributive hypervector space" on the field of real numbers.

Definition 2.2 ([7]). Let X be a weak hypervector space over F, $a \in F$ and $x \in X$. An essential point of $a \circ x$, denoted by $e_{a\circ x}$, for $a \neq 0$ is an element of $a \circ x$ such that $x \in a^{-1} \circ e_{a\circ x}$. For $a = 0$ we define $e_{a\circ x} = 0$.

Remark 2.3. As shown in the examples in the following sections, $e_{a\circ x}$ is not necessarily unique. Hence we denote the set of all essential points by $E_{a\circ x}$. In this note, $e_{a\circ x}$ represents an element in $E_{a\circ x}$.

Definition 2.4. A weak hypervector space X over the field F is said to be normal if for every $a, b \in F$ and $x, y \in X$ the following conditions hold:

1. $(E_{a\circ x} + E_{a\circ y}) \cap E_{a\circ(x+y)} \neq \emptyset$;
2. $E_{a\circ x} + E_{b\circ x} \subseteq E_{(a+b)\circ x}$.

Lemma 2.5 ([7])**.** *Let X be a weak hypervector space over F, $a, b \in F$ and $x \in X$. Then the following properties hold:*

1- $x \in E_{1\circ x}$.

2- If $b \neq 0$, then $a \circ e_{b\circ x} = ab \circ x$.

3- $E_{-a\circ x} = -E_{a\circ x}$.

4- If $a \neq 0$, then there exists a $y \in X$ such that $x \in E_{a\circ y}$.

5- If X is normal, then $E_{a\circ x}$ is a singleton.

Definition 2.6. A subset E of a weak hypervector space X is said to be hyperconvex if $E_{t\circ x} + E_{(1-t)\circ x} \subseteq E$, for every $x, y \in E$ and $t \in [0, 1]$.

Definition 2.7 ([6])**.** A subset E of a weak hypervector space X is said to be hyperabsorbing if for each $x \in X$, there is an $\varepsilon > 0$ such that $E_{\varepsilon\circ x} \subseteq E$.

Example 2.8. Consider $X = \mathbb{R}^2$ with usual addition and the following multi-valued product:

$$\begin{cases} \circ : \mathbb{R} \times X \to P_*(X) \\ a \circ (x, y) = \{ax\} \times ([0, +\infty) \cup \{y\}), \ a \in \mathbb{R}, \ (x, y) \in X. \end{cases}$$

Then $(X, +, \circ, \mathbb{R})$ is a weak hypervector space. The strip $E = [-1, 1] \times \mathbb{R}$ is hyperabsorbing and hyperconvex, but the set

$$A = \{(x, y) \in X | \ x^2 + y^2 \leq 1, \ x > 0, \ y > 0\}$$

is not. In fact, for $(x, y) \in X$, we choose $\varepsilon > 0$ such that $\varepsilon|x| < 1$. Since

$$E_{\varepsilon\circ(x,y)} = \begin{cases} \{\varepsilon x\} \times [0, \infty), & y \geq 0 \\ \{(\varepsilon x, y)\}, & y < 0 \end{cases}$$

then $E_{\varepsilon\circ(x,y)} \subseteq E$, therefore E is hyperabsorbing.

Definition 2.9 ([7])**.** An additive subgroup M of a weak hypervector space X is called a weak subhypervector space of X when $e_{a\circ x} \in M$ for every $a \in F$ and $x \in M$.

Definition 2.10 ([7])**.** Let X be a weak hypervector space over F. A map $f : X \to F$ is called a weak linear functional if f is additive and satisfies $f(e_{a\circ x}) = af(x)$, for every $a \in F$ and $x \in X$.

3. Ordered hypervector space

In this section, we introduce the concept of ordered weak hypervector spaces and give diverse examples of such hyperstructures. Moreover, we peresent the definition of positive weak linear functional, together with some examples of such functions. It is interesting that the order relation discussed in this hyperstructure is not necessarily antisymmetric, however, the corollaries of Han-Banach theorem can be proved for these theorems. We recall that Han-Banach theorem on weak hypervector spaces was first proved by Taghavi et al [8]. As a corollary of Han-Banach theorem, we here prove that each positive weak linear functional on a cofinal weak subhypervector space of an ordered weak hypervector space can be extended to the whole hypervector space.

