Asymptotic behavior of conformable fractional impulsive partial differential equations

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Abstract. In this article, we discuss the asymptotic behavior of conformable fractional impulsive partial differential equations. Some new sufficient conditions possessing a prescribed asymptotic behavior at infinity are derived by using riccati transform and impulsive differential inequalities. Our results extend a number of results reported in the literature. An example is also given to illustrate the validity of our results.

Keywords: oscillation, asymptotic, conformable differential equations, impulse, partial differential equations, forcing term.

1. Introduction

In recent years, fractional calculus has received increasing attention due to its applications in a variety of disciplines such as mechanics, physics, chemistry, biology, electrical engineering, control theory, material science, mathematical psychology. For more details, we refer the reader to the monographs. The field of fractional calculus is concerned with the generalization of the integer order differentiation and integration to an arbitrary real (or complex) order [1, 4, 5, 6, 8, 13, 17, 21]. Many events in diverse fields of engineering can be portrayed better and more accurately by differential equations of non-integer order.

Recently, a new definition of fractional derivative [12] that prominently compatible with the classical derivative was made by Khalil et al. Unlike other def-

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initions, this new definition satisfies formulas of the derivative of product and quotient of two functions and have a simpler chain rule.

In order to describe dynamics of populations subject to abrupt changes as well as other phenomena such as harvesting, diseases, and so forth, some authors have used impulsive differential systems to describe the model since the last century. For the basic theory on impulsive differential equations, the reader can refer to the monographs and references [2, 14, 18, 22, 25, 26]. In particular, the problem of asymptotic behavior of impulsive differential equations has been investigated by few authors, the references [3, 7, 9, 10, 11, 15, 19, 20, 23] cited therein. However, to the best of authors knowledge, no work has been reported on the asymptotic behavior of conformable fractional impulsive partial differential equations. Motivated by the above considerations we consider the following model of the form

$$(1)\qquad \left.\begin{array}{l} \dfrac{\partial^\alpha}{\partial t^\alpha}\left[p(t)\dfrac{\partial^\alpha}{\partial t^\alpha}\left(u(x,t)+c(t)u(x,\sigma(t))\right)\right]+q(x,t)f\left(u(x,\delta(t))\right) \\ = a(t)\Delta u(x,t)+\displaystyle\sum_{i=1}^{n} b_i(t)\Delta u(x,\rho_i(t))+F(x,t), \\ t\neq t_k,\ (x,t)\in\Omega\times\mathbb{R}_+\equiv G, \\ u(x,t_k^+)=c_k(x,t_k,u(x,t_k)), \\ \dfrac{\partial^\alpha}{\partial t^\alpha}\left(u(x,t_k^+)\right)=d_k(x,t_k,\dfrac{\partial^\alpha}{\partial t^\alpha}\left(u(x,t_k)\right)),\quad k=1,2,\cdots, \end{array}\right\}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a piecewise smooth boundary $\partial\Omega$, Δ is the Laplacian in the Euclidean space $\mathbb{R}^N$ and $\mathbb{R}_+=[0,+\infty)$.

Equation (1) is enhancement with the boundary condition

$$(2)\qquad u(x,t)=0,\qquad (x,t)\in\partial\Omega\times\mathbb{R}_+.$$

This work is planed as follows: In Section 2, we recall some basic definitions and preliminary results which will be used throughout this paper. In Section 3, we discussed the asymptotic behavior of the all solutions of the problem (1) and (2). In Section 4, we present an example is to illustrate our main results.

2. Preliminaries

In this paper, we assume that the following assumptions hold:

(C_1) $p(t)\in C^\alpha(\mathbb{R}_+,(0,+\infty))$ with $T_\alpha(p(t))>0$ and $\int_{t_0}^{+\infty} s^{\alpha-1}\dfrac{1}{p(s)}ds=+\infty$, $q(t)\in C(\mathbb{R}_+,\mathbb{R})$, $q(t)=\min_{x\in\bar{\Omega}} q(x,t)$, $c(t)\in C^{2\alpha}(\mathbb{R}_+,\mathbb{R}_+)$, $\sigma(t),\delta(t),\rho_i(t)\in C^\alpha(\mathbb{R}_+,\mathbb{R})$ and $\lim_{t\to+\infty}\sigma(t)=\lim_{t\to+\infty}\delta(t)=\lim_{t\to+\infty}\rho_i(t)=+\infty$, $i=1,2,\cdots,n$.

