More accurate Young, Heinz-Heron mean and Heinz inequalities for scalar and matrix

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Abstract. In this paper, we mainly give some refinements of Heinz mean-Heron mean inequality $F_{\alpha(v)}(a, b) \geq H_v(a, b)$ where $\alpha(v) = 1 - 4(v - v^2)$. More precisely, interpolation between them are established motivated by refinements of Young inequality. The matrix versions of these inequalities are also obtained in the last part.

Keywords: Young inequality, Heinz mean, Heron mean, matrix inequality.

1. Introduction

Nowadays, refinements, generalizations and extension to a multidimensional case of classical Young inequality, which states that if $a, b \geq 0$ and $v \in [0, 1]$, then

$$a\nabla_v b \geq a\sharp_v b \tag{1}$$

with equality if and only if $a = b$, have arisen researchers' attention. The weighted arithmetic and geometric means are defined, respectively, as follows $a\nabla_v b = (1 - v)a + vb$ and $a\sharp_v b = a^{1-v}b^v$. After discovering, the Young inequality was studied by numerous researchers, who either reproved it using various techniques, or applied and generalized it in many different directions.

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Replacing (v,b) by $(2v,\sqrt{ab})$ when $v\in[0,\frac{1}{2}]$ and (v,b) by $(2-2v,\sqrt{ab})$ when $v\in[\frac{1}{2},1]$ in (1) we got [1]

$$a\nabla_v b \ge a\sharp_v b + 2r_0(a\nabla b - a\sharp b), \tag{2}$$

where $r_0=\min\{v,1-v\}$.

Replacing (v,b), (v,a), (v,a) and (v,b) by $(2v,\sqrt{ab})$, $(1-2v,\sqrt{ab})$, $(2v-1,\sqrt{ab})$ and $(2-2v,\sqrt{ab})$ respectively when $v\in[0,\frac{1}{4}]$, $v\in(\frac{1}{4},\frac{1}{2}]$, $v\in(\frac{1}{2},\frac{3}{4}]$ and $v\in(\frac{3}{4},1]$ in (2) we got [2]

$$\begin{cases} a\nabla_v b \ge a\sharp_v b + 2v(a\nabla b - a\sharp b) + r_1(a + a\sharp b - 2a\sharp_{\frac{1}{4}} b), 0\le v\le \dfrac{1}{2} \\ a\nabla_v b \ge a\sharp_v b + 2(1-v)(a\nabla b - a\sharp b) + r_1(b + a\sharp b - 2a\sharp_{\frac{3}{4}} b), \dfrac{1}{2} < v \le 1 \end{cases}$$

where $r_1=\min\{r_0,1-r_0\}$.

Inequality (1) in multidimensional case has been obtained by M. Sababheh and D. Choi [3]

$$a\nabla_v b \ge a\sharp_v b + S_N(v;a,b),$$

where $S_N(v;a,b)=\sum_{j=1}^N s_j(v)\left(\sqrt[2^j]{a^{2^{j-1}-k_j(v)}b^{k_j(v)}} - \sqrt[2^j]{a^{2^{j-1}-k_j(v)-1}b^{k_j(v)+1}}\right)^2$, $s_j(v)=(-1)^{r_j(v)}2^{j-1}v+(-1)^{r_j(v)+1}[\frac{r_j(v)+1}{2}]$, $k_j(v)=[2^{j-1}v]$ and $r_j(v)=[2^j v]$. Here $[x]$ is the greatest integer less than or equal to x.

Although classical, the refinements of Young inequality is still of great interest to numerous authors.

M. Krnić [4] obtained that

$$a\nabla_v b \ge a\sharp_v b + 4v(1-v)(a\nabla b - a\sharp b) \tag{3}$$

when $(2v-1)(b-a)\ge 0$ and reverse inequality when $(2v-1)(b-a)\le 0$. Recently, M. Sababheh refined M. Krnić's result in [5] using the method of inductive quadratic interpolation.

Refining Young inequality or Young type inequalities can go forever, in the sense that one can find further refining terms for all refinements in the literature and extend it to matrix, norm and determinant inequality. We encourage the reader to figure different methods to obtain refinements of these inequalities.

So far the mentioned inequalities were involving a combination of a,b, i.e. $a\nabla_v b=(1-v)a+vb$. There exist some inequalities involving a combination of $a\nabla b, a\sharp b$. In the following some of them are listed.

