Orbit-maximal green sequences and general-maximal green sequences

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Abstract. In this paper we introduce the orbit-maximal green sequences for a skew symmetrizable matrix and set the relation between orbit-maximal green sequences and the maximal green sequences of its folding matrix. We also generalize the concept of maximal green sequences, that is, the so-called general-maximal green sequence and define the red size of this generalization.

Keywords: cluster algebra, maximal green sequence , orbit-maximal green sequence, general-maximal green sequence.

1. Introduction

Cluster algebra is a very important mathematical innovation, which was introduced by S. Fomin and A. Zalevensky in [5]. Its importance and effectiveness come from its connection with different branches of mathematics and mathematical physics such as Teichmuller theory, canonical basis, total positivity, Poisson-Lie groups, Calabi-Yau algebras and representation of finite dimensional algebras. One of the most interesting things in studying cluster algebra is that this study can be associated with combinatorial tools or geometrical description.

Maximal green sequences are a very important concept related to cluster algebras established by Keller [8] for the skew symmetric case, then and after the sign coherence property was proved for the skew-symmetrizable case [7], the definition was expanded to include the skew-symmetrizable matrices.

In this paper we approach this concept from different angles presenting new related definitions with some new results. We also give a generalization in a

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way that makes this generalized case a property owned by any given skew-symmetrizable matrix of finite size.

Firstly, we recall the definition of a group of automorphisms for a skew-symmetrizable matrix as mentioned in [4], [11]. Under the action of the group, mutations in directions of indices of the same orbit commute [4] which make it possible to introduce the so-called *orbit-mutation* in [4]. We firstly note that after mutating in direction of any orbit, all the columns in the C-matrix whose indices are of the same orbit have the same sign (Lemma 3.11). This allows us to define the *orbit-maximal green sequence*. We also prove that every orbit-maximal green sequence for a skew-symmetrizable matrix in a stable admissible pair corresponds to a maximal green sequence for its folding matrix (Theorem 3.19). Using this result we prove that the minimum length of any orbit-maximal green sequence for a skew-symmetrizable matrix is the number of the orbits (Corollary 3.22). Then, we study the relation between the orbit-maximal green sequences of a skew-symmetrizable matrix and the maximal green sequences of this matrix and prove that every orbit-maximal green sequence corresponds to a maximal green sequence (Theorem 3.25). The raised question here is when and how to create a corresponding orbit-maximal green sequence built from a given maximal green sequence? We discuss this question for maximal green sequences obtained from an admissible numbering by source for acyclic skew-symmetrizable matrices.

On the other hand, we introduce the *general-maximal green sequence* which produces a maximum number of red indices. This generalization makes every skew-symmetrizable matrix own a general-maximal green sequence. The interesting question here is that: how many red indices can be obtained maximally? We call this maximal number the *red size*. We prove that the red size of any decomposable skew-symmetrizable matrix is greater than or equals the sum of the red sizes of its skew-symmetrizale components (Theorem 4.8). Finally, we define *general orbit-maximal green sequences* and study the connection with general-maximal green sequences of a skew-symmetrizable matrix under the action of an automorphism group.

2. Preliminaries

Recall that $B = (b_{ij}) \in M_n(\mathbb{Z})$ is said to be a *skew-symmetrizable* matrix if there is a diagonal matrix D with positive integer entries such that DB is a skew-symmetric matrix. We call D the symmetrizer matrix.

Let $\widetilde{B} = (b_{ij}) \in M_{n+m,n}(\mathbb{Z})$ then $\widetilde{B}$ is said to be skew-symmetrizable if its principal part which is the square sub-matrix $B = (b_{i,j})$ for $1 \leq i, j \leq n$ is skew-symmetrizable. We call $\widetilde{B}$ the *extended matrix* of B.

The mutation of a matrix $\widetilde{B}$ in direction k where $1 \leq k \leq n$ is the matrix $\mu_k(B) = B' = (b'_{ij})$ where:

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i \text{ or } j = k, \\ b_{ij} + \frac{1}{2}(\mid b_{ik} \mid b_{kj} + b_{ik} \mid b_{kj} \mid), & \text{otherwise} \end{cases}.$$

This is called the *exchange relation of the mutation.* Given a skew symmetrizable matrix $\widetilde{B} = \begin{pmatrix} B \\ I_n \end{pmatrix} \in M_{2n \times n}(\mathbb{Z})$, let $\widetilde{B_\sigma} = \begin{pmatrix} B_\sigma \\ C_\sigma \end{pmatrix}$ be the matrix obtained from $\widetilde{B}$ by a composition of mutations $\mu_\sigma = \mu_{k_s}\mu_{k_{s-1}}....\mu_{k_1}$, $1 \leq k_j \leq n$ for every $1 \leq j \leq s$. The lower part of $\widetilde{B_\sigma}$ is called the $C-$ *matrix* of $\widetilde{B_\sigma}$. By the sign coherence property proved in [7], all the entries of a given column in C_σ are of the same sign.

Definition 2.1. An index k for $1 \leq k \leq n$ is called *green* (respectively, *red*) if the entries of the column indexed by k in C_σ are non-negative (respectively, non-positive).

Definition 2.2. A sequence of indices $(k_1, k_2, ..., k_s)$, where $1 \leq k_i \leq n$ for all $i \in \{1, 2, ..., s\}$, is called a *green sequence* if k_i is green in $\mu_{k_{i-1}}....\mu_{k_1}(\widetilde{B})$ for $1 \leq i \leq s$. Such sequence is called *maximal* if $\mu_{k_s}....\mu_{k_1}(\widetilde{B})$ does not have any green indices.

By convention, we denote $\mu_{k_0}(\widetilde{B}) = \widetilde{B}$.

3. Orbit-maximal green sequences

Suppose we have a skew-symmetrizable matrix equipped with a group of automorphisms, we will expand the sign-coherence idea to the orbit-sign coherence idea and introduce the orbit-maximal green sequences.