(C_2) $f\in C(\mathbb{R},\mathbb{R})$ is convex in $\mathbb{R}_+$ with $uf(u)>0$ and $\dfrac{f(u)}{u}\geq\epsilon>0$ for $u\neq 0$, $F\in C(\bar{G},\mathbb{R})$ with $\int_\Omega F(x,t)dx<0$.

(C_3) $a(t), b_i(t) \in PC(\mathbb{R}_+, \mathbb{R}_+)$, where PC represents the class of functions which are piecewise continuous in t with discontinuities of first kind only at $t = t_k$, and left continuous at $t = t_k$, $k = 1, 2, \cdots$.

(C_4) $u(x,t)$ and its derivative $\frac{\partial^\alpha}{\partial t^\alpha} u(x,t)$ are piecewise continuous in t with discontinuities of first kind only at $t = t_k$, $k = 1, 2, \cdots$, and left continuous at $t = t_k$, $u(x, t_k) = u(x, t_k^-)$, $\frac{\partial^\alpha}{\partial t^\alpha} u(x, t_k) = \frac{\partial^\alpha}{\partial t^\alpha} u(x, t_k^-)$, $k = 1, 2, \cdots$.

(C_5) $c_k, d_k \in PC(\bar{\Omega} \times \mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$ for $k = 1, 2, \cdots$, and there exist positive constants g_k, g_k^*, h_k, h_k^* such that $g_k^* \le g_k \le h_k^* \le h_k$ for $k = 1, 2, \cdots$,

$$g_k^* \le \frac{c_k\left(x, t_k, u(x, t_k)\right)}{u(x, t_k)} \le g_k, \quad h_k^* \le \frac{d_k\left(x, t_k, \frac{\partial^\alpha}{\partial t^\alpha}\left(u(x, t_k)\right)\right)}{\frac{\partial^\alpha}{\partial t^\alpha}\left(u(x, t_k)\right)} \le h_k.$$

Definition 2.1 ([26]). *A solution u of the problem* (1) *and* (2) *is a function $u \in C^{2\alpha}(\bar{\Omega} \times [t_{-1}, +\infty), \mathbb{R}) \cap C(\bar{\Omega} \times [\hat{t}_{-1}, +\infty), \mathbb{R})$ that satisfies* (1)*, where*

$$t_{-1} := \min\left\{0, \left\{\inf_{t \ge 0} \sigma(t)\right\}, \min_{1 \le i \le n}\left\{\inf_{t \ge 0} \rho_i(t)\right\}\right\}, \quad \hat{t}_{-1} := \min\left\{0, \inf_{t \ge 0} \delta(t)\right\}.$$

Definition 2.2. *The solution u of the problem $(1) - (2)$ is said to be oscillatory in the domain G, if it has arbitrary large zeros. Otherwise it is non-oscillatory.*

Definition 2.3 ([12]). *Let $f : [0, \infty) \to \mathbb{R}$. Then the conformable fractional derivative of f of order α is defined by*

$$T_\alpha(f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}$$

for all $t > 0, \alpha \in (0, 1]$.

If f is α-differentiable in some $(0, a), a > 0$ and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists, then define

$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t)$$

Definition 2.4. $I_\alpha^a(f)(t) = I_1^a(t^{\alpha-1} f) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx$, *where the integral is the usual Riemann improper integral, and $\alpha \in (0, 1)$.*

Definition 2.5 (Atangana et al. [1]). *f be a function with n variables $x_1, x_2, \cdots, x_n$. Then the conformable partial derivative of f of order $0 < \alpha \le 1$ in x_i is defined as follows*

$$\frac{\partial^\alpha}{\partial x_i^\alpha} f(x_1, x_2, \cdots, x_n)$$
$$= \lim_{\epsilon \to 0} \frac{f(x_1, x_2, \cdots, x_{i-1}, x_i + \epsilon x_i^{1-\alpha}, \cdots, x_n) - f(x_1, x_2, \cdots, x_n)}{\epsilon}.$$

Conformable fractional derivatives have the following properties :

Theorem 2.1. *Let $\alpha \in (0,1]$ and f, g be α- differentiable at some point $t > 0$. Then*

(i) $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, *for all* $a, b \in \mathbb{R}$.

