Characterization of 1-uniform dcsl graphs and learning graphs

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Abstract. A distance compatible set labeling (dcsl) of a connected graph G is an injective set assignment $f : V(G) \to 2^X$, X being a non empty ground set, such that the corresponding induced function $f^\oplus : E(G) \to 2^X \setminus \{\phi\}$ given by $f^\oplus(u,v) = f(u) \oplus f(v)$ satisfies $|f^\oplus(u,v)| = k^f_{(u,v)} d_G(u,v)$ for every pair of distinct vertices $u, v \in V(G)$, where $d_G(u,v)$ denotes the path distance between u and v and $k^f_{(u,v)}$ is a constant, not necessarily an integer, depending on the pair of vertices u, v chosen. A dcsl f of G is k-uniform if all the constants of proportionality with respect to f are equal to k, and if G admits such a dcsl then G is called a k-uniform dcsl graph. Let $\mathcal{F}$ be a family of subsets of a set X. A graph G is defined to be a learning graph, if it is a $\mathcal{F}$-induced graph of some learning space $\mathcal{F}$. In this paper, we characterize 1-uniform dcsl learning graphs and discuss the embedding problems.
Keywords: dcsl graphs, $1-$uniform dcsl graphs, wg-family of sets, learning graphs.

1. Introduction

Throughout this paper by a graph we mean a connected, finite, simple graph. Unless otherwise mentioned, for all terminology in graph theory the reader is referred to [4]. Acharya [1] introduced the notion of vertex set valuation as a set analogue of number valuation. For a graph $G = (V, E)$ and a non empty

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set X, Acharya defined a set valuation of G as an injective set valued function $f : V(G) \rightarrow 2^X$, and he defined a set-indexer as a set valuation such that the function $f^{\oplus} : E(G) \rightarrow 2^X \setminus \{\phi\}$ given by $f^{\oplus}(uv) = f(u) \oplus f(v)$ for every $uv \in E(G)$ is also injective, where 2^X is the set of all the subsets of X and $\oplus$ is the binary operation of taking the symmetric difference of subsets of X.

Acharya and Germina,[2] introduced the particular kind of set valuation for which a metric, especially the cardinality of the symmetric difference, is associated with each pair of vertices in proportion to the distance between them [2]. In other words, they discused the question whether one can determine those graphs $G = (V, E)$ that admit an injective function $f : V \rightarrow 2^X$, X being a non empty ground set such that the cardinality of the symmetric difference $f^{\oplus}(uv)$ is proportional to the usual path distance $d_G(u, v)$ between u and v in G, for each pair of distinct vertices u and v in G. They called f a *distance compatible set labeling* (dcsl) of G, and the ordered pair (G, f), a distance compatible set labeled (dcsl) graph. Thus

Definition 1.1 ([2]). Let $G = (V, E)$ be any connected graph. A distance compatible set labeling (dcsl) of a graph G is an injective set assignment $f : V(G) \rightarrow 2^X$, X being a non empty ground set, such that the corresponding induced function $f^{\oplus} : E(G) \rightarrow 2^X \setminus \{\phi\}$ given by $f^{\oplus}(u, v) = f(u) \oplus f(v)$ satisfies $|f^{\oplus}(u, v)| = k^f_{(u,v)} d_G(u, v)$ for every pair of distinct vertices $u, v \in V(G)$, where $d_G(u, v)$ denotes the path distance between u and v and $k^f_{(u,v)}$ is a constant, not necessarily an integer, depending on the pair of vertices u, v chosen.

A distance compatible set labeling f of G is k-uniform if all the constants of proportionality with respect to f in Definition 1.1 are equal to k, and if G admits such a distance compatible set labeling then G is called a k-uniform distance compatible set labeled graph.

Listed below are the definitions and known results which are used in this paper.

Definition 1.2 ([7]). A family of sets $\mathcal{F}$ is well-graded if any two sets in $\mathcal{F}$ can be connected by a sequence of sets formed by single element insertion and deletion, without redundant operations, such that all intermediate sets in the sequence belong to $\mathcal{F}$.