Both means, Heinz and Heron means, interpolate between the geometric and arithmetic means, defined respectively for $a,b>0$ as follows

$$H_v(a,b)=\frac{a\sharp_v b + a\sharp_{1-v} b}{2} \quad \text{and} \quad F_v(a,b)=(1-v)a\sharp b + va\nabla b.$$

It follows from inequality (1) when $v=\frac{1}{2}$ that the Heinz and Heron means interpolate between geometric mean and arithmetic mean

$$a\sharp b \le H_v(a,b) \le a\nabla b \quad \text{and} \quad a\sharp b \le F_v(a,b) \le a\nabla b.$$

R. Bhatia [6] proved that the Heinz and Heron means satisfy the following inequality

$$F_{\alpha(v)}(a,b) \geq H_v(a,b), \tag{4}$$

where $\alpha(v) = 1 - 4(v - v^2)$.

One of the questions that arises from this work is the following. Does this inequality between means of positive numbers lead to further refinements?

F. Kittaneh and Y. Manasrah [7] proved that

$$a\nabla b \geq H_v(a,b) + r_0(a\nabla b - a\sharp b). \tag{5}$$

W. Liao and P. Long [8] refined the above result as follows

$$a\nabla b \geq H_v(a,b) + 2r_0(a\nabla b - a\sharp b) + r_1(a\nabla b + a\sharp b - a\sharp_{\frac{1}{4}} b - a\sharp_{\frac{3}{4}} b)$$

and

$$(a\nabla b)^2 \geq H_v(a,b)^2 + 2r_0((a\nabla b)^2 - (a\sharp b)^2) + 4r_1[(a\nabla b)^2 - (a\nabla b)(a\sharp b)].$$

Further refinements of Heinz inequality have been obtained by numerous researchers. In what follows, we mainly present some refinements of general Heinz inequality and Heinz-Heron mean inequality motivated by the Young-type inequalities presented as above.

The organization of the paper is as follows. We firstly present several refinements of Young inequality and Heinz-Heron mean inequality with a series of similar inequalities are then presented. In the last part of this paper, these refinements are applied to matrix case.

2. Main results

In this section we mainly present some inequalities for scalars and then we present several matrix inequalities.

2.1 Scalar results

In this part of the paper, we mainly show some scalar results. The main approach we used in proving the scalar results is some delicate and tricky computations.

Theorem 2.1. *If $a, b > 0$ and $v \in [0,1]$, then*

$$a\nabla_v b \geq a\sharp_v b + 4v(1-v)(a\nabla b - a\sharp b) \tag{6}$$

holds for $(\frac{b}{a})^{v-\frac{1}{2}} \geq 1$.

Proof. If $a, b > 0$ and $v \in [0, 1]$, the inequality (6) for $a = 1, b = t > 0$, becomes

$$1\nabla_v t \geq 1\sharp_v t + 4v(1-v)(1\nabla t - 1\sharp t).$$

Let

$$f(t) = 1 - v + vt - t^v - 4v(1-v)(\frac{1+t}{2} - \sqrt{t}).$$

Then

$$f'(t) = v - vt^{v-1} - 4v(1-v)(\frac{1}{2} - \frac{1}{2\sqrt{t}})$$

and

$$f''(t) = v(1-v)t^{v-2} - v(1-v)t^{-\frac{3}{2}} = v(1-v)t^{-\frac{3}{2}}(t^{v-\frac{1}{2}} - 1).$$

In view of $t^{v-\frac{1}{2}} \geq 1$, then $t^{v-\frac{1}{2}} - 1 \geq 0$ which means $f''(t) \geq 0$. Hence, $f'(t)$ is increasing on $(0, \infty)$, $f'(1) = 0$ and $f(1) = 0$. So, $f(t)$ is decreasing on $(0, 1]$ and increasing on $(1, \infty)$. Consequently, $f(t) \geq 0$ for $v \in [0, 1]$ and $t^{v-\frac{1}{2}} \geq 1$.

This proved that

$$a\nabla_v b \geq a\sharp_v b + 4v(1-v)(a\nabla b - a\sharp b).$$

hold for $a, b > 0$, $v \in [0, 1]$ and $(\frac{b}{a})^{v-\frac{1}{2}} \geq 1$. □

Remark 2.2. In view of $4v(1-v) - 8v(1-2v) \leq 0$ when $v \in [0, \frac{1}{3}]$, it is natural to consider refining inequality (6). Using the same method of Theorem 2.1, we have

$$a\nabla_v b \geq a\sharp_v b + 8r_0(1-2r_0)(a\nabla b - a\sharp b)$$

for $a, b > 0$, $v \in [0, 1]$ and $(\frac{b}{a})^{v-\frac{1}{2}} \geq 2$.