(ii) $T_\alpha(t^p) = pt^{p-\alpha}$ *for all* $p \in \mathbb{R}$.

(iii) $T_\alpha(\lambda) = 0$ *for all constant functions* $f(t) = \lambda$.

(iv) $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.

(v) $T_\alpha\left(\frac{f}{g}\right) = \dfrac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$.

(vi) *If f is differentiable, then* $T_\alpha(f)(t) = t^{1-\alpha}\dfrac{df}{dt}(t)$.

It is identified that [24] the smallest eigenvalue $\lambda_0 > 0$ of the eigenvalue problem

$$\begin{aligned} \Delta\omega(x) + \lambda\omega(x) = 0, &\quad \text{in} \quad \Omega \\ \omega(x) = 0, &\quad \text{on} \quad \partial\Omega, \end{aligned}$$

and the consequent eigenfunction $\Phi(x) > 0$ in Ω.

For convenience, we introduce the following notations:

$$Y(t) = K_\Phi \int_\Omega u(x,t)\Phi(x)dx, \quad \textit{where} \quad K_\Phi = \left(\int_\Omega \Phi(x)dx\right)^{-1}.$$

We begin with some lemmas that will be used to prove our main results.

Lemma 2.1. *If X and Y are non negative, then*

$$\begin{aligned} X^\lambda - \lambda XY^{\lambda-1} + (\lambda-1)Y^\lambda \geq 0, &\qquad \lambda > 1 \\ X^\lambda - \lambda XY^{\lambda-1} - (1-\lambda)Y^\lambda \leq 0, &\qquad 0 < \lambda < 1 \end{aligned}$$

where the equality holds if and only if $X = Y$.

Lemma 2.2. *If the impulsive conformable fractional differential inequality*

$$\left.\begin{aligned} &T_\alpha(p(t)T_\alpha(z(t))) + c_0q(t)z(\delta(t)) \leq 0, \quad t \neq t_k \\ &g_k^* \leq \frac{Y(t_k^+)}{Y(t_k)} \leq g_k, \quad h_k^* \leq \frac{T_\alpha(Y(t_k^+))}{T_\alpha(Y(t_k))} \leq h_k, \quad k = 1, 2, \cdots. \end{aligned}\right\} \tag{3}$$

has no eventually positive solution, then every solution of the problem (1)-(2) *is oscillatory in G.*

Proof. Assume that there exists a non oscillatory solution $u(x,t)$ of the problem $(1)-(2)$ and $u(x,t)>0$, $(x,t)\in\Omega\times[t_0,+\infty), t_0\geq 0$. By assumption that there exists a $t_1>t_0$ such that

$$\begin{aligned}
u(x,\sigma(t))>0, &\qquad (x,t)\in\Omega\times[t_1,\infty)\\
u(x,\delta(t))>0, &\qquad (x,t)\in\Omega\times[t_1,\infty)\\
u(x,\rho_i(t))>0, &\qquad (x,t)\in\Omega\times[t_1,\infty), \qquad i=1,2,\cdots,n.
\end{aligned}$$

For $t\geq t_0$, $t\neq t_k$, $k=1,2,\cdots$, multiplying both sides of equation (1) by $K_\Phi\Phi(x)$ and integrating with respect to x over the domain Ω, we obtain

$$\left.\begin{aligned}
&t^{1-\alpha}\frac{d}{dt}\left[p(t)t^{1-\alpha}\frac{d}{dt}\left(K_\Phi\int_\Omega u(x,t)\Phi(x)dx+c(t)K_\Phi\int_\Omega u(x,\sigma(t))\Phi(x)dx\right)\right]\\
&+K_\Phi\int_\Omega q(x,t)f(u(x,\delta(t)))\Phi(x)dx=a(t)K_\Phi\int_\Omega\Delta u(x,t)\Phi(x)dx\\
&+\sum_{i=1}^n b_i(t)K_\Phi\int_\Omega\Delta u(x,\rho_i(t))\Phi(x)dx+K_\Phi\int_\Omega F(x,t)\Phi(x)dx.
\end{aligned}\right\}\tag{4}$$

