# Polynomial form fuzzy numbers and their application in linear programming with fuzzy variables

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**Abstract.** In this paper, a special class of fuzzy numbers, whose members are called polynomial form fuzzy numbers, is considered. Some properties of its members are presented. Furthermore, linear programming problems with 2-degree polynomial form fuzzy variables are studied. A method using a linear ranking function is proposed to solve the problem. The method generalizes the known methods based on linear ranking functions to solve linear programming problems with trapezoidal variables. To illustrate the method, a transportation problem with fuzzy values of the demands and the supplies is solved and the obtained results are discussed.

**Keywords:** polynomial form fuzzy numbers, fuzzy linear programming, fuzzy variables, ranking function, transportation problem.

## 1. Introduction

Fuzzy set theory has been applied to many disciplines such as control theory [7, 13], management sciences [14], and mathematical modeling [14]. In many applications, it is convenient to work with the same type of fuzzy numbers. Many authors have restricted attention to some specific class of fuzzy numbers, such as the class of triangular or trapezoidal fuzzy numbers, to use its many properties [6, 8, 10, 12]. There are many works which authors have tried to approximate a fuzzy number by a simpler one [1, 4, 5]. Obviously, if we use a defuzzification rule which replaces a fuzzy number by a crisp one, we generally loose too much important information. Interval approximations are also considered for fuzzy numbers. For instance, in [5], a fuzzy computation problem is converted into an interval arithmetic problem. But, in this case, we also loose the concept of fuzzy centrality.

This paper contains two main parts. In the first part, we consider a class of fuzzy numbers, called polynomial form fuzzy numbers (PFNs), which is also studied by some authors [1, 2, 3, 16, 20]. We present some new algebraic and analytical properties of the class. For instance, we prove that the class of polynomial form fuzzy numbers is a vector space on the real field and also, it is dense in the set of fuzzy numbers. The class also enjoys many other properties. So it can be a convenient candidate to approximate fuzzy numbers to its members.

The second part is about a method for solving a specific class of fuzzy programming problems. Let us first review the literature in this field. The concept of fuzzy mathematical programming was first proposed by Tanaka et al. [15]. The first formulation of fuzzy linear programming (FLP) is proposed by Zimmermann [22]. Afterwards, many authors have considered various kinds of FLP problems and have proposed several approaches for solving these problems [6, 12]. In fact, one of most convenient methods is based on the concept of comparison of fuzzy numbers by using linear ranking functions. At first, Maleki et al. in [9] proposed use of linear ranking function for solving fuzzy programming. Subsequently, many authors applied the concept of ranking functions in linear programming [10, 11].

In the second part, we consider a linear programming problem with 2-degree polynomial form fuzzy variables. Using a linear ranking function, we present a method to convert the problem into one with crisp variables. Since any trapezoidal fuzzy number is a PFN of degree 1, the proposed method is a generalization of the known methods to solve linear programming problems with trapezoidal variables by using ranking functions [8, 10].

The structure of the present paper is as follows: In Section 2, we give some preliminaries and notions. Definition of a PFN and its properties are presented in Section 3. In Section 4, we consider a linear programming problem with 2-degree polynomial form fuzzy variables and propose a method to solve the problem. In this section, we also solve a transportation problem with fuzzy values of the demands and the supplies to illustrate the proposed method. Finally, we conclude in Section 5.

## 2. Preliminaries

In this section, we present a review of the basic terminology used throughout this paper.

Let $X$ denote a universal set. A fuzzy subset $\tilde{a}$ of $X$ is defined by its membership function as follows:

$$\tilde{a} : X \to [0, 1],$$

which assigns to each element $x \in X$ a real number $\tilde{a}(x)$ where the value of $\tilde{a}(x)$ represents the degree of membership of $x$ in $\tilde{a}$. The fuzzy set $\tilde{a}$ on $X$ is said to be normal if there exists $x_0 \in X$ such that $\tilde{a}(x_0) = 1$. For $\alpha \in (0, 1]$, the $\alpha$-cut of a fuzzy set $\tilde{a}$ is the crisp set

$$\tilde{a}_\alpha = \{x \in X : \tilde{a}(x) \geq \alpha\}.$$

The following theorem states fuzzy sets can be obtained directly from its $\alpha$-cuts.

**Theorem 2.1** (Theorem 2.7 in [17]). *For every fuzzy set $\tilde{a}$,*

$$\tilde{a} = \bigcup_{\alpha \in (0,1]} \alpha \tilde{a}_\alpha.$$

is called a fuzzy number. The set of all fuzzy numbers is denoted by $\mathcal{F}(\mathbb{R})$. A fuzzy vector $\tilde{\mathbf{a}}$ is a member of $\mathcal{F}^n(\mathbb{R})$.

A fuzzy number $\tilde{a}$ is said to be an LR fuzzy number whenever its core is a singleton. We say that a fuzzy number $\tilde{a}$ is nonnegative, denoted by $\tilde{a} \geq 0$, if its support contains no negative number. A special type of fuzzy numbers is the linear trapezoidal fuzzy number (TFN). A TFN $\tilde{a}$ is denoted by $(a^1, a^2, a^3, a^4)$ where $a^1 \leq a^2 \leq a^3 \leq a^4$ and its membership function is as follows:

$$\tilde{a}(x) = \begin{cases} \frac{x-a^1}{a^2-a^1} & x \in [a^1, a^2], \\ 1 & x \in (a_2, a_3), \\ \frac{x-a^4}{a^3-a^4} & x \in [a^3, a^4], \\ 0 & \text{otherwise.} \end{cases}$$

Since any real number $c$ and any interval $[a, b]$ can be written respectively as $c = (c, c, c, c)$ and $[a, b] = (a, a, b, b)$, it is obvious that TFNs are an extension of real numbers and intervals. The LR case of a trapezoidal fuzzy number, in which $a_2 = a_3$, is called a triangular fuzzy number.