Definition 3.1. An ordered hypervector space is a pair $(X, \leq)$, where X is a weak hypervector space over the real line $\mathbb{R}$ and $\leq$ is a relation on X satisfying the following: $(x, y, z \in X,\ \alpha \in \mathbb{R})$

1. $x \leq x$.
2. If $x \leq y$ and $y \leq z$, then $x \leq z$.
3. If $x \leq y$ and $z \in X$, then $x + z \leq y + z$.
4. If $x \leq y$ and $\alpha \geq 0$, then $e_{\alpha \circ x} \leq e_{\alpha \circ y}$.

Note that it is not assumed that $\leq$ is antisymmetric. In other words, it is not assumed that if $x \leq y$ and $y \leq x$, then $x = y$.

From now on, for each x, y in the ordered weak hypervector space $(X, \leq)$, $x \geq y$ is equivalent to $y \leq x$.

Example 3.2. 1. Consider $X = \mathbb{R}^2$ with usual addition. For $0 \neq \lambda \in \mathbb{R}$, we define the hyperoperation of $\circ_\lambda : \mathbb{R} \times X \to P_*(X)$ as follows:

$$\forall \alpha \in \mathbb{R}\ ,\ x = (x_1, x_2) \in X,\ \alpha \circ_\lambda x :=\ {}_\lambda L_{\alpha(x_2 - \lambda x_1)},$$

where for $b \in \mathbb{R}$, ${}_\lambda L_b$ is a line in $\mathbb{R}^2$ whose slope and y-intercept are equal to λ and b respectively. In other words,

$${}_\lambda L_b = \{(x, y) \in \mathbb{R}^2 |\, y = \lambda x + b\}.$$

Then $(X, +, \circ_\lambda, \mathbb{R})$ is a weak hypervector space in which

$$\forall 0 \neq \alpha,\ x \in X,\ E_{\alpha \circ_\lambda x} = \alpha \circ_\lambda x.$$

On this hyperstructre, we now define the relation $\leq_\lambda$ as follows:
Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be two points of X, we set

$$x \leq_\lambda y\ \Leftrightarrow\ x_2 - \lambda x_1 \leq y_2 - \lambda y_1.$$

Intuitively, $x \leq_\lambda y$, if x is on or under the line ${}_\lambda L_{(y_2-\lambda y_1)}$. This is not an antisymmetric relation but satisfies the condition of Definition 3.1.

1, 2 and 3 are obvious. For 4. Let $\alpha > 0$, then if we suppose

$$e_{\alpha\circ_\lambda x} = (a_1, a_2),\ e_{\alpha\circ_\lambda y} = (b_1, b_2)$$

then from $x \leq_\lambda y$ and $\alpha > 0$ we have

$$a_2 - \lambda a_1 = \alpha(x_2 - \lambda x_1) \leq \alpha(y_2 - \lambda y_1) = b_2 - \lambda b_1$$

which shows that $e_{\alpha\circ_\lambda x} \leq_\lambda e_{\alpha\circ_\lambda y}$.

2. Consider $\mathbb{C}$, the set of complex numbers with usual addition and the following scalar hyperproduct:

$$a \circ z := \{re^{i\theta}|\ 0 \leq r \leq |a||z|,\ \theta = arg(z)\},\ a \in \mathbb{R},\ z \in \mathbb{C}.$$

Then according to (Example $4 \cdot 2$) in [7], $(\mathbb{C}, +, \circ, \mathbb{R})$ is a hypervector space with $E_{a\circ z} = \{|a|z\}$ for each $a \in \mathbb{R}$ and $z \in \mathbb{C}$. Together with each of the relations defined in the following, this hyperstructure is an ordered hypervectore space.

Let $z_1 = x_1 + ix_2$ and $z_2 = y_1 + iy_2$ be two elements in $\mathbb{C}$, we define:

$$\begin{aligned} &z_1 \leq_1 z_2 \Leftrightarrow x_1 \leq y_1,\, x_2 \leq y_2 \\ &z_1 \leq_2 z_2 \Leftrightarrow x_1 < y_1 \text{ or } (x_1 = y_1,\ x_2 < y_2) \\ &z_1 \leq_3 z_2 \Leftrightarrow (x_1 = y_1,\ x_2 = y_2) \text{ or } (x_1 < y_1,\ x_2 < y_2). \end{aligned}$$

3. The weak hypervector space defined in Example 2.8 is an ordered hypervector space with the following defined relation:

Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be two points in X, we define

$$x \preceq y \Leftrightarrow x_1 \geq y_1.$$

This relation in not antisymmetric but satisfies the conditions of Definition 3.1. To chek condition 4 of Definition 3.1, let $\alpha > 0$ be given, then $x \preceq y$ implies that $\alpha x_1 \geq \alpha y_1$. Then for each $t, s \in \mathbb{R}$, $(\alpha x_1, t) \preceq (\alpha y_1, s)$, which proves that $e_{\alpha\circ x} \preceq e_{\alpha\circ y}$.