Definition 1.3 ([7]). Let $\mathcal{F}$ be a family of subsets of a set X. A tight path between two distinct sets P and Q(or from P to Q) in $\mathcal{F}$ is a sequence $P_0 = P, P_1, P_2 \ldots P_n = Q$ in $\mathcal{F}$ such that $d(P, Q) = |P \oplus Q| = n$ and $d(P_i, P_{i+1}) = 1$ for $0 \leq i \leq n - 1$. The family $\mathcal{F}$ is well-graded family (or wg-family), if there is a tight path between any two of its distinct sets.

Definition 1.4 ([7]). Any family $\mathcal{F}$ of subsets of X defines a graph $G_{\mathcal{F}} = (\mathcal{F}, E_{\mathcal{F}})$, where $E_{\mathcal{F}} = \{\{P, Q\} \subseteq \mathcal{F} : |P \oplus Q| = 1\}$ and we call $G_{\mathcal{F}}$, an $\mathcal{F}$-induced graph.

Definition 1.5 ([7])**.** A family of sets $\mathcal{F}$ is closed under union or $\bigcup$−closed if for any nonempty $\mathcal{H} \subseteq \mathcal{F}$, we have $U\mathcal{H} \in \mathcal{F}$. The span of a family of sets $\mathcal{F}$ is the family $\mathcal{F}'$ containing any set which is the union of some nonempty subfamily of $\mathcal{F}$.

Definition 1.6 ([6])**.** The pair $(Q, \mathcal{K})$ in which Q is a nonempty set, and $\mathcal{K}$ is a family of subsets of Q, containing atleast Q and the empty set is called a learning space, if it satisfies the following conditions:

(1) For any two sets K, L such that $K \subset L$, there exists a finite chain of states

$$K = K_0 \subset K_1 \subset \ldots \subset K_p = L$$

such that $|K_i \setminus K_{i-1}| = 1$, for $1 \leq i \leq p$ and so $|L \setminus K| = p$.

(2) If K, L are two sets satisfying $K \subset L$ and q is an item such that $K \cup \{q\} \in \mathcal{K}$, and $q \notin L$, then $L \cup \{q\} \in \mathcal{K}$.

One may note that a well-graded family which is closed under union and containing the empty set, is a learning space. Germina et al. [10] defined a learning graph as

Definition 1.7. A graph G is said to be a learning graph, if it is an $\mathcal{F}$-induced graph of some learning space $\mathcal{F}$.

In this paper, we discuss the structure of a 1-uniform dcsl graph whose collection of vertex labeling induces a learning space. That is, we analyse those classes of 1-uniform dcsl graphs that are also learning graphs. Following result has been established by Germina and Jinto [9].

Theorem 1.8 ([9])**.** *The path P_n on n vertices is a learning graph.*

One may note that the 1- uniform dcsl f of a graph G need not necessarily be unique. For example, the 1- uniform dcsl f of a path P_n is not unique. It is also to be noted that not all 1-uniform labeling of path is a learning space. Consider the 1- uniform dcsl labeling of a path $P_5 = u_1, u_2, u_3, u_4, u_5$ with the assignments of the vertices respectively as $\{1\}$, $\{1, 2\}$, $\{1, 2, 3\}$, $\{1, 2, 3, 4\}$, $\{1, 2, 3, 4, 5\}$ which is a 1 -uniform dcsl. However, since empty set is not being assigned to any vertex, with this labeling P_5 is not a learning graph. Again, consider the labeling of the path P_5 with the assignments of the vertices respectively as $\{1, 2\}, \{1\}, \{\emptyset\}, \{3\}, \{3, 4\}$ which is a 1 - uniform dcsl, but, is not a learning graph with respect to this labeling as collection of vertex labeling is not $\bigcup$-closed. However, consider the 1-uniform dcsl for a path $P_n = u_1, u_2, u_3, \ldots, u_n$ given as, $\{\emptyset\}$ $\{1\}$, $\{1, 2\}$, $\{1, 2, 3\}$, $\ldots$, $\{1, 2, 3, 4, \ldots, n-1\}$. With this assignments, path P_n is a learning graph. One may note that, there are many other classes of 1- uniform graphs that are not learning graphs. For example, even cycles C_{2n}, are 1-uniform but, $C_{2n}, (n \geq 3)$ is not a learning graph.