Using Green's formula and boundary condition (2), we see that

$$\begin{aligned}
K_\Phi\int_\Omega\Delta u(x,t)\Phi(x)dx=&\ K_\Phi\int_{\partial\Omega}\left[\Phi(x)\frac{\partial u}{\partial\gamma}-u\frac{\partial\Phi(x)}{\partial\gamma}\right]dS\\
&+K_\Phi\int_\Omega u(x,t)\Delta\Phi(x)dx=\ -\lambda_0Y(t)\leq 0
\end{aligned}\tag{5}$$

and for $i=1,2,\cdots,n$, we have

$$\begin{aligned}
K_\Phi\int_\Omega\Delta u(x,\rho_i(t))\Phi(x)dx=&\ K_\Phi\int_{\partial\Omega}\left[\Phi(x)\frac{\partial u(x,\rho_i(t))}{\partial\gamma}-u(x,\rho_i(t))\frac{\partial\Phi(x)}{\partial\gamma}\right]dS\\
&+K_\Phi\int_\Omega u(x,\rho_i(t))\Delta\Phi(x)dx\\
=&\ -\lambda_0Y(\rho_i(t))\leq 0,
\end{aligned}\tag{6}$$

where dS is surface element on $\partial\Omega$. Applying Jensen's inequality and from (C_2), it follows that

$$K_\Phi\int_\Omega q(x,t)f(u(x,\delta(t)))\Phi(x)dx\geq\epsilon q(t)K_\Phi\int_\Omega u(x,\delta(t))\Phi(x)dx.\tag{7}$$

Also, we have

$$F(t)=K_\Phi\int_\Omega F(x,t)\Phi(x)dx\leq 0.\tag{8}$$

In view of (4)-(8), we obtain

$$T_\alpha\left(p(t)T_\alpha(z(t))\right)+\epsilon q(t)Y(\delta(t))\leq 0,\tag{9}$$

where $z(t) = Y(t) + c(t)Y(\sigma(t))$. It is easy to obtain that $z(t) > 0$ for $t \geq t_1$. Next we prove that $T_\alpha(z(t)) > 0$ for $t \geq t_2$. Assume the contrary that there exists $K \geq t_2$ such that $T_\alpha(z(t)) \leq 0$, so we get

$$T_\alpha(p(t)T_\alpha(z(t))) \leq 0, \quad t \geq K. \tag{10}$$

From (10), we have

$$p(t)T_\alpha(z(t)) \leq p(K)T_\alpha(z(K)), \qquad t \geq K.$$

Then we get

$$z(t) \leq z(K) + p(K)T_\alpha(z(K)) \int_K^t \frac{ds}{s^{1-\alpha}p(s)}, \qquad t \geq K. \tag{11}$$

From the hypothesis (C_1), we get $\lim_{t\to+\infty} z(t) = -\infty$. This contradicts that $z(t) > 0$ for $t \geq 0$. Thus $T_\alpha(z(t)) > 0$ and $\sigma(t) \leq t$ for $t \geq t_1$, we have

$$\begin{aligned} Y(t) =& z(t) - c(t)Y(\sigma(t)) \geq z(t) - c(t)z(t), \\ Y(t) \geq& z(t)(1 - c(t)) \geq c_0 z(t), \end{aligned}$$

where $c_0 = 1 - c(t)$ is a positive constant. Therefore from (9), we have

$$T_\alpha\left(p(t)T_\alpha(z(t))\right) + k_0 q(t) z(\delta(t)) \leq 0,$$

where $k_0 = \epsilon c_0$. For $t \geq t_0, t = t_k,\ k = 1, 2, \cdots$, multiplying both sides of the equation (1) by $K_\Phi \Phi(x)$, integrating with respect to x over the domain Ω, and from (C_5), we obtain

$$g_k^* \leq \frac{Y(t_k^+)}{Y(t_k)} \leq g_k, \qquad h_k^* \leq \frac{T_\alpha(Y(t_k^+))}{T_\alpha(Y(t_k))} \leq h_k$$

and

$$g_k^* \leq \frac{z(t_k^+)}{z(t_k)} \leq g_k, \qquad h_k^* \leq \frac{T_\alpha(z(t_k^+))}{T_\alpha(z(t_k))} \leq h_k.$$

Hence we obtain that $z(t)$ is a positive solution of impulsive conformable fractional inequality (3). This completes the proof. $\square$