Definition 3.1. Let $B \in M_n(\mathbb{Z})$ be a skew-symmetrizable matrix and S_n be the group of permutations on the set $\{1, 2, .., n\}$. Let $g \in S_n$, then g is said to be an *automorphism* of B if $b_{gi,gj} = b_{ij}$ for every $1 \leq i, j \leq n$.

Definition 3.2. Let G be a subgroup of S_n, G is said to be an *automorphism group* of B if for every $g \in G$, g is an automorphism of B.

We denote the G-orbits by $\bar{i}$ for $1 \leq i \leq n$.

Definition 3.3. An automorphism g of a skew-symmetrizable matrix B is said to be *admissible* if:

$b_{i_1 j} b_{i_2 j} \geq 0$, $b_{i_1 i_2} = 0$ for any i_1, i_2 falling in the same G-orbit and $1 \leq j \leq n$.

Definition 3.4. Let G be a group of automorphisms of a skew-symmetrizable matrix B, the group G is said to be *an admissible automorphism group* and the pair (B, G) is said to be *admissible* if g is an admissible automorphism for every $g \in G$.

By [4, Lemma 2.12], we note that mutations in directions falling in the same orbit commute whenever we have an admissible pair (B, G), $\mu_{i_2}\mu_{i_1}(B) = \mu_{i_1}\mu_{i_2}(B)$, for every $i_1, i_2 \in \bar{i}$. Thus, the so-called orbit-mutation can be defined as follows:

Definition 3.5. Let (B, G) be an admissible pair, the *orbit-mutation* in direction $\bar{i}$ is the composition $\mu_{\bar{i}}(B) = \prod_{k\in\bar{i}} \mu_k(B)$.

Remark 3.6. It is clear that $c_{ij} = c_{gi,gj}$ for any $g \in S_n$ and any $c_{ij} \in I_n$. Thus any subgroup of S_n can be considered as an automorphism group for the identity matrix, even though I_n is not skew-symmetrizable.

Remark 3.7. The sign-coherence property for the $C-$matrices obtained from the extended matrix $\widetilde{B} = \begin{pmatrix} B \\ I_n \end{pmatrix}$ by orbit-mutations, can be regarded as a suitable substitution of the admissibility condition to be used in the results of this paper.

An admissible automorphism group of an extended skew-symmetrizable matrix $\widetilde{B} = \begin{pmatrix} B \\ I_n \end{pmatrix} \in M_{2n\times n}(\mathbb{Z})$ is defined to be any admissible automorphism group of its principle part B.

We denote by $\widetilde{B}^{\bar{k}}$ the matrix obtained from $\widetilde{B}$ by the orbit-mutation in direction $\bar{k}$, that is,

$$\widetilde{B}^{\bar{k}} = \mu_{\bar{k}}(\widetilde{B}) = \begin{pmatrix} B^{\bar{k}} \\ C^{\bar{k}} \end{pmatrix}$$

Assume $(\widetilde{B}, G)$ is an admissible pair. Then, by induction, it follows from [4], [11] that the orbit-mutation of a skew-symmetrizable matrix $\widetilde{B} = \begin{pmatrix} B \\ I_n \end{pmatrix}$ in direction $\bar{k}$ for $1 \leq k \leq n$ is given by the following form:

$$b_{ij}^{\bar{k}} = \begin{cases} -b_{ij}, & \text{if } i \text{ or } j \in \bar{k} \\ b_{ij} + \frac{1}{2}\sum\limits_{t\in\bar{k}}(\mid b_{it} \mid b_{tj} + b_{it} \mid b_{tj} \mid), & \text{otherwise} \end{cases} \tag{1}$$

$$c_{ij}^{\bar{k}} = \begin{cases} -c_{ij}, & \text{if } j \in \bar{k} \\ c_{ij} + \frac{1}{2}\sum\limits_{t\in\bar{k}}(\mid c_{it} \mid b_{tj} + c_{it} \mid b_{tj} \mid), & \text{otherwise} \end{cases} \tag{2}$$

Remark 3.8. Let $(\widetilde{B}, G)$ be an admissible pair, it is clear that $i \in \bar{k}$ if and only if $g(i) \in \bar{k}$ for every $g \in G$ and $1 \leq i, k \leq n$. If i or $j \in \bar{k}$ then $b_{ij}^{\bar{k}} = -b_{ij}$, $c_{ij}^{\bar{k}} = -c_{ij}$, and $b_{g(i)g(j)}^{\bar{k}} = -b_{g(i)g(j)}$, $c_{g(i)g(j)}^{\bar{k}} = -c_{g(i)g(j)}$. Hence $b_{ij}^{\bar{k}} = b_{g(i)g(j)}^{\bar{k}}$ and $c_{ij}^{\bar{k}} = c_{g(i)g(j)}^{\bar{k}}$ in this case.

Clearly if t runs over $\bar{k}$, then $g(t)$ runs over $\bar{k}$ for every $g \in G$, then: $\frac{1}{2}\sum_{t\in\bar{k}}(\mid b_{it} \mid b_{tj} + b_{it} \mid b_{tj} \mid) = \frac{1}{2}\sum_{f=g(t)\in\bar{k}}(\mid b_{g(i)f} \mid b_{fg(j)} + b_{g(i)f} \mid b_{fg(j)} \mid)$

$\frac{1}{2}\sum_{t\in\bar{k}}(\mid c_{it}\mid b_{tj}+c_{it}\mid b_{tj}\mid)=\frac{1}{2}\sum_{f=g(t)\in\bar{k}}(\mid c_{g(i)f}\mid b_{fg(j)}+c_{g(i)f}\mid b_{fg(j)}\mid)$ thus $b^{\bar{k}}_{ij}=b^{\bar{k}}_{g(i)g(j)}$ and $c^{\bar{k}}_{ij}=c^{\bar{k}}_{g(i)g(j)}$ for every $k\in\{1,2,...,n\}$ and $g\in G$, in case we mutate in the direction $\bar{k}$ where i and j are not in $\bar{k}$.