Germina et.al. proved the following results [10]:

Theorem 1.9 ([10])**.** *The star graph $K_{1,n}, n \geq 3$ is not a learning graph.*

Theorem 1.10 ([10])**.** *If G is a 1-uniform dcsl graph with the collection of vertex labeling $\mathcal{F}$, contains empty set, then the graph $G_{Span\mathcal{F}}$ is a learning graph.*

Since all learning graphs are trivially 1-uniform, the classes of 1-uniform graphs which are learning graphs seems to be an interesting problem.

2. Main results

Towards discussing this problem, we first study the structure of $\mathcal{B}$-induced 1-uniform distance compatible set labeled graph with $\mathcal{B}$ as the basis and, the structure of the respective $\mathcal{F}$-induced graph.

David Eppstein et. al. [5] defined the base of a $\bigcup$-closed family of sets as follows.

Definition 2.1 ([5])**.** A base of a $\bigcup$-closed family of sets $\mathcal{F}$ is a minimal subfamily $\mathcal{B}$ of $\mathcal{F}$, spanning $\mathcal{F}$, where minimal is meant with respect to set inclusion. That is, if $Span(\mathcal{H}) = \mathcal{F}$ for some $\mathcal{H} \subseteq \mathcal{B}$, then $\mathcal{H} = \mathcal{B}$.

Correspondingly, Germina and Jinto [9] defined the span of a 1- uniform dcsl graph G as

Definition 2.2 ([9])**.** The span of a 1- uniform dcsl graph G is the graph induced by the span of the vertex labelings of G, denoted by $SpanG$.

One may note that a family $\mathcal{B}$, spanning a family $\mathcal{F}$, is a base of $\mathcal{F}$, if and only if, none of the sets in $\mathcal{B}$ is the union of some other sets in $\mathcal{B}$. Let $\mathcal{F}$ be a $\bigcup$-closed wg-family, and $\mathcal{B}$ be a base of $\mathcal{F}$. Then $\mathcal{B}$ need not necessarily be a wg-family and hence $G_{\mathcal{B}}$, the $\mathcal{B}$-induced graph in general, need not necessarily be a 1- uniform dcsl graph. In particular, the $\mathcal{B}$-induced graph may even be disconnected so that it is not 1-uniform dcsl graph.

Theorem 2.3. A tree T is a learning graph if and only if the tree is isomorphic to path P_n.

Proof. Let T be a learning tree. Let $\mathcal{B}$, be the basis so that none of the sets in $\mathcal{B}$ is the union of some other sets in $\mathcal{B}$ and let $\mathcal{F}$ be a $\bigcup$-closed wg-family of sets with $\mathcal{B}$ as basis. Let $G_{\mathcal{F}}$, be the $\mathcal{F}$-induced graph, which is a well graded graph and is 1-uniform dcsl, say with 1 - uniform dcsl f. $G_{\mathcal{F}}$, being a well graded graph, there exists a tight path between every pair of vertices u and v of T. Consider any such tight path say $P_k = u_1, u_2, \ldots, u_k$ of length k in a tree T. Hence, the labeling of P_k with respect to which P_k, is a tight path is either of the following:

(i) $f(u_1) = \{\emptyset\}$, $f(u_2)$ is any singleton set, $f(u_3)$ is any two element set containing $f(u_2)$. Proceeding in this order with $f(u_k)$ is assigned with a set of order k in such a way that, $f(u_k)$ contains $f(u_{k-1})$ and the assignment is minimal with respect to set inclusion.

Or (ii) $f(u_1)$ is assigned any singleton set, $f(u_2)$ any two element set containing $f(u_1)$; $f(u_3)$ is any three element set containing $f(u_2)$ …, in the order with $f(u_k)$ is assigned with a set of order k in such a way that, $f(u_k)$ contains $f(u_{k-1})$ and the assignment is minimal with respect to set inclusion.