Lemma 2.3. *Assume that conditions $(C_1) - (C_5)$ hold and let $u(x,t)$ be a positive solution of (1) and (2). Then for sufficiently large t, either*

(I) $\quad z(t) > 0,\ T_\alpha(z(t)) > 0,\ T_\alpha\left(p(t)T_\alpha(z(t))\right) < 0$ *or*
(II) $\quad z(t) > 0,\ T_\alpha(z(t)) < 0,\ T_\alpha\left(p(t)T_\alpha(z(t))\right) < 0.$

Lemma 2.4. *Assume that conditions* $(C_1)-(C_5)$ *hold and let* $u(x,t)$ *be an eventually positive solution of* (1) *and* (2) *with* $z(t)$ *satisfying case* (II) *of Lemma 2.3. If*

$$\int_{t_1}^{\infty} \frac{1}{p(y)y^{1-\alpha}} \int_{y}^{\infty} \frac{q(s)}{s^{1-\alpha}} z(\delta(s))dsdy = \infty, \tag{12}$$

then $\lim_{t\to\infty} z(t) = 0$.

Proof. Let u(x,t) be an eventually positive solution of (1) and (2). Then z(t) satisfies the inequality (3) and

$$T_\alpha\left(p(t)T_\alpha(z(t))\right) \le -k_0 q(t) z(\delta(t)) \le 0, \quad t \ge t_2.$$

By Lemma 2.3, there exists a constant m such that

$$\lim_{t\to\infty} z(t) = m < \infty. \tag{13}$$

Integrating the above inequality from t to ∞, we get

$$p(t)T_\alpha(z(t)) \ge \int_t^\infty k_0 \frac{q(s)}{s^{1-\alpha}} z(\delta(s))ds$$
$$T_\alpha(z(t)) \ge \frac{k_0}{p(t)} \int_t^\infty \frac{q(s)}{s^{1-\alpha}} z(\delta(s))ds.$$

Again integrating from t_1 to ∞, we get

$$z(t) \ge k_0 \int_{t_1}^{\infty} \frac{1}{p(y)y^{1-\alpha}} \int_{y}^{\infty} \frac{q(s)}{s^{1-\alpha}} z(\delta(s))dsdy,$$

which contradicts (13), and so we have m = 0. Therefore $\lim_{t\to\infty} z(t) = 0$. This complete the proof. □

3. Main results

In this section, by using Riccati transformation and impulsive differential inequality, we investigate the asymptotic behavior of all solutions of nonlinear partial differential equations with impulse effects and obtained the following two theorems.

Theorem 3.1. *Assume that* (C_1)-(C_5) *holds and there exists* $\eta(t)\in C^\alpha([t_0,\infty),\mathbb{R})$ *such that for all sufficiently large* K *and for* $t_1 \ge K$,

$$\limsup_{t\to\infty} \int_{t_1}^{t} \prod_{t_0\le t_k\le s} \left(\frac{h_k}{g_k^*}\right)^{-1} \left\{ k_0 s^{\alpha-1}\eta(s)q(s) - \frac{s^{1-\alpha}T_\alpha(\eta(s))^2 p(s)}{4\eta(s)} \right\} ds = \infty, \tag{14}$$

then any solution $u(x,t)$ *of* (1) *and* (2) *is either oscillatory or converges to zero as* $t \to \infty$.

Proof. Let $u(x,t)$ be a non oscillatory solution of (1) and (2). Without loss of generality, we may assume that there exists $t_1 \geq t_0$ such that $u(x,t) > 0$, $u(x,\sigma(t)) > 0$ and $u(x,\delta(t)) > 0$ for $t \geq t_1$. Also $z(t)$ satisfies either case (I) or case (II) for $t \geq t_1$.

Assume that case (I) and define

$$w(t) = \eta(t)\frac{p(t)T_\alpha(z(t))}{z(\delta(t))} \quad for \quad t \geq t_1,$$

$$T_\alpha(w(t)) \leq -k_0\eta(t)q(t) - \frac{w^2(t)}{\eta(t)p(t)} + \frac{T_\alpha(\eta(t))}{\eta(t)}w(t). \tag{15}$$