Let $\tilde{a} = (a^1, a^2, a^3, a^4)$ and $\tilde{b} = (b^1, b^2, b^3, b^4)$ be TFNs and $c \in \mathbb{R}$. The following formulas for addition of two TFNs and multiplication of a TFN by a scalar are drawn from the extension principle of Zadeh [21].

- $c\tilde{a} = (ca^1, ca^2, ca^3, ca^4), \qquad \text{if } c \geq 0;$
- $c\tilde{a} = (ca^4, ca^3, ca^2, ca^1), \qquad \text{if } c < 0;$
- $\tilde{a} + \tilde{b} = (a^1 + b^1, a^2 + b^2, a^3 + b^3, a^4 + b^4);$
- $\tilde{a} - \tilde{b} = (a^1 - b^4, a^2 - b^3, a^3 - b^2, a^4 - b^1).$

When we use fuzzy numbers for practical applications, it is sometimes needed to compare them. There exist several methods to compare fuzzy numbers. Most convenient one of these methods is based on the concept of comparing fuzzy numbers by ranking functions [9]. In fact, an efficient approach for ordering the elements of $\mathcal{F}(\mathbb{R})$ is to define a ranking function $\mathcal{R} : \mathcal{F}(\mathbb{R}) \to \mathbb{R}$ which maps any fuzzy number into the real line. Let us define an order on fuzzy numbers by a ranking function $\mathcal{R}$. Suppose $\tilde{a}$ and $\tilde{b}$ are two fuzzy numbers. $\tilde{a}$ is less than or equal to $\tilde{b}$, denoted by $\tilde{a} \precsim \tilde{b}$, if the value of its ranking function is less than or equal to that of $\tilde{b}$, i.e. $\mathcal{R}(\tilde{a}) \leq \mathcal{R}(\tilde{b})$. Similarly, $\tilde{a}$ and $\tilde{b}$ in $\mathcal{F}(\mathbb{R})$ are equal, denoted by $\tilde{a} \approx \tilde{b}$, whenever the values of their ranking functions are the same, i.e. $\mathcal{R}(\tilde{a}) = \mathcal{R}(\tilde{b})$.

## 3. Polynomial form fuzzy numbers

In this section, we introduce polynomial form fuzzy numbers and present some properties of them.

**Definition 3.1** ([1, 2])**.** *A fuzzy number $\tilde{a}$ with $\alpha$-cut $\tilde{a}_\alpha = [p^-_{\tilde{a}}(\alpha), p^+_{\tilde{a}}(\alpha)]$, $\alpha \in (0,1]$, is $k$-degree polynomial form if $p^-_{\tilde{a}}$ and $p^+_{\tilde{a}}$ are polynomials of degree at most $k$.*

It is obvious that any trapezoidal fuzzy number is a polynomial form fuzzy number of degree 1. Let $\mathcal{PF}^{\mathrm{k}}(\mathbb{R})$ be the set of all $k$-degree polynomial form fuzzy numbers and $\mathcal{PF}(\mathbb{R})$ be the set of all polynomial form fuzzy numbers. Obviously, we have $\mathcal{PF}^{\mathrm{k}}(\mathbb{R}) \subseteq \mathcal{PF}^{\mathrm{k+1}}(\mathbb{R})$ and $\mathcal{PF}(\mathbb{R}) = \cup_{k=0}^{\infty} \mathcal{PF}^{\mathrm{k}}(\mathbb{R})$. The following lemma is about the arithmetic of polynomial form fuzzy numbers.

**Lemma 3.1.** *Let $\tilde{a} \in \mathcal{PF}^k(\mathbb{R})$, $\tilde{b} \in \mathcal{PF}^l(\mathbb{R})$ and $c \in \mathbb{R}\backslash\{0\}$. The following statements hold.*

($a$) *$\tilde{a} + \tilde{b}$ is a polynomial form fuzzy number of degree* $\max\{k, l\}$.

($b$) *$c \cdot \tilde{a}$ is a polynomial form fuzzy number of degree $k$.*

($c$) *If $\tilde{a} \geq 0$, then $\tilde{a} \times \tilde{b}$ is a polynomial form fuzzy number of degree $k + l$.*

($d$) *The fuzzy number $\tilde{a}$ is determined uniquely by its $(k + 1)$ $\alpha$-cuts.*