From the previous discussion, we note that if G is an automorphism group for $\widetilde{B}=\begin{pmatrix}B\\I_n\end{pmatrix}$, then it is also an automorphism group of $\widetilde{B}^{\bar{k}}=\mu_{\bar{k}}(\widetilde{B})=\begin{pmatrix}B^{\bar{k}}\\C^{\bar{k}}\end{pmatrix}$ for $1\le k\le n$.

In the following discussion we need to deal with a composition of orbit-mutations, but first a condition needs to be established to guarantee the admissibility after such composition.

Definition 3.9. An admissible pair $(\widetilde{B},G)$ is said to be *stable* if for any sequence of orbit-mutations $(\bar{k}_1,\bar{k}_2.......,\bar{k}_s)$, the pair $(\mu_{\bar{k}_j}\mu_{\bar{k}_{j-1}}.........\mu_{\bar{k}_1}(\widetilde{B}),G)$ is an admissible pair for every $1\le j\le s$.

Remark 3.10. By [4], [11], the relations (1) and (2) are used to obtain an orbit-mutation for any skew-symmterizable matrix in a stable admissible pair.

Let $\mu_{\sigma_s}(\widetilde{B})$ be the matrix obtained by a composition of orbit-mutations where $\sigma_s:=(\bar{k}_1,\bar{k}_2,...,\bar{k}_s)$. Because the orbit-mutation is nothing but a composition of mutations in directions of all indices falling in this specific orbit, it is clear that the property of sign-coherence proved in [7, Theorem 3] is well preserved for the C-matrix C^{σ_s} of the extended skew-symmetrizable matrix $\widetilde{B}^{\sigma_s}=\begin{pmatrix}B^{\sigma_s}\\C^{\sigma_s}\end{pmatrix}$.

Lemma 3.11. Let $(\widetilde{B},G)$ be a stable admissible pair and let $\widetilde{B}^{\bar{k}}=\mu_{\bar{k}}(\widetilde{B})=\begin{pmatrix}B^{\bar{k}}\\C^{\bar{k}}\end{pmatrix}$ be the orbit-mutation in direction $\bar{k}$ for $1\le k\le n$. If i_1,i_2 are of the same orbit, the columns indexed by i_1,i_2 in the C-matrix $C^{\bar{k}}$ have the same sign.

Proof. For any i_1,i_2 falling in the same orbit, there is $g\in G$ such that $g(i_1)=i_2$. Note that g is an automorphism for $\mu_{\bar{k}}(\widetilde{B})$, thus $c^{\bar{k}}_{li_1}=c^{\bar{k}}_{g(l)g(i_1)}=c^{\bar{k}}_{g(l)i_2}$ for any $1\le l\le n$. By the sign coherence property the two columns i_1,i_2 in the C-matrix $C^{\bar{k}}$ have the same sign since they have two equal entries. □

In the light of the previous lemma we denote the sign of an orbit $\bar{i}$ by $sgn(\bar{i})$ for any $i\in\{1,2,...,n\}$. Now we can establish the definitions of the orbit-maximal green sequence.

Definition 3.12. Let $(\widetilde{B},G)$ be a stable admissible pair and let C^{σ_s} be the $C-matrix$ of $\widetilde{B}^{\sigma_s}$ obtained from $\widetilde{B}$ by a sequence of orbit-mutations, $\widetilde{B}^{\sigma_s}=\mu_{\sigma_s}(\widetilde{B})$ where $\sigma_s:=(\bar{k}_1,\bar{k}_2,...,\bar{k}_s)$ such that $1\le k_j\le n$ and $1\le j\le s$, then the orbit $\bar{i}$ is said to be *green* (respectively, *red*) if $sgn(\bar{i})$ is nonnegative (respectively, nonpositive) in C^{σ_s}.

Definition 3.13. Let $(\widetilde{B}, G)$ be a stable admissible pair, $\sigma_s := (\bar{k}_1, \bar{k}_2, ..., \bar{k}_s)$ be a sequence of orbit-mutations and let C^{σ_j} be the *C-matrix* of $\widetilde{B}^{\sigma_j}$ obtained from $\widetilde{B}$ by the sequence of orbit-mutations $(\bar{k}_1, \bar{k}_2, ..., \bar{k}_j)$ for $1 \leq j \leq s$, then $(\bar{k}_1, \bar{k}_2 ...., \bar{k}_s)$ is said to be *an orbit-green sequence* if for every $1 \leq j \leq s$, $\bar{k}_j$ is a green orbit in $C^{\sigma_{j-1}}$. The sequence $(\bar{k}_1, \bar{k}_2 ...., \bar{k}_s)$ is said to be *orbit-maximal green sequence* if $\widetilde{B}^{\sigma_s}$ doesn't have green orbits.

Now we define the folding of a matrix as mentioned in [4] and [11].

Definition 3.14. Let $(\widetilde{B}, G)$ be a stable admissible pair, the folding matrix of $\widetilde{B}$ is given by $\pi(\widetilde{B}) = \begin{pmatrix} \pi(B) \\ \pi(C) \end{pmatrix} \in M_{2m \times m}(\mathbb{Z})$, where :

$$\pi(B) = (b_{\bar{i}\bar{j}}) \qquad \text{such that, } b_{\bar{i},\bar{j}} = \sum_{f \in \bar{i}} b_{fj} \tag{3}$$

$$\pi(C) = (c_{\bar{i}\bar{j}}) \qquad \text{such that, } c_{\bar{i},\bar{j}} = \sum_{f \in \bar{i}} c_{fj} \tag{4}$$

and m is the number of the G-orbits.