Also

$$w(t_k^+) \leq \frac{h_k}{g_k^*}w(t_k). \tag{16}$$

Define

$$V(t) = \prod_{t_0 \leq t_k < t}\left(\frac{h_k}{g_k^*}\right)^{-1} w(t).$$

In fact, $w(t)$ is continuous on each interval $(t_k, t_{k+1}]$ and it follows that for $t \geq t_0$,

$$V(t_k^+) = \prod_{t_0 \leq t_j \leq t_k}\left(\frac{h_k}{g_k^*}\right)^{-1} w(t_k^+) \leq \prod_{t_0 \leq t_j < t}\left(\frac{h_k}{g_k^*}\right)^{-1} w(t_k) = V(t_k),$$

and for all $t \geq t_0$

$$V(t_k^-) = \prod_{t_0 \leq t_j \leq t_{k-1}}\left(\frac{h_k}{g_k^*}\right)^{-1} w(t_k^-) \leq \prod_{t_0 \leq t_k < t}\left(\frac{h_k}{g_k^*}\right)^{-1} w(t_k) = V(t_k),$$

which implies that $V(t)$ is continuous on $[t_0, +\infty)$, from (15), we get

$$\begin{aligned}
&\prod_{t_0 \leq t_k < t}\left(\frac{h_k}{g_k^*}\right) T_\alpha(V(t)) \leq -k_0\eta(t)q(t) \\
&\quad - \prod_{t_0 \leq t_k < t}\left(\frac{h_k}{g_k^*}\right)^2 \frac{V^2(t)}{\eta(t)p(t)} + \frac{T_\alpha(\eta(t))}{\eta(t)} \prod_{t_0 \leq t_k < t}\left(\frac{h_k}{g_k^*}\right) V(t), \\
&T_\alpha(V(t)) \leq - \prod_{t_0 \leq t_k < t}\left(\frac{h_k}{g_k^*}\right) \frac{V^2(t)}{\eta(t)p(t)} + \frac{T_\alpha(\eta(t))}{\eta(t)}V(t) \\
&\quad - \prod_{t_0 \leq t_k < t}\left(\frac{h_k}{g_k^*}\right)^{-1} k_0\eta(t)q(t).
\end{aligned} \tag{17}$$

Applying Lemma 2.1, we have

$$X = \sqrt{\prod_{t_0 \leq t_k < t}\left(\frac{h_k}{g_k^*}\right) \frac{1}{\eta(t)p(t)}}V(t) \quad \text{and} \quad Y = \frac{T_\alpha(\eta(t))}{2}\sqrt{\prod_{t_0 \leq t_k < t}\left(\frac{h_k}{g_k^*}\right)^{-1} \frac{p(t)}{\eta(t)}}.$$

We have

$$\prod_{t_0 \le t_k < t} \left(\frac{h_k}{g_k^*}\right) \frac{V^2(t)}{\eta(t)p(t)} + \frac{T_\alpha(\eta(t))}{\eta(t)} V(t) \le \prod_{t_0 \le t_k < t} \left(\frac{h_k}{g_k^*}\right)^{-1} \frac{T_\alpha^2(\eta(t))p(t)}{4\eta(t)}.$$

Thus

$$T_\alpha(V(t)) \le - \prod_{t_0 \le t_k < t} \left(\frac{h_k}{g_k^*}\right)^{-1} \left[k_0\eta(t)q(t) - \frac{T_\alpha^2(\eta(t))p(t)}{4\eta(t)}\right].$$

Integrating both sides from t_1 to t, we have

$$V(t) \le V(t_1) - \int_{t_1}^{t} \prod_{t_0 \le t_k < t} \left(\frac{h_k}{g_k^*}\right)^{-1} \left[k_0 s^{\alpha-1}\eta(s)q(s) - \frac{s^{\alpha-1}T_\alpha(\eta(s))^2 p(s)}{4\eta(s)}\right] ds.$$

Letting $t \to \infty$, from (14), we have $\lim_{t\to\infty} V(t) = -\infty$, which is a contradiction. If case (II) holds, then using Lemma 2.4, we have $\lim_{t\to\infty} z(t) = 0$. Since $0 < Y(t) < z(t)$ on (t_1, ∞), we get $\lim_{t\to\infty} u(x,t) = 0$. The proof of the theorem is complete. □