**Proof.** By the interval arithmetic, for every $\alpha \in [0,1]$, we know that

$$\begin{aligned}
&(\tilde{a} \odot \tilde{b})_\alpha = [p^-_{\tilde{a}\odot\tilde{b}}(\alpha), p^+_{\tilde{a}\odot\tilde{b}}(\alpha)] \\
&= [\min\{p^-_{\tilde{a}}(\alpha) \odot p^-_{\tilde{b}}(\alpha), p^+_{\tilde{a}}(\alpha) \odot p^-_{\tilde{b}}(\alpha), p^-_{\tilde{a}}(\alpha) \odot p^+_{\tilde{b}}(\alpha), p^+_{\tilde{a}}(\alpha) + p^+_{\tilde{b}}(\alpha)\}, \\
&\ \max\{p^-_{\tilde{a}}(\alpha) \odot p^-_{\tilde{b}}(\alpha), p^+_{\tilde{a}}(\alpha) \odot p^-_{\tilde{b}}(\alpha), p^-_{\tilde{a}}(\alpha) \odot p^+_{\tilde{b}}(\alpha), p^+_{\tilde{a}}(\alpha) + p^+_{\tilde{b}}(\alpha)\}],
\end{aligned}$$

where $\odot \in \{+, -, \times\}$. Therefore,

$$\begin{aligned}
&[p^-_{\tilde{a}+\tilde{b}}(\alpha), p^+_{\tilde{a}+\tilde{b}}(\alpha)] = [p^-_{\tilde{a}}(\alpha) + p^-_{\tilde{b}}(\alpha), p^+_{\tilde{a}}(\alpha) + p^+_{\tilde{b}}(\alpha)], \\
&[p^-_{c\tilde{a}}(\alpha), p^+_{c\tilde{a}}(\alpha)] = [cp^-_{\tilde{a}}(\alpha), cp^+_{\tilde{a}}(\alpha)] \quad \forall c \geq 0, \\
&[p^-_{c\tilde{a}}(\alpha), p^+_{c\tilde{a}}(\alpha)] = [cp^+_{\tilde{a}}(\alpha), cp^-_{\tilde{a}}(\alpha)] \quad \forall c < 0, \\
&[p^-_{\tilde{a}\times\tilde{b}}(\alpha), p^+_{\tilde{a}\times\tilde{b}}(\alpha)] = [p^-_{\tilde{a}}(\alpha) \times p^-_{\tilde{b}}(\alpha), p^+_{\tilde{a}}(\alpha) \times p^+_{\tilde{b}}(\alpha)] \quad \forall \tilde{a} \geq 0.
\end{aligned}$$

These together with Theorem 2.1 complete the proofs of (a), (b) and (c). For proving (d), note that the $\alpha$-cut of any $\tilde{a} \in \mathcal{PF}^{\mathrm{k}}(\mathbb{R})$ is $[p^-_{\tilde{a}}(\alpha), p^+_{\tilde{a}}(\alpha)]$ in which $p^-_{\tilde{a}}(\alpha)$ and $p^+_{\tilde{a}}(\alpha)$ are polynomials of degree $k$. Hence, if $k+1$ number of $\alpha$-cuts of $\tilde{a}$ are given, then we can obtain uniquely $p^-_{\tilde{a}}(\alpha)$ and $p^+_{\tilde{a}}(\alpha)$ by interpolating endpoints of $\alpha$-cuts. This together with Theorem 2.1 imply the result. $\square$

From Lemma 3.1, we imply that $\mathcal{PF}^{\mathrm{k}}(\mathbb{R})$ and $\mathcal{PF}(\mathbb{R})$ are vector spaces over the field $\mathbb{R}$. Using this fact, any $\tilde{a} \in \mathcal{PF}^{\mathrm{k}}(\mathbb{R})$ with $\alpha$-cut $\tilde{a}_\alpha = [p^-_{\tilde{a}}(\alpha), p^+_{\tilde{a}}(\alpha)]$ can be expressed in the coordinate representation as follows:

$$\tilde{a} = (a^-_0, a^-_1, \ldots, a^-_k, a^+_0, a^+_1, \ldots, a^+_k) \in \mathbb{R}^{2k+2}$$

in which

$$p^-_{\tilde{a}}(\alpha) = a^-_0 + a^-_1\alpha + a^-_2\alpha^2 + \ldots + a^-_k\alpha^k,$$
$$p^+_{\tilde{a}}(\alpha) = a^+_0 + a^+_1\alpha + a^+_2\alpha^2 + \ldots + a^+_k\alpha^k.$$

Therefore, we can imagine that $\mathcal{PF}^{\mathrm{k}}(\mathbb{R})$ is a vector subspace of $\mathbb{R}^{2k+2}$.

There exist various methods based on $\alpha$-cuts to rank fuzzy numbers (for details see [17]). Yager in [19] proposed a ranking function which is based on $\alpha$-cuts. For any fuzzy number $\tilde{a}$, this function is the integral of the mean of its $\alpha$-cuts. In order to compare polynomial form fuzzy numbers, we used the ranking function introduced by Yager [19].

**Definition 3.2.** *Let $\tilde{a} \in \mathcal{F}(\mathbb{R})$ with $\alpha$-cut $\tilde{a}_\alpha = [p^-_{\tilde{a}}(\alpha), p^+_{\tilde{a}}(\alpha)]$. Yager's ranking function $\mathcal{R} : \mathcal{F}(\mathbb{R}) \to \mathbb{R}$ is defined as follows:*

$$\mathcal{R}(\tilde{a}) = \frac{1}{2}\int_0^1 (p^-_{\tilde{a}}(\alpha) + p^+_{\tilde{a}}(\alpha))d\alpha. \tag{1}$$

We say that the fuzzy number $\tilde{a}$ is less than or equal to the fuzzy number $\tilde{b}$, denoted by $\tilde{a} \leq_{\mathcal{R}} \tilde{b}$, if and only if $\mathcal{R}(\tilde{a}) \leq \mathcal{R}(\tilde{b})$. Similarly, $\geq_{\mathcal{R}}$ and $=_{\mathcal{R}}$ are defined. It is obvious that the ranking function $\mathcal{R}$ has the property of linearity, i.e.