In the following, we refine the Heinz and Heron means inequality using the same method as above. As $a\sharp b \leq H_v(a, b) \leq F_\alpha(a, b) \leq a\nabla b$, we are interested in finding an upper bound or a lower bound of $F_\alpha(a, b) - H_v(a, b)$.

Theorem 2.3. *The Heinz and Heron means satisfy*

$$F_\alpha(a, b) \geq H_v(a, b) + 4v(1-v)(a\nabla b - a\sharp b) \tag{7}$$

for $a, b > 0$, $v \in [0, 1]$, $\alpha = 1 - 4(v - v^2)$ *and* $(\frac{b}{a})^{v-\frac{1}{2}} + (\frac{b}{a})^{\frac{1}{2}-v} \geq 4$.

Proof. Inequality (7), in expanded form, says

$$4(v-v^2)\sqrt{ab} + (1-4(v-v^2))\frac{a+b}{2} \geq \frac{a^{1-v}b^v + a^v b^{1-v}}{2} + 4v(1-v)(\frac{a+b}{2} - \sqrt{ab}).$$

Put $a = 1, b = t$,

$$4(v-v^2)\sqrt{t} + (1-4(v-v^2))\frac{1+t}{2} \geq \frac{t^v + t^{1-v}}{2} + 4v(1-v)(\frac{1+t}{2} - \sqrt{t}).$$

Let

$$f(t) = 4(v - v^2)\sqrt{t} + (1 - 4(v - v^2))\frac{1 + t}{2} - \frac{t^v + t^{1-v}}{2} - 4v(1 - v)(\frac{1 + t}{2} - \sqrt{t}).$$

Then

$$f'(t) = 4(v-v^2)\frac{1}{2\sqrt{t}}+(1-4(v-v^2))\frac{1}{2}-\frac{vt^{v-1} + (1 - v)t^{-v}}{2}-4v(1-v)(\frac{1}{2}-\frac{1}{2\sqrt{t}})$$

and

$$\begin{aligned} f''(t) &= 4(v - v^2)(-\frac{1}{4})t^{-\frac{3}{2}} - \frac{v(v - 1)t^{v-2} + v(v - 1)t^{-v-1}}{2} - v(1 - v)t^{-\frac{3}{2}} \\ &= v(1 - v)t^{-\frac{3}{2}}[\frac{t^{v-\frac{1}{2}} + t^{\frac{1}{2}-v}}{2} - 2]. \end{aligned}$$

In view of $t^{v-\frac{1}{2}} + t^{\frac{1}{2}-v} \geq 4$, we have $f''(t) \geq 0$. Then $f(1) = f'(1) = 0$ which means that $f(t)$ is decreasing on $(0, 1]$ and increasing on $(1, \infty)$, respectively. Consequently, $f(t) \geq 0$ for $v \in [0, 1]$ and $t^{v-\frac{1}{2}} + t^{\frac{1}{2}-v} \geq 4$. This proved that

$$F_\alpha(a, b) \geq H_v(a, b) + 4v(1 - v)(a\nabla b - a\sharp b)$$

holds for $a, b > 0$, $v \in [0, 1]$, $\alpha = 1 - 4(v - v^2)$ and $(\frac{b}{a})^{v-\frac{1}{2}} + (\frac{b}{a})^{\frac{1}{2}-v} \geq 4$. □

We emphasize that the significance of this result is to lead to as many refining inequalities as we wish. Provided that $f(v) = 4v(1-v)(a\nabla b-a\sharp b)$, we can refine $f(v)$ to find an upper bound or a lower bound. Follow the same logic of Theorem 2.3, if we choose $f(v) = 8v(1 - 2v)$, then we can have the following corollary.

Corollary 2.4. *If $a, b > 0$ and $v \in [0, 1]$, then we can have*

$$F_\alpha(a, b) \geq H_v(a, b) + 8v(1 - 2v)(a\nabla b - a\sharp b)$$

for $\alpha = 1 - 4(v - v^2)$ and $(\frac{b}{a})^{v-\frac{1}{2}} + (\frac{b}{a})^{\frac{1}{2}-v} \geq 6$.

Put

$$f(t) = 4(v - v^2)\sqrt{t} + (1 - 4(v - v^2))\frac{1 + t}{2} - \frac{t^v + t^{1-v}}{2} - 8v(1 - 2v)(\frac{1 + t}{2} - \sqrt{t}),$$

the detail proof of Corollary 2.4 is similar to the above theorems. Here we leave it to the reader.

Remark 2.5. It is obvious that inequality (7) is a refinement of inequality (4). But it is not a general refinement since inequality (7) constraints $(\frac{b}{a})^{v-\frac{1}{2}} + (\frac{b}{a})^{\frac{1}{2}-v} \geq 4$. One way to refine inequality (4) without constraints of $\frac{b}{a}$ is to weaken α which we present as follows.