By [4, Lemma 2.5], the folding matrix of a skew-symmetrizable matix is skew-symmetrizable.

Lemma 3.15. Let $(\widetilde{B}, G)$ be a stable admissible pair, such that $\widetilde{B} = \begin{pmatrix} B \\ I_n \end{pmatrix} \in M_{2n \times n}(\mathbb{Z})$, then $\pi(\widetilde{B}) = \begin{pmatrix} \pi(B) \\ I_m \end{pmatrix}$, where m is the number of the G-orbits.

Proof. Let $c_{\bar{i}\bar{j}} =\in \pi(I_n)$. By the definition of the folding matrix and the nature of the the entries of I_n, we have:

$$c_{\bar{i}\bar{j}} = \begin{cases} 1, & \text{if } \quad \bar{i} = \bar{j} \\ 0, & \text{if } \quad \bar{i} \neq \bar{j} \end{cases}$$

□

Since the folding of an extended skew-symmetrizable matrix $\widetilde{B} = \begin{pmatrix} B \\ I_n \end{pmatrix} \in M_{2n \times n}(\mathbb{Z})$ in a stable admissible pair $(\widetilde{B}, G)$ is of the form $\pi(\widetilde{B}) = \begin{pmatrix} \pi(B) \\ I_m \end{pmatrix}$, where $\pi(B)$ is a skew-symmetrizable matrix, the natural question here is: what is the relation between orbit-maximal green sequences for the matrix $\widetilde{B}$ and maximal green sequences for its folding matrix $\pi(\widetilde{B})$.

The following Proposition shows the entries of the folding C-matrix after mutation in direction $\bar{k}$ as an index.

Proposition 3.16 ([11, Proposition 5.8]). Let B be any skew-symmetrizable integer matrix and let σ be a stable admissible automorphism of B. The matrix $\pi(C)$ which is the C-matrix of the matrix $\pi(\widetilde{B})$ satisfies the recursion relation:

$$c'_{\overline{i}\overline{j}} = \begin{cases} -c_{\overline{i}\overline{j}}, & \text{if} \quad \overline{j} = \overline{k} \\ c_{\overline{i}\overline{j}} + \frac{1}{2}(\mid c_{\overline{i}\overline{k}} \mid b_{\overline{k}\overline{j}} + c_{\overline{i}\overline{k}} \mid b_{\overline{k}\overline{j}} \mid), & \text{otherwise} \end{cases} \tag{5}$$

The following Lemma is a generalization of Theorem 2.24 in [4].

Lemma 3.17. Let $(\widetilde{B}, G)$ be a stable admissible pair and B is of the size $n \times n$, then $\pi(\mu_{\bar{k}}(\widetilde{B})) = \mu_{\bar{k}}(\pi(\widetilde{B}))$ for every $1 \leq k \leq n$.

Proof. As for the principal part of the extended matrix $\widetilde{B}$, it was proved in [4, Theorem 2.24].

For the C-matrix, we have the following points:

- the orbit-mutation of this matrix which is given in (2)
- the folding of this matrix which is given in (4)
- the mutation of the folding C-matrix which is given in (5).
- the sign coherence property.

By the previous points, we get the desired result following the same steps of the proof in Theorem 2.24 [4]. □

Remark 3.18. According to the familiar method of indexing a matrix and to avoid ambiguity, the matrix $\pi(B) = (b_{\overline{i}\overline{j}})$ for $1 \leq i, j \leq n$ will be referred to as $F = (F_{i^f j^f})$ for $1 \leq i^f, j^f \leq m$ such that i^f represents the order of the row or column indexed by $\overline{i}$ in $\pi(B)$ and m is the number of the orbits. Here, i^f is called the *natural corresponding index* of $\overline{i}$.

Theorem 3.19. Let $(\widetilde{B}, G)$ be a stable admissible pair. Then $(\bar{k}_1, \bar{k}_2......., \bar{k}_s)$ is an orbit-maximal green sequence for B if and only if the natural corresponding sequence $(k_1^f, k_2^f......., k_s^f)$ is a maximal green sequence for its folding matrix $\pi(B) = F$.

Proof. Let $\mu_{k_j^f}\mu_{k_{j-1}^f}.....\mu_{k_1^f}(\pi(\widetilde{B})) = \begin{pmatrix} (\pi(B))^{\sigma_j} \\ (\pi(C))^{\sigma_j} \end{pmatrix}$, and $\pi(\mu_{\bar{k}_j}\mu_{\bar{k}_{j-1}}.....\mu_{\bar{k}_1}(\widetilde{B})) = \pi \begin{pmatrix} B^{\sigma_j} \\ C^{\sigma_j} \end{pmatrix} = \begin{pmatrix} \pi(B^{\sigma_j}) \\ \pi(C^{\sigma_j}) \end{pmatrix}$ for every $1 \leq j \leq s$.

By Lemma 3.17 we have:

$$\begin{pmatrix} (\pi(B))^{\sigma_j} \\ (\pi(C))^{\sigma_j} \end{pmatrix} = \begin{pmatrix} \pi(B^{\sigma_j}) \\ \pi(C^{\sigma_j}) \end{pmatrix}.$$

That is, $(\pi(C))^{\sigma_j}$ can be considered as the folding of the matrix C^{σ_j}, by the definition of the folding matrix:

$$c^{\sigma_j}_{\bar{i}k_{j+1}} = \sum_{t\in\bar{i}} c^{\sigma_j}_{tk_{j+1}} \quad \text{such that} \quad c^{\sigma_j}_{tk_{j+1}} \in C^{\sigma_j}.$$

By the sign coherence property, we have that $c^{\sigma_j}_{\bar{i}k_{j+1}}$ is non-negative (non-positive) if and only if $c^{\sigma_j}_{tk_{j+1}}$ is non-negative (non-positive) for every $t \in \bar{i}$ and $1 \le j \le s-1$. That is, the column indexed by k^f_{j+1} in $(\pi(\widetilde{B}))^{\sigma_j}$ is green (respectively, red) if and only if the column indexed by k_{j+1} is green (respectively, red) in $\widetilde{B}^{\sigma_j}$ and as a result the orbit $\bar{k}_{j+1}$ is also green (respectively, red) for every $1 \le j \le s-1$ by Lemma 3.11. Moreover, $(\pi(\widetilde{B}))^{\sigma_s}$ has no green indices if and only if $\widetilde{B}^{\sigma_s}$ has no green orbits, and the result follows. □

The result which determines the minimal length of any maximal green sequence for a skew-symmetric matrix [2, Corollary 2.18] is clearly true in the skew-symmetrizable case.