$$\mathcal{R}(c\tilde{a} + \tilde{b}) = c\mathcal{R}(\tilde{a}) + \mathcal{R}(\tilde{b}), \quad \forall \tilde{a}, \tilde{b} \in \mathcal{F}(\mathbb{R}),\ \forall c \in \mathbb{R}. \tag{2}$$

**Lemma 3.2.** *Suppose that $\tilde{a} = (a^-_0, a^-_1, \ldots, a^-_k, a^+_0, a^+_1, \ldots, a^+_k) \in \mathcal{PF}^k(\mathbb{R})$. Then*

$$\mathcal{R}(\tilde{a}) = \sum_{i=0}^{k} \frac{a^-_i + a^+_i}{2i+2}.$$

**Proof.**

$$\mathcal{R}(\tilde{a}) = \frac{1}{2}\int_0^1 (p^-_{\tilde{a}}(\alpha) + p^+_{\tilde{a}}(\alpha))d\alpha = \frac{1}{2}\int_0^1 \sum_{i=0}^{k}(a^-_i\alpha^i + a^+_i\alpha^i)d\alpha = \sum_{i=0}^{k} \frac{a^-_i + a^+_i}{2i+2}.$$

□

### 3.1 Approximating fuzzy numbers

In many applications of fuzzy mathematics it is more convenient to work with the fuzzy numbers belonging to one specific class due to use of its properties. If a problem has not this property, then one can approximate its fuzzy parameters with members of a class. The class $\mathcal{PF}(\mathbb{R})$ is suitable because it provides us many properties (see [2, 3]). In [1], the authors approximate fuzzy numbers with polynomial form fuzzy numbers by converting them into some linear programming problem. Here, using the well-known Weierstrass Theorem [18], we also present a theoretical result. The Weierstrass Theorem [18] shows that the continuous real-valued functions on a compact interval can be uniformly approximated by polynomials. At first, it is needed to introduce the notion of convergence for a sequence of fuzzy numbers.

**Definition 3.3.** *A sequence of fuzzy numbers* $\{\tilde{p}_n\}$ *with* $(\tilde{p}_n)_\alpha = [p^-_{\tilde{p}_n}(\alpha), p^+_{\tilde{p}_n}(\alpha)]$ *is convergent to* $\tilde{a}$ *with* $\tilde{a}_\alpha = [p^-_{\tilde{a}}(\alpha), p^+_{\tilde{a}}(\alpha)]$, *if the sequences of functions* $\{p^-_{\tilde{p}_n}\}$ *and* $\{p^+_{\tilde{p}_n}\}$ *are respectively convergent to* $p^-_{\tilde{a}}$ *and* $p^+_{\tilde{a}}$.

**Theorem 3.1.** *Let* $\tilde{a}$ *be a fuzzy number with* $\alpha$*-cut* $\tilde{a}_\alpha = [p^-_{\tilde{a}}(\alpha), p^+_{\tilde{a}}(\alpha)]$ *where the functions* $p^-_{\tilde{a}}$ *and* $p^+_{\tilde{a}}$ *are continues on* $[0, 1]$. *Then there is a sequence* $\{\tilde{p}_n\} \subseteq \mathcal{PF}(\mathbb{R})$ *that converges to* $\tilde{a}$.

**Proof.** Since the functions $p^-_{\tilde{a}}$ and $p^+_{\tilde{a}}$ are continues, according to the Weierstrass Theorem, it follows that there exist sequences $\{p^-_n\}$ and $\{p^+_n\}$ such that $p^-_n$ and $p^+_n$ are convergent to $p^-_{\tilde{a}}$ and $p^+_{\tilde{a}}$, respectively. We know that the functions $p^-_{\tilde{a}}$ and $p^+_{\tilde{a}}$ are increasing and decreasing, respectively. Then we can assume that the members of $\{p^-_n\}$ and $\{p^+_n\}$ are also increasing and decreasing, respectively. Define the fuzzy number $\tilde{p}_n$, $n \in \mathbb{N}$, with $\alpha$-cut $(\tilde{p}_n)_\alpha = [p^-_n(\alpha), p^+_n(\alpha)]$. Obviously, $\tilde{p}_n \in \mathcal{PF}(\mathbb{R})$ and the sequence $\{\tilde{p}_n\}$ converges to $\tilde{a}$. □

The following immediate result from Theorem 3.1 is so important to motivate use of polynomial form fuzzy numbers.

**Corollary 3.1.** *Polynomial form fuzzy numbers are dense in the set of continuous fuzzy numbers.*

## 4. Linear programming with fuzzy 2-degree polynomial form variables

In this section we consider a linear programming problem with polynomial form fuzzy variables formulated as follows:

$$\min \quad \mathbf{c}^T\tilde{\mathbf{x}}, \tag{3a}$$

$$\text{s.t.} \quad A\tilde{\mathbf{x}} =_{\mathcal{R}} \tilde{\mathbf{b}}, \tag{3b}$$

$$\mathbf{l} \le \tilde{\mathbf{x}} \le \mathbf{u}, \tag{3c}$$

where $A = (a_{ij}) \in \mathbb{R}^{m\times n}$, $\tilde{\mathbf{b}} = [\tilde{b}_1, \dots, \tilde{b}_m]^T \in (\mathcal{PF}^2(\mathbb{R}))^m$, $\mathbf{c} = [c_1, \dots, c_n]^T, \mathbf{l} = [l_1, \dots, l_n]^T \in \mathbb{R}^n, \mathbf{u} = [u_1, \dots, u_n]^T \in \mathbb{R}^n$ are given and the fuzzy vector $\tilde{\mathbf{x}} = [\tilde{x}_1, \dots, \tilde{x}_n]^T \in (\mathcal{PF}^2(\mathbb{R}))^n$ is to be determined.