Theorem 2.6. *The Heinz and Heron means satisfy*

$$F_\beta(a,b) \geq H_v(a,b) + 4v(1-v)(a\nabla b - a\sharp b) \tag{8}$$

for $a,b>0,\ v\in[0,1]$ *and* $\beta = 1-8\max\{v,1-v\}\min\{1-2v,2v-1\}$.

Proof. If $a,b>0$ and $v\in[\frac{1}{2},1]$, inequality (8) for $a=1, b=t>0$, becomes

$$8v(1-2v)\sqrt{t} + (1-8v(1-2v))\frac{1+t}{2} \geq \frac{t^v+t^{1-v}}{2} + 4v(1-v)(\frac{1+t}{2} - \sqrt{t}).$$

Let

$$f(t) = 8v(1-2v)\sqrt{t} + (1-8v(1-2v))\frac{1+t}{2} - \frac{t^v+t^{1-v}}{2} - 4v(1-v)(\frac{1+t}{2} - \sqrt{t}).$$

Then

$$f'(t) = 8v(1-2v)\frac{1}{2\sqrt{t}} + (1-8v(1-2v))\frac{1}{2} - \frac{vt^{v-1}+(1-v)t^{-v}}{2} - 4v(1-v)(\frac{1}{2} - \frac{1}{2\sqrt{t}})$$

and

$$\begin{aligned} f''(t) &= -2v(1-2v)t^{-\frac{3}{2}} + \frac{v(1-v)t^{v-2}+v(1-v)t^{-v-1}}{2} - v(1-v)t^{-\frac{3}{2}} \\ &= v(1-v)t^{-\frac{3}{2}}(\frac{t^{v-\frac{1}{2}}+t^{\frac{1}{2}-v}}{2} + 2\frac{v}{1-v} - 3). \end{aligned}$$

Since $\frac{2v}{1-v} - 3 \geq -1$ for $v\in[\frac{1}{2},1]$, then $f''(t)\geq 0$, $f'(1)=f(1)=0$ which means that $f(t)$ is decreasing on $(0,1]$ and increasing on $(1,\infty)$, respectively. Hence, we have $f(t)\geq 0$ for $v\in[\frac{1}{2},1]$. Consequently,

$$F_\beta(a,b) \geq H_v(a,b) + 4v(1-v)(a\nabla b - a\sharp b) \quad (*)$$

holds for $v\in[\frac{1}{2},1]$ and $\beta = 1-8\max\{v,1-v\}\min\{1-2v,2v-1\}$.

Replacing v by $1-v$ and exchanging a and b in $(*)$, then we have inequality (8) holds for $v\in[0,\frac{1}{2}]$ and $\beta = 1-8\max\{v,1-v\}\min\{1-2v,2v-1\}$. □

In view of $a\nabla b \geq F_\alpha(a,b)$, we take inequality (8) as a refinement of inequality (5). Next, we present a squared version refinement of inequality (5).

Theorem 2.7. *If* $a,b>0$ *and* $v\in[0,1]$, *then we can have*

$$(a+b)^2 \geq 4(H_v(a,b))^2 + 8v(1-v)(a^2\nabla b^2 - a^2\sharp b^2) \tag{9}$$

for $(2v-1)(b^2-a^2)\geq 0$.

Proof. From inequality (8), we have

$$\frac{a+b}{2} \geq H_v(a,b) + 4v(1-v)(a\nabla b - a\sharp b).$$

By virtue of replacing a by a^2 and b by b^2, respectively, then

$$\begin{aligned}(a+b)^2 - 4H_v(a,b)^2 &= a^2 + b^2 - (a^{1-v}b^v)^2 - (a^v b^{1-v})^2 \\ &= (1-v)a^2 + vb^2 - (a^{1-v}b^v)^2 + va^2 + (1-v)b^2 - (a^v b^{1-v})^2 \\ &\geq 8v(1-v)(a^2\nabla b^2 - a^2\sharp b^2) \quad by (1.3)\end{aligned}$$

holds for $(2v-1)(b^2-a^2) \geq 0$. Hence,

$$(a+b)^2 \geq 4(H_v(a,b))^2 + 8v(1-v)(a^2\nabla b^2 - a^2\sharp b^2).$$

□

2.2 Matrix inequalities

To state our results in this part, we introduce the following matrix theories.