Definition 3.3. If X is a real weak hypervector space, a hypercone is a nonempty subset P of X such that

1. If $x, y \in P$ then $x + y \in P$.

2. If $x \in P$ and $\alpha \geq 0$ then $E_{\alpha\circ x} \subseteq P$.

For instance, the set $P = \{re^{i\theta}| r \geq 0,\ \frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}\}$ is a hypercone of the hypervectore space given in Example 2.8.

Proposition 3.4. 1. *If* $(X, \leq)$ *is an ordered hypervector space such that*

$$\forall x, y \in X,\ a \in \mathbb{R}, \qquad E_{a\circ x} + E_{a\circ y} \subseteq E_{a\circ(x+y)} \tag{3.1}$$

and

$$P = \{x \in X | x \geq 0\},$$

then P *is a hypercone.*

2. *If* P *is a hypercone in a real weak hypervector space* X *that satisfies* (3.1) *and* $\leq$ *is defined on* X *declared as "*$x \leq y$ *if and only if* $y - x \in P$*", then* $(X, \leq)$ *is an ordered hypervector space.*

Proof. 1. Let $x \in P$ and $\alpha \geq 0$ be given, since X satisfies (3.1), then $e_{\alpha\circ x} \geq e_{\alpha\circ 0} = 0$ and hence $e_{\alpha\circ x} \in P$.

(Note that $0 \in E_{a\circ 0} - E_{a\circ 0} \subseteq E_{a\circ 0}$).

2. Let $x, y \in X$, $\alpha \geq 0$ and $x \leq y$, then $y - x \geq 0$, since P is a hypercone then (3.1) implies that:

$$e_{\alpha\circ y} - e_{\alpha\circ x} \in E_{\alpha\circ y} - E_{\alpha\circ x} \subseteq E_{\alpha\circ(y-x)} \subseteq P,$$

therefore $e_{\alpha\circ x} \leq e_{\alpha\circ y}$. □

If $(X, \leq)$ is an ordered hypervectore space, $P = \{x \in X | \, x \geq 0\}$ is called the hypercone of positive elements of X.

For instance, for the ordered weak hypervectore space given in Example 3.2 -1, $P = \{(x, y) \in \mathbb{R}^2 | \, y \geq \lambda x\}$. Furthermor, if we let $-P := \{x \in X | \ -x \geq 0\}$, then in Example 3.2- 1 we have $P \cap (-P) = {}_{\lambda}L_0$.

The following proposition gives a condition equivalent to antisymmetric property according to the point of hypercone. The proof has been omitted due to simplicity.

Proposition 3.5. *If* $(X, \leq)$ *is an ordered normal hypervector space and* P *is the hypercone of positive elements of* X*, then* $\leq$ *is antisymmetric if and only if* $P \cap (-P) = \{0\}$.

Definition 3.6. If $(X, \leq)$ is an ordered hypervector space, a subset A of X is said to be cofinal if for every $x \geq 0$ in X, there is an $a \in A$ such that $x \leq a$. An element e of X is said to be an order unit if for every $x \in X$ there exists a positive integer n such that $e_{(-n)\circ e} \leq x \leq e_{n\circ e}$.

If e is an order unit, clearly the set $\{e_{n\circ e} : n \geq 1\}$ is cofinal.

The set $\{(r, s) | \, r > 0,\ s \in \mathbb{R}\}$ is a set of order unit elements for the ordered hypervector space given in Example 3.2-3. Moreover, the set $\{(0, r) | \ r > 0\}$ is also a set of order unit elements for the ordered hypervector space given in Example 3.2 -1.

Definition 3.7. If $(X, \leq)$ is an ordered hypervector space, then a weak linear functional f on X is said to be positive if $f(x) \geq 0$ for every $x \geq 0$.