The constraints (3b) are in the form of the soft equalities, namely, their left side fuzzy number is approximately equal to the right side one. Depending on the problem type, one can replace them with constraints in the form of the inequalities $\le_{\mathcal{R}}$ and $\ge_{\mathcal{R}}$. The constraints (3c) containing the hard inequality $\le$ are called the bound constraints. These constraints guarantee that the support of $\tilde{x}_j$, $j = 1, \dots, n$, belongs to $[l_j, u_j]$. To define the soft equalities and inequalities, we use a ranking function $\mathcal{R} : \mathcal{PF}^2(\mathbb{R}) \to \mathbb{R}$ defined as

$$\mathcal{R}(\tilde{x}) = r^-_0 x^-_0 + r^-_1 x^-_1 + r^-_2 x^-_2 + r^+_0 x^+_0 + r^+_1 x^+_1 + r^+_2 x^+_2, \tag{4}$$

for every $\tilde{x} = (x^-_0, x^-_1, x^-_2, x^+_0, x^+_1, x^+_2) \in \mathcal{PF}^2(\mathbb{R})$ in which $r^-_1, r^-_2, r^-_3, r^+_1, r^+_2, r^+_3$ are nonnegative real numbers. It is easy to see that $\mathcal{R}$ has the linearity property, that is, $\mathcal{R}(\tilde{x}+c\tilde{y})=\mathcal{R}(\tilde{x})+c\mathcal{R}(\tilde{y})$ for every $\tilde{x},\tilde{y}\in\mathcal{PF}^2(\mathbb{R})$ and every $c\in\mathbb{R}$. A specific case of the function $\mathcal{R}$ is Yager's ranking function defined as

$$\mathcal{R}(\tilde{x})=\frac{1}{2}x_0^-+\frac{1}{4}x_1^-+\frac{1}{6}x_2^-+\frac{1}{2}x_0^++\frac{1}{4}x_1^++\frac{1}{6}x_2^+, \tag{5}$$

for every $\tilde{x}=(x_0^-,x_1^-,x_2^-,x_0^+,x_1^+,x_2^+)\in\mathcal{PF}^2(\mathbb{R})$ (see Lemma 3.1). From the viewpoint of the linear algebra, the ranking function $\mathcal{R}$ is a linear functional on the vector space $\mathcal{PF}^2(\mathbb{R})$.

To obtain an optimal solution of the problem (3), we will reduce the problem to a linear programming problem with crisp variables. Before stating details, we need some initial notions of the linear programming.

**Definition 4.1.** *A fuzzy vector $\tilde{\mathbf{x}}\in(\mathcal{PF}^2(\mathbb{R}))^n$ is called a fuzzy feasible solution to the problem (3), if $\tilde{\mathbf{x}}$ satisfies the constraints (3b)-(3c).*

**Definition 4.2.** *A fuzzy feasible solution $\tilde{\mathbf{x}}^*\in(\mathcal{PF}^2(\mathbb{R}))^n$ is an optimal solution of the problem (3) if*

$$\mathcal{R}(c\tilde{\mathbf{x}}^*)\le\mathcal{R}(c\tilde{\mathbf{x}})$$

*for any fuzzy feasible solution $\tilde{\mathbf{x}}\in(\mathcal{PF}^2(\mathbb{R}))^n$.*

Since any decision variable $\tilde{x}_j$, $j=1,2,\dots,n$, is a fuzzy 2-degree polynomial form number, it can be denoted by

$$\tilde{x}_j=((x_j)_0^-,(x_j)_1^-,(x_j)_2^-,(x_j)_0^+,(x_j)_1^+,(x_j)_2^+), \tag{6}$$

whose $\alpha$-cut is $(\tilde{x}_j)_\alpha=[(x_j)_0^-+(x_j)_1^-\alpha+(x_j)_2^-\alpha^2,(x_j)_0^++(x_j)_1^+\alpha+(x_j)_2^+\alpha^2]$ for every $\alpha\in[0,1]$. Based on the linearity property of the ranking function $\mathcal{R}$, notice that a feasible solution $\tilde{\mathbf{x}}^*\in(\mathcal{PF}^2(\mathbb{R}))^n$ is optimal if

$$\sum_{j=1}^n c_j\mathcal{R}(\tilde{x}_j^*)\le\sum_{j=1}^n c_j\mathcal{R}(\tilde{x}_j)$$

or equivalently,

$$\begin{aligned}&\sum_{j=1}^n c_j(r_0^-(x_j^*)_0^-+r_1^-(x_j^*)_1^-+r_2^-(x_j^*)_2^-+r_0^+(x_j^*)_0^++r_1^+(x_j^*)_1^++r_2^+(x_j^*)_2^+)\\ &\le\sum_{j=1}^n c_j(r_0^-(x_j)_0^-+r_1^-(x_j)_1^-+r_2^-(x_j)_2^-+r_0^+(x_j)_0^++r_1^+(x_j)_1^++r_2^+(x_j)_2^+),\end{aligned} \tag{7}$$