Since T is a learning tree, $\emptyset$ should necessarily be assigned to one of the vertices of T.

Case (i): One possibility is that $\emptyset$ is assigned to a pendent vertex say, u_1 of T. That is $f(u_1) = \{\emptyset\}$. Now consider the vertices in the 1-neighbourhood of vertex u_1. T being a learning graph, all these vertices in the 1-neighbourhood of u_1 should necessarily be assigned with singleton sets, say $\{1\}, \{2\}, \ldots \{k\}$, where k is the degree of u_1. Then, all the vertices in the 2- neighbourhood of u_1 should necessarily be assigned with sets of cardinality two containing the singleton set to which they are adjacent. Denote these vertices that are adjacent to vertices with assignment as singleton sets, by $\{u_{11}, u_{12}, \ldots, u_{1x_i}\}$. Among these vertices that are adjacent to the vertex with labeling $\{1\}$ shall necessarily receive the assignments as $\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \ldots, \{1, x_j\}$, where $x_j = deg\ (u_{11}) - 1$;

Those vertices that are adjacent to the vertex with labeling $\{2\}$ should necessarily receive the assignments as $\{2,2\}, \{2,3\}, \{2,4\}, \{2,5\}, \ldots, \{2, x_l\}$, where $x_l = deg\ (u_{12}) - 1$;

Those vertices that are adjacent to the vertex with labeling $\{3\}$ shall receive the assignments as $\{3,4\}, \{3,5\}, \{3,6\}, \ldots, \{3, x_k\}$, where $x_k = deg\ (u_{13}) - 1$.

Proceeding this way of assignment to the vertices in the immediate neighbourhoods shall necessarily receive the labeling with respect to the pre-assigned vertex. Also note that the assignment followed is minimal with respect to set inclusion. With this assignment that occurred in a natural way, we reach a contradiction either to the fact that the set assignment is not U-closed or the symmetric difference between the set assignments of pair of vertices in T is not equal to the distance between the vertices. Hence, in this case, for T to be a learning graph, $T \cong P_n$.

Case (ii): $\emptyset$ is assigned to a non pendent vertex say, v of T. As in Case(i), we should necessarily give an assignment to the vertices in the first neighbourhood of v, with singleton sets; to the vertices in the second neighbourhood of v with sets of cardinality two containing the singleton set, the assignment of the vertex to which it is adjacent. And then proceed to the next immediate neighbourhood vertices and give the assignments following the rule that the assignment followed is minimal with respect to set inclusion. In this case also, we reach a contradiction either to the fact that the set assignment is not $\bigcup$-closed or the symmetric difference between the set assignments of pair of vertices in T is not equal to the distance between the vertices. Hence, for T to be a learning graph, then, $T \cong P_n$.

Conversely, let $T \cong P_n$, a path of length n. Let $P_n = u_1, u_2, \ldots, u_n$. Then assign the vertices $\{u_1, u_2, \ldots, u_n\}$ respectively as $\{\emptyset\}$, $\{1\}$, $\{1,2\}$, …, $\{1,2,3,\ldots,n\}$, with respect to which P_n is a learning graph. □

Theorem 2.4. A learning tree T is isometrically embeddable in a partial cube of dimension equal to the length of the eccentric path of the tree T.

Proof. Let T be a learning tree. Let $\mathcal{B}$ be a basis and $\mathcal{G}_{\mathcal{B}}$ be the $\mathcal{B}-$ induced graph. Hence, $\mathcal{B}$ contains all unions and is minimal with respect to inclusion. Invoking Theorem 2.3, $\mathcal{G}_{\mathcal{B}}$ is a 1-uniform dcsl path. Being a 1-uniform dcsl graph, $\mathcal{G}_{\mathcal{B}}$ is bipartite and so is the span $\mathcal{F}$. The span $\mathcal{F}$ is always 1-uniform and there exists a unique vertex u such that $f(u) = \{\emptyset\}$ and such that all $u - v$ paths are tight. Now, if T is a learning graph, there exists exactly one such tight path and hence is embedable in a partial cube. Now consider the path of maximum length in T, which is nothing but the eccentric path of tree T. By choosing the initial vertex of the eccentric path and assigning it with $\emptyset$ and proceeding as in Theorem 2.3, we get an assignment with respect to which the tree T is a learning graph. Hence, a learning tree T can be isometrically embedded in a partial cube of dimension equal to the length of the eccentric path of the tree. □