Next we obtain some new asymptotic results for (1) and (2), by using integral average condition of Philos type [16]. Let $D = \{(t,s) : t_0 \le s \le t\}, H \in C^\alpha(D, \mathbb{R})$. If $H \in \mathbb{H}$, then $H(t,t) = 0$ and $H(t,s) > 0$ for $t > s$ and $h \in L_{loc}(D, \mathbb{R})$ such that

$$\frac{\partial H(t,s)}{\partial t} = h(t,s)\sqrt{H(t,s)}, \quad \frac{\partial H(t,s)}{\partial s} = -h(t,s)\sqrt{H(t,s)}.$$

Theorem 3.2. *Assume that conditions (C_1)-(C_5) hold and there exist $\phi(t), \psi(t) \in C^\alpha([0,\infty),(0,+\infty))$, if*

$$\limsup_{t\to+\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \prod_{t_0 \le t_k < s} \left(\frac{h_k}{g_k^*}\right)^{-1} k_0\eta(s)q(s)\psi(s) \Big[(1-\alpha)s^{-\alpha}\sqrt{H(t,s)}$$

$$-s^{1-\alpha}h(t,s) + s^{1-\alpha}\sqrt{H(t,s)}\frac{\psi'(s)}{\psi(s)}\Big]^2 ds = +\infty, \tag{18}$$

then every solution of the boundary value problem (1) *and* (2) *is oscillatory or converges to zero as $t \to \infty$.*

Proof. Assume that the boundary value problem (1) and (2) has a non oscillatory solution $u(x,t)$. Without loss of generality, assume that $u(x,t) > 0, (x,t) \in \Omega \times [0,+\infty)$. As in the proof of the Theorem 3.1, we obtain

$$T_\alpha(V(t)) \le - \prod_{t_0 \le t_k < t} \left(\frac{h_k}{g_k^*}\right) \frac{V^2(t)}{\eta(t)p(t)} + \frac{T_\alpha(\eta(t))}{\eta(t)} V(t) - \prod_{t_0 \le t_k < t} \left(\frac{h_k}{g_k^*}\right)^{-1} k_0\eta(t)q(t).$$

Multiplying the above inequality by $H(t,s)\psi(s)$ for $t \geq s \geq K$, and integrating from K to t, we have

$$\int_K^t T_\alpha(V(t))H(t,s)\psi(s)ds \leq -\int_K^t \prod_{t_0 \leq t_k < s} \left(\frac{h_k}{g_k^*}\right) \frac{V^2(s)}{\eta(s)p(s)} H(t,s)\psi(s)ds + \int_K^t \frac{T_\alpha(\eta(s))}{\eta(s)} V(t)H(t,s)\psi(s)ds - \int_K^t \prod_{t_0 \leq t_k < s} \left(\frac{h_k}{g_k^*}\right)^{-1} k_0\eta(s)q(s)H(t,s)\psi(s)ds,$$

Consider

$$\begin{aligned} &\int_K^t \prod_{t_0 \leq t_k < t} \left(\frac{h_k}{g_k^*}\right)^{-1} k_0\eta(s)q(s)H(t,s)\psi(s)ds \\ &\leq H(t,K)\psi(K)T^{1-\alpha}V(K) + \int_K^t \Big[(1-\alpha)s^{-\alpha}H(t,s)\psi(s) \\ &-s^{1-\alpha}h(t,s)\sqrt{H(t,s)}\psi(s)+s^{1-\alpha}H(t,s)\psi'(s)-\frac{T_\alpha(\eta(s))}{\eta(s)}H(t,s)\psi(s)\Big]V(s)ds \\ &-\int_K^t \prod_{t_0 \leq t_k < s} \left(\frac{h_k}{g_k^*}\right) \frac{V^2(s)}{\eta(s)p(s)} H(t,s)\psi(s)ds. \end{aligned} \tag{19}$$

From this,

$$\begin{aligned} &\int_{t_0}^t \prod_{t_0 \leq t_k < s} \left(\frac{h_k}{g_k^*}\right)^{-1} \Big(k_0\eta(s)q(s)H(t,s)\psi(s) \\ &- \frac{\eta(s)p(s)\psi(s)}{4}\Big[(1-\alpha)s^{-\alpha}\sqrt{H(t,s)} - s^{1-\alpha}h(t,s) \\ &+ s^{1-\alpha}\sqrt{H(t,s)}\frac{\psi'(s)}{\psi(s)}\Big]^2\Big)ds \leq V(t)H(t,K)\psi(K)K^{1-\alpha}. \end{aligned} \tag{20}$$