Proposition 3.20 ([2, Corollary 2.18])**.** If B is a skew-symmetrizable matrix of the size $n \times n$ which admits a maximal green sequence, then the minimal length of any maximal green sequence for B is n.

Definition 3.21. The length of an orbit-maximal green sequence is the number of the orbits in this sequence.

Corollary 3.22. Let $(\widetilde{B}, G)$ be a stable admissible pair, the minimum length of any orbit-maximal green sequence is the number of G-orbits.

Proof. By Theorem 3.19 every orbit-maximal sequence for $\widetilde{B}$ corresponds to a maximal green sequence for its folding matrix $\pi(\widetilde{B}) \in M_{m\times m}(\mathbb{Z})$ where m is the number of the orbits. By Proposition 3.20 the minimum length of any such maximal green sequence is m, and this is the required. □

Example 3.23. $\widetilde{B} = \begin{pmatrix} 0 & 2 & 2 & 0 \\ -2 & 0 & 0 & 2 \\ -2 & 0 & 0 & 2 \\ 0 & -2 & -2 & 0 \\ & --- & & \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is an acyclic skew symmetric matrix, the pair $\left(\widetilde{B}, G\right)$ is an admissible pair where $G = \langle (2,3) \rangle$, by [4, Theorem 2.23] this pair is stable admissible.

$$\mu_{\bar{1}}\mu_{\bar{2}}\mu_{\bar{4}}\left(\widetilde{B}\right) = \begin{pmatrix} 0 & 2 & 2 & 0 \\ -2 & 0 & 0 & 2 \\ -2 & 0 & 0 & 2 \\ 0 & -2 & -2 & 0 \\ & --- & & \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$, thus $(\bar{4}, \bar{2}, \bar{1})$ is an orbit-maximal green sequence. The folding matrix $\pi(\widetilde{B}) = \begin{pmatrix} 0 & 2 & 0 \\ -4 & 0 & 4 \\ 0 & -2 & 0 \\ & --- & \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. The natural corresponding sequence for the sequence $(\bar{4}, \bar{2}, \bar{1})$ is $(3, 2, 1)$. It is easy to check that $(3, 2, 1)$ is a maximal green sequence of the folding matrix.

Clearly, the existence of orbit-maximal green sequences for a given skew-symmetrizable matrix B is related to two factors; the matrix entries and the group G itself for which the pair (B, G) is a stable admissible pair. We will prove that every orbit-maximal green sequence is corresponding to a maximal green sequence.

Lemma 3.24. Let $(\widetilde{B}, G)$ be an admissible pair, the entries of the column i_j in the $C-$matrix are invariant under any mutation in direction $i_l \neq i_j$, where i_l, i_j are of the same orbit $\bar{i}$.

Proof. Let c'_{ki_j} be one of these entries then $c'_{ki_j} = c_{ki_j} + \frac{1}{2}(|c_{ki_l}|b_{i_li_j} + c_{ki_l}|b_{i_li_j}|)$ for every $1 \leq k \leq n$. Since the pair is admissible, we have $b_{i_li_j} = 0$ and thus $c'_{ki_j} = c_{ki_j}$. □

In the following Theorem, we denote the indices of the orbit $\bar{i_s}$ by $i_{s1}, i_{l2}, ..., i_{sl_s}$ where $1 \leq s \leq m$ and m is the number of the orbits, that is l_s denotes the number of the indices in the orbit $\bar{i_s}$.

Theorem 3.25. Let $(\widetilde{B}, G)$ be a stable admissible pair, then every orbit-green sequence (respectively, orbit-maximal green sequence) of $(\widetilde{B}, G)$ corresponds to a green sequence (respectively, maximal green sequence) of $\widetilde{B}$.

Proof. (i) Suppose $(\bar{i_1}, \bar{i_2}, ..., \bar{i_k})$ is an orbit-green sequence, then $\bar{i_s}$ is a green orbit in the C-matrix of the extended matrix $\mu_{\bar{i}_{s-1}}...\mu_{\bar{i_2}}\mu_{\bar{i_1}}(\widetilde{B})$ for every $1 \leq s \leq k$. By the definition of orbit mutation, the stability of the admissible condition and Lemma 3.24 the column indexed by i_{fg} is green in the $C-$matrix of the extended matrix $\mu_{i_{fg-1}}...\mu_{i_{f1}}...\mu_{i_{1l_1}}...\mu_{i_{11}}(\widetilde{B})$ for every $1 \leq f \leq k$ and every $1 \leq g \leq l_f$. Hence the sequence $(i_{11}, ..., i_{1l_1}..., i_{k1}, ..., i_{kl_k})$ is a green sequence.