Example 3.8. 1. Let $(X, \leq_\lambda)$ be the ordered hypervector space given in Example 3.2-1. Then the additive map

$$\begin{cases} f : X \to \mathbb{R} \\ f(x, y) = y - \lambda x \end{cases}$$

is a positive weak linear functional on X. Let $\alpha \in \mathbb{R}$ and $(x, y) \in X$, then for $\alpha \neq 0$,

$$\begin{aligned} & f(e_{\alpha \circ_\lambda (x,y)}) \in f(E_{\alpha \circ_\lambda (x,y)}) = f({}_\lambda L_{\alpha(y-\lambda x)}) = \{\alpha(y - \lambda x)\} \\ & \Rightarrow f(e_{\alpha \circ (x,y)}) = \alpha(y - \lambda x) = \alpha f(x, y). \end{aligned}$$

Moreover, if $(x, y) \geq_\lambda 0$, then $f(x, y) = y - \lambda x \geq 0$.

2. Let X be the ordered hypervectore space given in Example 3.2-3. The additive map

$$\begin{cases} f : X \to \mathbb{R} \\ f(x, y) = -x \end{cases}$$

is a positive weak linear functional on X.

The following theorem is the main result of this paper.

Theorem 3.9. *Let $(X, \leq)$ be an ordered normal hypervector space and let M be a weak subhypervector space of X, and let f be a weak linear functional on M. Then the following are equivalent:*

1. *The functional f can be extended to a positive weak linear functional on X.*
2. *There is a hyperconvex and hyperabsorbing set $A \subseteq X$ such that*

$$f(x) \leq 1 \quad \textit{whenever } x \in M \textit{ and } (\exists y \in A,\ x \leq y). \tag{3.2}$$

Furthermore, when the statement 2 is satisfied, an extension $\tilde{f}$ can be chosen such that $\tilde{f}(x) \leq 1$ whenever $x \leq y$ for some $y \in A$.

Proof. $1 \to 2$: Assume that a positive weak linear functional $\tilde{f}$ is an extension of f and set $A = \{x \in X | \tilde{f}(x) \leq 1\}$. Then A is hyperconvex and hyperabsorbing. Moreover, A satisfies the condition (3.2), for if $x \in M$ such that $x \leq y$ for some $y \in A$, then $f(x) = \tilde{f}(x) \leq \tilde{f}(y) \leq 1$.

$2 \to 1$: Consider the set $C = A - P$, where P is the hypercone of positive elements of X. since A and P are hypercovex, then C is hyperconvex. Furthermore, C is hyperabsorbing too, because if $x \in X$ then since A is hyperabsorbing, $e_{\varepsilon \circ x} \in A$, for some $\varepsilon > 0$. Since $0 \in P$, then $e_{\varepsilon \circ x} = e_{\varepsilon \circ x} - 0 \in A - P = C$. Now since $0 \in C$, Lemma 3.1 of [3] implies that the functional

$$\begin{cases} q : X \to \mathbb{R} \\ q(x) = \inf\{\alpha > 0 |\ e_{\alpha^{-1} \circ x} \in C\} \end{cases}$$

have the following properties,

$$q(x+y) \leq q(x) + q(y), \qquad x, y \in X \tag{3.3}$$

$$q(e_{\lambda \circ x}) = \lambda q(x), \qquad x \in X,\ \lambda > 0. \tag{3.4}$$

By the hypothesis, $f(x) \leq 1$ whenever $x \in M \cap C$, because if $x \in M \cap C$ then $x = a - p$ for some $a \in A$, $p \in P$, but $p \geq 0$ implies $x = a - p \leq a$, therefore $x \in M$ and $x \leq a$ for $a \in A$, hence $f(x) \leq 1$.

Now let $x \in M$ be given, if $\alpha > 0$ and $e_{\alpha^{-1} \circ x} \in C$, then since $e_{\alpha^{-1} \circ x} \in M$, $f(e_{\alpha^{-1} \circ x}) \leq 1$, hence $f(x) \leq \alpha$, so $f(x) \leq q(x)$. Therefore, by (Theorem $3 \cdot 8$) of [8] there is a weak linear functional $\tilde{f}$ on X which is an extension of f, such that $\tilde{f}(x) \leq 1$ for all $x \in C$. The functional $\tilde{f}$ is necessarily positive. This is because if we take $x \in P$, then for all positive numbers t, $e_{(-t) \circ x} = 0 - e_{t \circ x} \in A - P = C$, hence $1 \geq \tilde{f}(e_{(-t) \circ x}) = -t\tilde{f}(x)$, for all $t > 0$, therefore

$$\forall t > 0 \qquad \tilde{f}(x) \geq \frac{-1}{t},$$

letting $t \to \infty$, we have $\tilde{f}(x) \geq 0$. This completes the proof. □