Letting $t \to \infty$, we have

$$\begin{aligned} &\limsup_{t\to+\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \prod_{t_0 \leq t_k < s} \left(\frac{h_k}{g_k^*}\right)^{-1} \Big\{k_0\eta(s)q(s)H(t,s)\psi(s) \\ &- \frac{\eta(s)p(s)\psi(s)}{4}\Big[(1-\alpha)s^{-\alpha}\sqrt{H(t,s)} - s^{1-\alpha}h(t,s) \\ &+ s^{1-\alpha}\sqrt{H(t,s)}\frac{\psi'(s)}{\psi(s)}\Big]^2\Big\}ds < +\infty \end{aligned} \tag{21}$$

which is a contradiction with (18).

Let $z(t)$ satisfies case (II) of Lemma 2.3, then using Lemma 2.4, we have $\lim_{t\to\infty} z(t) = 0$. Since $0 < Y(t) < z(t)$ on (t_1,∞), we get $\lim_{t\to\infty} u(x,t) = 0$. The proof of the theorem is complete. □

4. Example

In this section, we present an example to illustrate our main results established in Section 3.

Example 4.1. Consider the following impulsive neutral partial differential equation is of the form

$$\left.\begin{aligned}
&\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}\left[\sqrt{t}\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}\left(u(x,t)+\frac{1}{3}u\left(x,\frac{t}{2}\right)\right)\right]+\frac{2}{15}u\left(x,\frac{t}{3}\right)\\
&=\frac{2}{3}\Delta u(x,t)+\frac{1}{5}\Delta u\left(x,\frac{t}{3}\right)+F(x,t),\ t\neq 2^k,(x,t)\in\Omega\times\mathbb{R}_+\equiv G,\\
&u(x,t_k^+)=\frac{k}{k+1}u(x,2^k),\\
&\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}u(x,t_k^+)=\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}u(x,2^k),\quad k=1,2,\cdots,
\end{aligned}\right\}\tag{22}$$

for $(x,t)\in(0,\pi)\times\mathbb{R}_+$, with the boundary condition

$$u(0,t)=u(\pi,t)=0,\quad (x,t)\in\partial\Omega\times\mathbb{R}_+.\tag{23}$$

Here $p(t)=\sqrt{t}$, $\alpha=1/2$, $f(u)=2u$, $q(t)=\frac{1}{15}$, $a(t)=\frac{2}{3}$, $b_1(t)=\frac{1}{5}$, $c(t)=1/3$, $\sigma(t)=\frac{t}{2}$, $\delta(t)=\rho_1(t)=\frac{t}{3}$, $F(x,t)=\frac{5}{3}(1+\frac{1}{\sqrt{t}})\frac{\sin x}{t}$. Let $g_k=g_k^*=\frac{k}{k+1}$, $h_k=h_k^*=1$, $t_0=1$, $t_k=2^k$, $\eta(t)=\frac{t}{2}$, $\epsilon=\frac{1}{2}$. Then hypotheses $(C_1)-(C_5)$ hold, and moreover

$$\begin{aligned}
\lim_{t\to+\infty}\int_{t_0}^{t}\prod_{t_0\leqslant t_k<s}\left(\frac{h_k}{g_k^*}\right)^{-1}ds &= \int_1^{+\infty}\prod_{1<t_k<s}\frac{k}{k+1}ds\\
&= \int_1^{t_1}\prod_{1<t_k<s}\frac{k}{k+1}ds+\int_{t_1^+}^{t_2}\prod_{1<t_k<s}\frac{k}{k+1}ds+\cdots\\
&= 1+\frac{1}{2}\times 2+\frac{1}{2}\times\frac{2}{3}\times 2^2+\cdots\\
&= \sum_{n=0}^{\infty}\frac{2^n}{n+1}=+\infty.
\end{aligned}$$

Thus,

$$\int_1^{+\infty}\prod_{1<t_k<s}\frac{k}{k+1}\left[\frac{s}{90}-\frac{s^{1/2}}{4}\right]ds=+\infty.$$

Here all the conditions of Theorem 3.1 are satisfied. Hence every solution $u(x,t)$ of the problem (22) and (23) is either oscillatory or converges to zero as $t\to\infty$. In fact $u(x,t)=\frac{\sin x}{t}$ is such a solution.

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