(ii) Suppose $(\bar{i_1}, \bar{i_2}, ..., \bar{i_k})$ is an orbit-maximal green sequence, then by the proof of the part (i), $(i_{11}, ..., i_{1l_1}..., i_{k1}, ..., i_{kl_k})$ is a green sequence which changes the matrix $\begin{pmatrix} B \\ I_n \end{pmatrix}$ into the matrix $\mu_{i_{kl_k}}...\mu_{i_{k1}}....\mu_{i_{1l_1}}...\mu_{i_{11}} \begin{pmatrix} B \\ I_n \end{pmatrix} \simeq \begin{pmatrix} B \\ -I_n \end{pmatrix}$ and thus this green sequence is maximal. □

We do not think that it is always possible to construct an orbit-maximal green sequence from a given maximal green sequence for a skew-symmetrizable matrix $\widetilde{B}$ since a stable admissible pair $(\widetilde{B}, G)$ may not exist. Let us suppose that such pair does exist and $\widetilde{B}$ admits a maximal green sequence, still we do not think that when changing index mutation into orbit-mutation, the green indices are guaranteed to remain green. This raises the following question:

Question 3.26. When can a maximal green sequence for a skew-symmetrizable matrix $\widetilde{B}$ in a stable admissible pair $(\widetilde{B}, G)$ be reconstructed as an orbit-maximal green sequence?

In an attempt to answer this question, we will discuss a special case of both skew-symmetrizable matrices and maximal green sequences.

Definition 3.27. A skew-symmetrizable matrix B is said to be *acyclic* if there are no entries $i_1, ..., i_k$ such that $b_{i_{j+1}i_j} > 0$ for every $1 \leq j \leq k-1$ and $i_k = i_1$.

Definition 3.28. A *source* in a skew-symmetrizable matrix B of the size $n \times n$ is an index i where $1 \leq i \leq n$ and $b_{ki} \geq 0$ for all $1 \leq k \leq n$.

Definition 3.29. An *admissible numbering by sources* of a skew-symmetrizable matrix $\widetilde{B}$ is an n-tuple $(i_1, i_2, ..., i_n)$ such that the indices of B are $\{i_1, ..., i_n\}$ with i_1 a source in $\widetilde{B}$, and for any $2 \leq k \leq n$, the vertex i_k is a source in $\mu_{i_{k-1}}....\mu_{i_1}(\widetilde{B})$.

In [2, Lemma 2.20] the following Proposition was considered for quivers (in the skew-symmetric case). It can be easily checked that it is true for the skew-symmetrizable case following the same steps.

Proposition 3.30 ([2, Lemma 2.20])**.** Let $\widetilde{B} = \begin{pmatrix} B \\ I_n \end{pmatrix} \in M_{2n\times n}(\mathbb{Z})$ be a skew-symmetrizable matrix. Then any admissible numbering by sources of $\widetilde{B}$ is a maximal green sequence.

We will call such maximal green sequence an *admissible numbering maximal green sequence.*

Lemma 3.31. If i is a source in a skew-symmetrizable matrix $\widetilde{B} = \begin{pmatrix} B \\ I_n \end{pmatrix} \in M_{2n\times n}(\mathbb{Z})$ and suppose j is another green (respectively, red) source in this matrix, then j will remain a green (respectively, red) source in the matrix $\mu_i(B)$.

Proof. The result follows by substituting $b_{ij} = 0$ in the exchange relations. □

Theorem 3.32. Let $(\widetilde{B}, G)$ be a stable admissible pair such that $\widetilde{B} = \begin{pmatrix} B \\ I_n \end{pmatrix} \in M_{2n\times n}(\mathbb{Z})$ is an acyclic skew-symmetrizable matrix and let $s = (s_1, s_2, ..., s_n)$ be an admissible numbering maximal green sequence of $\widetilde{B}$.

(i) We can always re-arrange this sequence in a way such that the indices of the orbit represented by s_1 are successive and are followed by the indices of the orbit represented by s_2 (in case they are not of the same orbit) and so on, to produce a new admissible numbering maximal green sequence of $\widetilde{B}$.

(ii) This new admissible numbering maximal green sequence corresponds to an orbit-maximal green sequence.

Proof. (i) If the indices of s are arranged in the way described above, there is nothing to do. Suppose that there are tow consecutive indices s_k, s_{k+1} which do not meet the above description and they are the first indices to do that. We denote the orbits of G by $\bar{i_1}, \bar{i_2}, ..., \bar{i_m}$ and the indices of each orbit by $\bar{i_g} = \{i_{g1}, ..., i_{gl_g}\}$ for $1 \leq g \leq m$. Without losing of generality we suppose the first index of the orbit $\bar{i_t}$ in s is $i_{t1}, s_k = i_{tw}, s_{k+1} = i_{d1}$, with $t \neq d$ and $1 \leq w < l_t$. Mutations before starting mutating in directions of indices of the orbit $\bar{i_t}$ can be regarded as orbit-mutations. We denote the matrix obtained by this orbit-mutations B'. i_{t1} is a source in the matrix B'. Since G is a group of automorphisms, $b_{qi_{t1}} \geq 0$ if and only if $b_{qi_{tf}} \geq 0$ for every $1 \leq q \leq n$ and every $1 \leq f \leq l_t$. Hence i_{tf} is also a source in the matrix B' for every $1 \leq f \leq l_t$. Since B' is obtained by a composition of orbit-mutations and i_{t1} is green, then by Lemma 3.11 all the indices of the orbit $\bar{i_t}$ are green. By Lemma 3.31 i_{tf} will remain a green source in the matrix $\mu_{i_{tf-1}}..\mu_{i_{t1}}(B')$ for every $1 \leq f \leq l_t$. $s_{k+1} = i_{d1}$ is a green source in $\mu_{i_{tw}}..\mu_{i_{t1}}(B')$ again by By Lemma 3.31 it will remain a green source in $\mu_{\bar{i_t}}(B')$. By iterating the same previous discussion we can put the indices of the orbit $\bar{i_d}$ together such that every i_{dr} is a green source in the matrix $\mu_{i_{dr-1}}..\mu_{i_{d1}}\mu_{\bar{i_t}}(B')$ for every $1 \leq r \leq l_d$. We do the same for all orbits whose indices are not successive. Finally we get a new admissible numbering maximal green sequence arranged as required.
(ii) By the first part of the proof, we can transform s into a new admissible numbering maximal green sequence s' where the indices of each orbit are successive $s' = (i_{11}, ..., i_{1l_1}, ..., i_{m1}, ..., i_{ml_m}) = (\bar{i}_1, ..., \bar{i}_m)$, where $\bar{i_g}$ is a green orbit in $\mu_{\bar{i}_{g-1}}..\mu_{\bar{i_1}}(B)$ for every for $1 \leq g \leq m$ and $\mu_{\bar{i}_m}......\mu_{\bar{i_1}}(\widetilde{B})$ has no green orbits. By the definition of the orbit-maximal green sequence, the result follows. □

4. General-maximal green sequences

It is not necessary for a skew symmetrizable matrix to have a maximal green sequence, this fact is a motivation to set a more general concept.