for every feasible solution $\tilde{\mathbf{x}}\in(\mathcal{PF}^2(\mathbb{R}))^n$. Similarly, the linearity of the ranking function $\mathcal{R}$ implies that a solution $\tilde{\mathbf{x}}\in(\mathcal{PF}^2(\mathbb{R}))^n$ satisfies the constraints (3b) if

$$\sum_{j=1}^n a_i\mathcal{R}(\tilde{x}_j)=\mathcal{R}(\tilde{b}_i),\quad i=1,\dots,m,$$

where $a_i$ is the $i$th row of the matrix $A$. On the other hand, it is notable that the variable $\tilde{x}_j$ satisfies its bound constraint if $l_j \leq (x_j)_0^-$ and $(x_j)_0^+ \leq u_j$. Hence, $\tilde{\mathbf{x}} \in (\mathcal{PF}^2(\mathbb{R}))^n$ is a feasible solution of the problem (3) if

$$\begin{aligned}
&\sum_{j=1}^{n} a_{ij}(r_0^-(x_j)_0^- + r_1^-(x_j)_1^- + r_2^-(x_j)_2^- + r_0^+(x_j)_0^+ + r_1^+(x_j)_1^+ + r_2^+(x_j)_2^+)\\
&= r_0^-(b_i)_0^- + r_1^-(b_i)_1^- + r_2^-(b_i)_2^- + r_0^+(b_i)_0^+ + r_1^+(b_i)_1^+ + r_2^+(b_i)_2^+ \quad i = 1, \dots, m,\\
&(x_j)_0^- \geq l_j \quad j = 1, 2, \dots, n,\\
&(x_j)_0^+ \leq u_j \quad j = 1, 2, \dots, n.
\end{aligned}$$

Therefore, solving the problem (3) is equivalent to solving the following optimization problem.

$$\min \sum_{j=1}^{n} c_j(r_0^-(x_j)_0^- + r_1^-(x_j)_1^- + r_2^-(x_j)_2^- + r_0^+(x_j)_0^+ + r_1^+(x_j)_1^+ + r_2^+(x_j)_2^+), \tag{8a}$$

$$\text{s.t.} \sum_{j=1}^{n} a_{ij}(r_0^-(x_j)_0^- + r_1^-(x_j)_1^- + r_2^-(x_j)_2^- + r_0^+(x_j)_0^+ + r_1^+(x_j)_1^+ + r_2^+(x_j)_2^+) \tag{8b}$$

$$= r_0^-(b_i)_0^- + r_1^-(b_i)_1^- + r_2^-(b_i)_2^- + r_0^+(b_i)_0^+ + r_1^+(b_i)_1^+ + r_2^+(b_i)_2^+ \quad i = 1, \dots, m, \tag{8c}$$

$$\begin{aligned}
&(x_j)_0^- \geq l_j \quad j = 1, 2, \dots, n,\\
&(x_j)_0^+ \leq u_j \quad j = 1, 2, \dots, n,
\end{aligned} \tag{8d}$$

$$\tilde{x}_j = ((x_j)_0^-, (x_j)_1^-, (x_j)_2^-, (x_j)_0^+, (x_j)_1^+, (x_j)_2^+) \in \mathcal{PF}^2(\mathbb{R}) \quad j = 1, 2, \dots, n. \tag{8e}$$

The problem (8) is a linear programming problem with the additional conditions (8e). These conditions guarantee that a fuzzy 2-degree polynomial form number $\tilde{x}_j$ can be constructed from any $((x_j)_0^-, (x_j)_1^-, (x_j)_2^-, (x_j)_0^+, (x_j)_1^+, (x_j)_2^+) \in \mathbb{R}^6$. Note that if we eliminate the conditions (8e), then the obtained optimal solution may not be a member of $\mathcal{PF}^2(\mathbb{R})$ and consequently not be an optimal solution of the problem (3). Now we try that replace the conditions (8e) with some linear constraints.

**Lemma 4.1.** *$(x_0^-, x_1^-, x_2^-, x_0^+, x_1^+, x_2^+)$ is a member of $\mathcal{PF}^2(\mathbb{R})$ if and only if the following constraints hold.*

$$x_1^- \geq 0, \tag{9}$$

$$x_1^+ \leq 0, \tag{10}$$

$$x_1^- + 2x_2^- \geq 0, \tag{11}$$

$$x_1^+ + 2x_2^+ \leq 0, \tag{12}$$

$$x_0^- + x_1^- + x_2^- \leq x_0^+ + x_1^+ + x_2^+. \tag{13}$$

**Proof.** We know that $(x_0^-, x_1^-, x_2^-, x_0^+, x_1^+, x_2^+)$ is a member of $\mathcal{PF}^2(\mathbb{R})$ iff the polynomials $p^-(\alpha) = x_0^- + x_1^-\alpha + x_2^-\alpha^2$ and $p^+(\alpha) = x_0^+ + x_1^+\alpha + x_2^+\alpha^2$ are respectively nondecreasing and nonincreasing on $[0, 1]$, and equivalently, the values of the first derivatives of the functions $p^-(.)$ and $p^+(.)$ are respectively nonnegative and nonpositive on the unit interval $[0, 1]$. Since the first derivatives of the functions are linear, it is sufficient that the conditions (9)-(12) are satisfied. Furthermore, the condition (13) guarantees that the left spread always appear before the right spread. □