As we have already discussed, not all 1-uniform dcsl graphs are learning graphs. The 1- uniform dcsl f of a graph G need not necessarily be unique. Hence, we expect 1- uniform dcsl for a given graph, with respect to which the assignment may or may not induce a learning graph. Hence, a 1-uniform graph with a 1-uniform dcsl f is a learning graph only when one of the vertices receives emptyset as an assignment and the given assignment is union closed. That is, learning graph not only depend on the structure of a graph, but also, depend on the specific assignment chosen. Hence, given a 1-uniform dcsl f for a given graph G, characterizing the 1-uniform dcsl f with respect to which G is a learning graph seems to be an interesting problem.

Germina et. al. [10] proved the following theorems

Theorem 2.5. *The base $\mathcal{B}-$induced graph $G_{\mathcal{B}}$ is* 1- *uniform distance compatible set labeled graph if and only if for every pair of vertices $u, v \in V(G_{\mathcal{B}})$ with $f(u) = \{\emptyset\}$, there exist a tight path between u and v.*

Theorem 2.6. If G is a 1-uniform dcsl graph. Then, G is a learning graph if and only if there exists a collection $\{\mathcal{F}\}$, of vertex labeling of G, with $\emptyset \in \{\mathcal{F}\}$ which is well graded.

Proof. Setting, $\{f(V(G)\} = \{\mathcal{F}\}$, where f is the 1-uniform dcsl of G, a learning graph. Then G being a learning graph, $\{\mathcal{F}\}$ is well graded and necessarily, $\emptyset \in \{\mathcal{F}\}$.

Conversely, let if $\emptyset \in \{\mathcal{F}\}$ and $\{\mathcal{F}\}$, is a well graded family of sets. Being well graded family of sets, for every pair of vertices (u, v) in G, there exists a tight $u - v$ path, and $\{\mathcal{F}\}$, is $\bigcup$-closed. Hence, setting $\{f(V(G)\} = \{\mathcal{F}\}$, G is a learning graph. □

Theorem 2.7. Let $\mathcal{B}$ be a basis with $\emptyset \in \mathcal{B}$ and let $\mathcal{G}_{\mathcal{B}}$ be the induced graph induced by the basis $\mathcal{B}$. Then, classes of $\mathcal{F}$-induced graph $G_{\mathcal{F}}$ in which the base

$\mathcal{B}$-induced graph is a learning graph and is isomorphic to a bipartite graph that are embedable in a partial cube.

Proof. Let $\mathcal{B}$ be a basis with $\emptyset \in \mathcal{B}$ and $\mathcal{G}_{\mathcal{B}}$ be the $\mathcal{B}-$ induced graph. Hence, $\mathcal{B}$ contains all unions and is minimal with respect to inclusion. Assume $\mathcal{G}_{\mathcal{B}}$ is 1-uniform dcsl. Being a 1-uniform dcsl, $\mathcal{G}_{\mathcal{B}}$ is bipartite and so is the span $G_{\mathcal{F}}$ with $\{\emptyset\}$ as an assignment of one of the vertex. The span $\mathcal{F}$ always induces a 1-uniform dcsl and there exists a unique vertex u such that $f(u) = \{\emptyset\}$ and that all $u - v$ paths are tight. Also, the $\mathcal{F}$ induced graph $G_{\mathcal{F}}$ which is 1-uniform is a bipartite graph. Also, if $G_{\mathcal{F}}$ is a learning graph, invoking Theorem 2.6, and Theorem 2.7, G is isometrically embedable in a partial cube. □

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