Definition 4.1. Let $\widetilde{B} = \begin{pmatrix} B \\ I_n \end{pmatrix} \in M_{2n\times n}(\mathbb{Z})$ be a skew-symmetrizable matrix and let $\tau = (k_1, .., k_{s-1}, k_s)$ be a green sequence, such sequence is called a *general-maximal green sequence* if $\mu_{k_s}\mu_{k_{s-1}}.....\mu_{k_1}(\widetilde{B})$ has the maximum number of red indices among the red indices obtained by any other green sequences.

Definition 4.2. Let $\widetilde{B} = \begin{pmatrix} B \\ I_n \end{pmatrix} \in M_{2n\times n}(\mathbb{Z})$ be a skew-symmetrizable matrix and let $\tau = (k_1, ..., k_{s-1}, k_s)$ be a general-maximal green sequence for this matrix, the maximum number of red indices obtained by τ is called the *red size* of $\widetilde{B}$. A general-maximal green sequence becomes a maximal green sequence when the red size of $\widetilde{B}$ is n.

A general-maximal green sequence which is not a maximal green sequence is called a *proper general-maximal green sequence.*

Proposition 4.3 ([10, Theorem 12])**.** Let $Q_{a,b,c}$ a cyclic quiver with three vertices $1, 2, 3$, such that there are a-many arrows from 1 to 2 and b-many arrows from 2 to 3 and c-many arrows from 3 to 1. If $a, b, c \geq 2$, then the quiver $Q_{a,b,c}$ does not admit a maximal green sequence

Now we can find a general-maximal green sequence for 3×3 quivers which do not admit a maximal green sequence.

Example 4.4. $B = \begin{pmatrix} 0 & -3 & 4 \\ 3 & 0 & -3 \\ -4 & 3 & 0 \end{pmatrix}$. By Proposition 4.3 this matrix does not admit a maximal green sequence thus its red size is at most 2.

$$\begin{pmatrix} 0 & -3 & 4 \\ 3 & 0 & -3 \\ -4 & 3 & 0 \\ & --- & \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \overset{\mu_2}{\rightsquigarrow} \begin{pmatrix} 0 & 3 & -5 \\ -3 & 0 & 3 \\ 5 & -3 & 0 \\ & --- & \\ 1 & 0 & 0 \\ 3 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \overset{\mu_1}{\rightsquigarrow} \begin{pmatrix} 0 & -3 & 5 \\ 3 & 0 & -12 \\ -5 & 12 & 0 \\ & --- & \\ -1 & 3 & 0 \\ -3 & 8 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\overset{\mu_3}{\rightsquigarrow} \begin{pmatrix} 0 & 57 & -5 \\ -57 & 0 & 12 \\ 5 & -12 & 0 \\ & --- & \\ -1 & 3 & 0 \\ -3 & 8 & 0 \\ 0 & 12 & -1 \end{pmatrix}.$$

Thus the green sequence $(2, 1, 3)$ is a general-maximal green sequence since it gives two red indices.

Since we deal with matrices of finite size, a general-maximal green sequences always exists, yet determining the red size of a given skew-symmetrizable matrix is not easy, especially in the infinite-mutation case. In the following we will discuss the red size of a reducible skew symmetrizable matrix. For more detailed information and results about reducibility we refer to [3]. We recall that an integer matrix is said to be a *column (row) sign coherent* if the nonzero elements which are of the same column (row) have the same sign. An integer matrix $B_2 \in M_{m\times n}(\mathbb{Z})$ is said to be *uniformly column (row) sign coherent* with respect to a skew symmetrizable matrix $B_1 \in M_{n\times n}(\mathbb{Z})$, if for any composition of mutations $\mu_{k_s}......\mu_{k_2}\mu_{k_1}$ where $1 \le k_i \le n$ for $1 \le i \le s$, the lower part of the matrix $\mu_{k_s}.....\mu_{k_1}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ is still column (row) sign coherent.

Proposition 4.5 ([3, Corollary 3.3])**.** Any integer matrix with non-negative entries is uniformly column sign coherent with respect to any skew symmetrizable matrix which shares the same number of columns.

Definition 4.6. Let B be a skew symmetrizable matrix. We say that B is *reducible*, if B can be written as a block matrix as follows

$$B = \begin{pmatrix} B_1 & B_3 \\ B_2 & B_4 \end{pmatrix}$$

such that $B_1 \in M_{n_1\times n_1}(\mathbb{Z}), B_4 \in M_{n_2\times n_2}(\mathbb{Z})$, and the proper submatrix $B_2 \in M_{n_2\times n_1}(\mathbb{Z})$ has non-negative entries.

Proposition 4.7 ([3, Proposition 3.7])**.** Let $B = \begin{pmatrix} B_1 & B_3 \\ B_2 & B_4 \end{pmatrix}$ be a skew-symmetrizable matrix, with $B_1 \in M_{n\times n}(\mathbb{Z})$ and $B_4 \in M_{m\times m}(\mathbb{Z})$. Then B_2 is uniformly sign coherent with respect to B_1 if and only if B_4 is invariant under any composition of mutations $\mu_{k_s}\mu_{k_{s-1}}.....\mu_{k_1} := \mu_{\sigma_s}$, where $1 \le k_i \le n$, $1 \le i \le s$.