**Theorem 4.1.** *Solving the problem (3) is equivalent to solving the problem*

$$\min \sum_{j=1}^{n} c_j(r_0^-(x_j)_0^- + r_1^-(x_j)_1^- + r_2^-(x_j)_2^- + r_0^+(x_j)_0^+ + r_1^+(x_j)_1^+ + r_2^+(x_j)_2^+), \tag{14a}$$

$$s.t. \sum_{j=1}^{n} a_{ij}(r_0^-(x_j)_0^- + r_1^-(x_j)_1^- + r_2^-(x_j)_2^- + r_0^+(x_j)_0^+ + r_1^+(x_j)_1^+ + r_2^+(x_j)_2^+) \tag{14b}$$

$$= r_0^-(b_i)_0^- + r_1^-(b_i)_1^- + r_2^-(b_i)_2^- + r_0^+(b_i)_0^+ + r_1^+(b_i)_1^+ + r_2^+(b_i)_2^+ \quad i = 1, \ldots, m, \tag{14c}$$

$$(x_j)_0^- \geq l_j \quad j = 1, 2, \ldots, n, \tag{14d}$$

$$(x_j)_0^+ \leq u_j \quad j = 1, 2, \ldots, n, \tag{14e}$$

$$(x_j)_1^- \geq 0 \quad j = 1, \ldots, n, \tag{14f}$$

$$(x_j)_1^+ \leq 0 \quad j = 1, \ldots, n, \tag{14g}$$

$$(x_j)_1^- + 2(x_j)_2^- \geq 0 \quad j = 1, \ldots, n, \tag{14h}$$

$$(x_j)_1^+ + 2(x_j)_2^+ \leq 0 \quad j = 1, \ldots, n, \tag{14i}$$

$$(x_j)_0^- + (x_j)_1^- + (x_j)_2^- \leq (x_j)_0^+ + (x_j)_1^+ + (x_j)_2^+ \quad j = 1, \ldots, n, \tag{14j}$$

*where* $\tilde{x}_j = ((x_j)_0^-, (x_j)_1^-, (x_j)_2^-, (x_j)_0^+, (x_j)_1^+, (x_j)_2^+)$ *for* $i = 1, .., n$.

**Proof.** Based on Lemma 4.1 and the above argument, the proof is trivial. □

**Remark 4.1.** It is notable that any optimal solution $\tilde{\mathbf{x}}^* \in (\mathcal{PF}^2(\mathbb{R}))^n$ of the problem is an LR fuzzy number whenever the constraints (14j) are in the equality form.

**Remark 4.2.** If some constraints (3c) are in the form of $\geq_{\mathcal{R}}$ or $\leq_{\mathcal{R}}$, then the corresponding constraints (14b) are respectively in the form of $\geq$ and $\leq$.

A defect of the methods based on ranking functions for solving the linear programming problem with fuzzy variables is that the corresponding crisp linear programming problem has usually many optimal solutions because the number of its decision variables is several times greater than that of the corresponding fuzzy linear programming problem. As an instance, the problem (3) contains $n$ fuzzy variables while the problem (14) has $6 \times n$ crisp variables as well as some additional constraints. Therefore, we have to choose the optimal solutions of the crisp problem (14) more carefully to be able to construct the fuzzy solutions with more quality. A remedy is to we restrict the support of fuzzy variables. Let $s_{max}$ be the maximum length of the supports of the fuzzy right side numbers $\tilde{b}_i$, i.e.,

$$s_{max} = \max\{(b_i)_0^+ - (b_i)_0^- : i = 1, \ldots, m\}.$$

To obtain the fuzzy variables $\tilde{x}_j$ which are rather like right side numbers, we can add the constraints

$$(x_j)_0^+ - (x_j)_0^- \leq s_{\max} \quad j = 1, 2, \ldots, n, \tag{15}$$

to the problem (14) to restrict the length of the support of $\tilde{x}_j$'s.

| | Market 1 | Market 2 | Market 3 |
|---|---|---|---|
| Source 1 | 22 | 20 | 25 |
| Source 2 | 18 | 21 | 24 |

Table 1: The unit cost of shipment from sources to markets.

### 4.1 An applicable example

Here, we consider an instance of the well-known transportation problem in which the values of demands and supplies are polynomial form fuzzy numbers. Since the technological coefficients of the problem are 0 or 1, it follows that the decision variables have to take fuzzy values. Hence, this problem is an instance of the problems which can be formulated only with fuzzy variables.

Let us state the problem. A lumber company has two sources of wood and 3 markets where wood is demanded. The approximate quantity of wood available in the next month in the two sources of supply is $\tilde{78} = (76, 0, 2, 80, -1, -1)$ and $\tilde{69} = (67, 1.5, 0.5, 71, -1, -1)$ million board feet, respectively. The amount that can be sold at the 3 markets is approximately equal to $\tilde{28} = (26, 0, 2, 30, -1, -1)$, $\tilde{46} = (44.5, 1, 0.5, 48.5, -1.5, -1)$, and $\tilde{37} = (35, 1.5, 0.5, 39, -0.5, -1.5)$ million board feet, respectively. Figure 2 shows the fuzzy values of the demands and the supplies. The company currently transports all of the wood by trains whose capacity is 35 million board feet. The unit cost of shipment is described in Table 1. The management needs to decide to how ship the wood from the sources to the markets with minimum cost. The problem can be formulated as an instance of the problem (3) in the following form.