Corollary 3.10. *Let M be a cofinal weak subhypervector space of an ordered normal hypervector space X. Then each positive weak linear functional on M can be extended to a positive weak linear functional on X.*

Proof. without loss of generality we can assume that $X = M + P - P$, where P is the hypercone of positive elements of X. Because if we let $X_1 = M + P - P$, then X_1, is a weak subhypervector space of X, Now if there is a positive weak linear functional $g : X_1 \to \mathbb{R}$ that extends the positive weak linear functional $f : M \to \mathbb{R}$, then each weak linear extenstion $\tilde{f} : X \to \mathbb{R}$ of g is positive, that is because if $x \geq 0$, then $x \in P \subseteq X_1$, which implies $\tilde{f}(x) = g(x) \geq 0$.

Thus we may assume that $X = M + P - P$. Assume that f is a positive weak linear functional on M, set

$$A = \{x \in X \mid \exists y \in M \quad x \leq y \, and \, f(y) \leq 1\},$$

then we claim that

(I) A is hyperconvex,

(II) A is hyperabsorbing,

(III) (3.2) is satisfied.

(III) is trivial. For (I), let $x_1, x_2 \in A$ and $t \in [0, 1]$ be given, then there are $y_1, y_2 \in M$ such that $x_1 \leq y_1$, $x_2 \leq y_2$ and $f(y_1) \leq 1$, $f(y_2) \leq 1$. Since M is a weak subhypervector space of X, then $y = e_{t \circ y_1} + e_{(1-t) \circ y_2} \in M$ and also $e_{t \circ x_1} + e_{(1-t) \circ x_2} \leq y$. Furthermore,

$$f(y) = tf(y_1) + (1-t)f(y_2) \leq 1.$$

Therefore $e_{t\circ x_1} + e_{(1-t)\circ x_2} \in A$.

For (II), first note that we have $X = M - P$, for if $x \in X$, then $x = m + p_1 - p_2$ for some $m \in M$, $p_1, p_2 \in P$. Since M is cofinal, there is $m_1 \in M$ such that $p_1 \leq m_1$, hence $p_1 = m_1 - (m_1 - p_1) \in M - P$, thus $x = m - p_2 + p_1 \in (M - P) + (M + P) \subseteq M - P$.

Now let $x \in X$ be given, then $x = m - p$ for some $m \in M$, $p \in P$. If $e_{\lambda o m} \in P \cap M$ for some $\lambda > 0$, then $m = e_{\lambda^{-1} o e_{\lambda o m}} \in P \cap M$ and hence $m \geq 0$, therefore, $f(m) \geq 0$ ($f \geq 0\, on\, M$). So $\varepsilon = \frac{1}{f(m)+1} > 0$ and since $x \leq m$, then $e_{\varepsilon o x} \leq e_{\varepsilon o m}$, and $e_{\varepsilon o m} \in M$ and

$$f(e_{\varepsilon o m}) = \varepsilon f(m) = \frac{f(m)}{1 + f(m)} \leq 1.$$

If $e_{\lambda o m} \in (-P) \cap M$ for some $\lambda > 0$, then $-m \geq 0$ and hence $x \leq m \leq 0 \leq -m$, $\varepsilon = \frac{1}{1-f(m)} > 0$ and $e_{\varepsilon o x} \leq e_{\varepsilon \circ (-m)}$ and $e_{\varepsilon \circ (-m)} \in M$ and

$$f(e_{\varepsilon \circ (-m)}) = \frac{-f(m)}{1 - f(m)} \leq 1.$$

So $e_{\varepsilon o x} \in A$ and therefore A is hyperabsorbing.

Hence the assertion of this corollary is a consequence of Theorem 3.9. □

Corollary 3.11. *Let $(X, \leq)$ be an ordered normal hypervector space and let M be a weak subhypervector space of X. If X has an order unit $e \in M$, then each positive weak linear functional on M can be extended to a positive weak linear functional on X.*

Proof. It's enough to prove that M is cofinal. Let $x \geq 0$ be given in X, since e is an order unit, then $x \leq e_{n o e}$ for some integer $n \geq 1$, but since $e \in M$, then $e_{n o e} \in M$ and hence M is cofinal. Now the assertion follows from the last corollary. □

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