Let $M_n(C)$ be the algebra of $n \times n$ complex matrices. For Hermitian matrices $A, B \in M_n(C)$, we write that $A \geq 0$ if A is positive semidefinite, $A > 0$ if A if positive definite and $A \geq B$ if $A - B \geq 0$.

In this part, the weighted arithmetic and geometric matrix means are defined for $A, B \geq 0$ as follows

$$A\nabla_v B = (1-v)A + vB \quad \text{and} \quad A\sharp_v B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^v A^{\frac{1}{2}},$$

while we use the standard notations $A\nabla B$ and $A\sharp B$ for $A\nabla_{\frac{1}{2}} B$ and $A\sharp_{\frac{1}{2}} B$, respectively. The scalar Young inequality $a\nabla_v b \geq a\sharp_v b$ can be used to prove the matrix Young inequality

$$A\nabla_v B \geq A\sharp_v B$$

holds for $A, B \geq 0$ and $v \in [0,1]$. The matrix version of the Heinz means is defined by

$$H_v(A,B) = \frac{A\sharp_v B + A\sharp_{1-v} B}{2},$$

where $A, B \geq 0$ and $v \in [0,1]$. The matrix version of the Heron means is defined by

$$F_v(A,B) = (1-v)A\sharp B + vA\nabla B$$

for $0 \leq v \leq 1$.

To reach inequality for bounded self-adjoint matrix on Hilbert space, we shall use the following monotonicity property for matrix functions: If $X \in M_n(C)$ with a spectrum $S_p(X)$ and f, g are continuous real-valued functions on an interval containing $S_p(X)$, then

$$f(t) \geq g(t), t \in S_p(X) \Rightarrow f(X) \geq g(X).$$

For more details about this property, the reader is referred to [9].

Now we are in a position to give our first matrix inequality.

Theorem 2.8. *Let $A, B > 0$ and $v \in [0,1]$, then we can have*

$$F_\beta(A,B) \geq H_v(A,B) + 4v(1-v)(A\nabla B - A\sharp B) \tag{10}$$

for $\beta = 1 - 8\max\{v, 1-v\}\min\{1-2v, 2v-1\}$.

For $A = (a_{ij}) \in M_n(C)$, the Hilbert-Schmidt norm of A is defined by $\|A\|_2 = \sqrt{\sum_{i,j=1}^n |a_{ij}|^2}$. $\|\cdot\|_2$ has the unitarily invariant property: $\|UAV\|_2 = \|A\|_2$ for all $A \in M_n(C)$ and all unitary matrices $U, V \in M_n(C)$.

Theorem 2.9. *Suppose that two matrices A, B and positive real numbers m, m', M, M' satisfy either of the following conditions:*
(i) $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$;
(ii) $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$.
Then

$$\|AX + XB\|_2^2 \geq \|A^{1-v}XB^v + A^vXB^{1-v}\|_2^2 + 4v(1-v)\|AX - XB\|_2^2. \tag{11}$$

for $A, B, X > 0$, $v \in [0,1]$ and $(2v-1)((M')^2 - (m')^2) \geq 0$.

Proof. Since $A, B > 0$, then by spectral theorem there are unitary matrices U, V and diagonal matrices $D_1 = diag(\lambda_i)$ and $D_2 = diag(\mu_i)$ for $i = 1, \cdots, n$, such that

$$A = UD_1U^* \quad \text{and} \quad B = VD_2V^*.$$

For our computations, let $Y = U^*XV = [y_{ij}]$. Then we have

$$AX + XB = U[(\lambda_i + \mu_j)y_{ij}]V^*.$$

If $v \in [0,1]$, utilizing (9) and the unitarily invariant property of $\|\cdot\|_2$, we have

$$\begin{aligned}
\|AX + XB\|_2^2 &= \sum_{i,j=1}^n (\lambda_i + \mu_j)^2 |y_{ij}^2| \\
&\geq \sum_{i,j=1}^n [(\lambda_i^{1-v}\mu_j^v + \lambda_i^v\mu_j^{1-v})^2 + 8v(1-v)(\frac{\lambda_i^2 + \mu_j^2}{2} - \sqrt{\lambda_i^2\mu_j^2})]|y_{ij}^2| \\
&= \|A^{1-v}XB^v + A^vXB^{1-v}\|_2^2 + 4v(1-v)\|AX - XB\|_2^2
\end{aligned}$$

for $A, B > 0$, $v \in [0,1]$ and $(2v-1)((M')^2 - (m')^2) \geq 0$. That is

$$\|AX + XB\|_2^2 \geq \|A^{1-v}XB^v + A^vXB^{1-v}\|_2^2 + 4v(1-v)\|AX - XB\|_2^2.$$

□

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