$$
\begin{aligned}
\min \quad & z = 22\tilde{x}_{11} + 20\tilde{x}_{12} + 25\tilde{x}_{13} + 18\tilde{x}_{21} + 21\tilde{x}_{22} + 24\tilde{x}_{23} \\
s.t. \quad & \tilde{x}_{11} + \tilde{x}_{12} + \tilde{x}_{13} \leq_{\mathcal{R}} \tilde{78}, \\
& \tilde{x}_{21} + \tilde{x}_{22} + \tilde{x}_{23} \leq_{\mathcal{R}} \tilde{69}, \\
& \tilde{x}_{12} + \tilde{x}_{22} =_{\mathcal{R}} \tilde{28}, \\
& \tilde{x}_{12} + \tilde{x}_{22} =_{\mathcal{R}} \tilde{46}, \\
& \tilde{x}_{12} + \tilde{x}_{22} =_{\mathcal{R}} \tilde{37}, \\
& 0 \leq \tilde{x}_{ij} \leq 35 \quad i = 1, 2, \ j = 1, 2, 3.
\end{aligned}
$$

Since the right side numbers are LR fuzzy numbers, it is natural that variables are also LR fuzzy number. So we converted the problem into the corresponding crisp problem with the constraints (14j) in the equality form. We applied Yager's ranking function defined as (5) to solve the problem. We added the constraints (15) to the problem for $s_{\max} = 4$. We used the software Matlab, version 2016a, to solve the crisp linear programming problem by 5 different algorithms named "interior point legacy", "interior point", "active set", "simplex", and "dual-simplex". The optimal solutions obtained are depicted in Figure 2. It is notable that the optimal solution obtained by the simplex algorithm has crisp values since the simplex algorithm , looks for an optimal basic solution, that is, an optimal solution with the most number of zero variables.

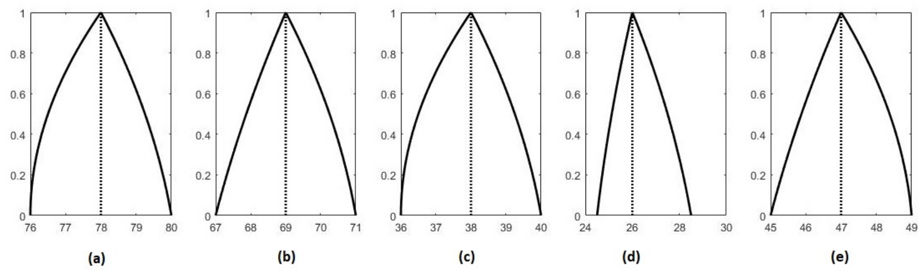


Figure 1: The fuzzy values of demands and supplies: (a) the supply of the first source (b) the supply of the second source; (c) the demand of the first market; (d) the demand of the second market; (e) the demand of the third market.

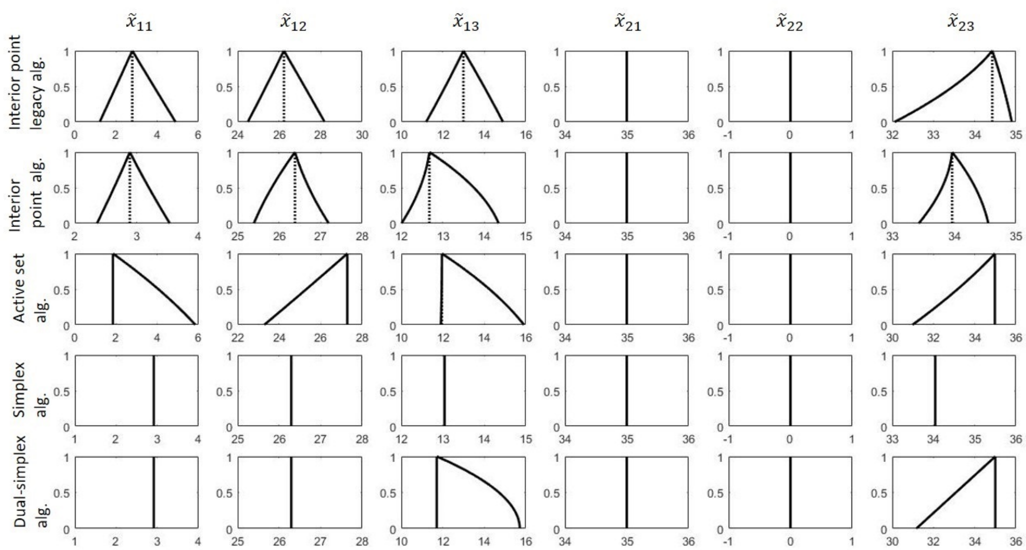


Figure 2: Optimal solutions obtained by Matlab.

## 5. Conclusion

In this paper, a special class of fuzzy numbers, whose members are called polynomial form fuzzy numbers, is investigated. Some properties of the class are presented. Specially, it is shown that its members are dense in the set of continuous fuzzy numbers. Furthermore, a linear programming problem with fuzzy

2-degree polynomial form variables is studied. To solve the problem, a method based on Yager's ranking function is proposed. Since any fuzzy trapezoidal number is also a fuzzy 2-degree polynomial form number, the proposed method generalizes the known methods using this ranking function to solve linear programming problems with trapezoidal variables [8].

Even though, the class of polynomial form fuzzy numbers contains many algebraic properties, it is less interested by authors (specially in the field of fuzzy optimization). As future works, we propose use of this class in different fields of fuzzy optimization such as fuzzy game theory and fuzzy regression.

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