In [3], it was proved that for a reducible matrix $B = \begin{pmatrix} B_1 & B_3 \\ B_2 & B_4 \end{pmatrix}$, k (respectively, j) is a maximal green sequence for B_1 (respectively, B_4) if and only if $\widetilde{k} = (kj)$ is a maximal green sequence for B. In the following theorem, we find the relation between the red size of a reducible skew-symmetrizable matrix and the red sizes of the skew symmetrizable matrices which appear in its decomposition.

Theorem 4.8. Let $B = \begin{pmatrix} B_1 & B_3 \\ B_2 & B_4 \end{pmatrix}$ be a reducible skew-symmetrizable matrix with $B_1 \in M_{n_1\times n_1}(\mathbb{Z})$ and $B_4 \in M_{n_2\times n_2}(\mathbb{Z})$, $\sigma_s := (k_1, ..., k_s)$ is a general-maximal green sequence for B_1 with red size k, and $\tau_r := (l_1, ..., l_r)$ is a general-maximal green sequence for B_4 with red size l. Then m the red size of B is greater than or equal to $k + l$.

Proof. When applying the sequence of mutations σ_s, we will get k red indices. Because of the non negative entries of the submatrix $\begin{pmatrix} B_2 \\ I_{n_1} \\ 0 \end{pmatrix}$ it is uniformly sign coherent with respect to B_1 by Proposition 4.5. By Proposition 4.7, the submatrix $\begin{pmatrix} B4 \\ 0 \\ I_{n_2} \end{pmatrix}$ is mutation invariant in directions $1 \leq k_s, ...k_1 \leq n$. As a result when applying the sequence of mutations τ_r we will get l new red indices while the C-matric $\mu_{k_s}..\mu_{k_1}\begin{pmatrix} I_{n_1} \\ 0 \end{pmatrix} = \begin{pmatrix} C^{\sigma_s} \\ 0 \end{pmatrix}$ remains invariant since $\begin{pmatrix} 0 \\ I_{n_2} \end{pmatrix}$ is uniformly column sign coherent with respect to the skew symmetrizable B_4 because of its non-negative entries. Then the total number of the red indices that we get after applying the composition $(\tau_r\sigma_s)$ is $k+l$ and the result follows. □

And here we raise the following question:

Question 4.9. Does the red size of a reducible skew-symmetrizable matrix equal the sum of the red sizes of the skew-symmetrizable matrices which appear in its decomposition ?

Corollary 4.10. Let $B = \begin{pmatrix} B_1 & B_3 \\ B_2 & B_4 \end{pmatrix}$ be a reducible skew-symmetric matrix of the size $n \times n$ with $B_1 \in M_{n_1 \times n_1}(\mathbb{Z})$ and $B_4 \in M_{n_2 \times n_2}(\mathbb{Z})$, $\sigma_s := (k_1, ..., k_s)$ is a general-maximal green sequence for B_1 with red size k, and $\tau_r := (l_1, ..., l_r)$ is a general-maximal green sequence for B_4 with red size l. If σ_s or τ_r is proper, then the red size m of B is determined by $l + k \leq m < n$.

Proof. The left part of the inequality is obtained from Theorem 4.8. If $m = n$, then B admits a maximal green sequence. By [10, Theorem 1.4.1] any induced sub-matrix also admits a maximal green sequence. Hence, $k = n_1$ and $l = n_2$ which contradicts with the hypothesis that one the general-maximal sequences σ_s,τ_r is proper. □

Since every orbit-maximal green sequence in a stable admissible pair corresponds to a maximal green sequence, see Theorem 3.25, some skew-symmetrizable matrices admit orbit-maximal green sequences while others do not. By following the same method of maximal green sequence generalization, we define the general orbit-maximal green sequence.

Definition 4.11. For a stable admissible pair $(\widetilde{B}, G)$ and an orbit-green sequence $\sigma = (\bar{k}_1, ..., \bar{k}_{s-1}, \bar{k}_s)$, this sequence is said to be a *general orbit-maximal green sequence* if $\mu_\sigma(\widetilde{B})$ has the maximum number of red orbits among the red orbits obtained by any other orbit-green sequences. Similarly the number of these red orbits (which are obtained maximally) is called the *orbit-red size of* $\widetilde{B}$.

It is clear that every orbit-maximal green sequence for a skew-symmetrizable matrix $\widetilde{B} = \begin{pmatrix} B \\ I_n \end{pmatrix} \in M_{2n\times n}(\mathbb{Z})$ in a stable admissible pair $(\widetilde{B}, G)$ is a general orbit-maximal green sequence whose orbit red size is the number of orbits obtained under the action of G .

Lemma 4.12. Let B be a skew-symmetrizable matrix of the size $n = 3 \times 3$ and (B, G) be a stable admissible pair with G not trivial (G is not the identity or S_3), then every general orbit-maximal green sequence has an orbit red size $m = 2$ (that is, it is an orbit-maximal green sequence).

Proof. Since G is not trivial, G has two orbits $j = \{i_1, i_2\}$ and $m = \{i_3\}$. By the admissibility $b_{i_1 i_2} = 0$ and $b_{i_1 i_3}.b_{i_2 i_3} \geq 0$. Hence B is acyclic with i_3 is the only source and i_1,i_2 are two sinks or i_3 is the only sink and i_1,i_2 are two sources. Thus (m, j), (j, m) are two orbit-maximal green sequence and $m = 2$ is the maximum number of red orbits to be obtained. □

In analogy to Theorem 3.25, we get the following result:

Corollary 4.13. For a stable admissible pair $(\widetilde{B}, G)$, every general orbit-maximal green sequence corresponds to a general-maximal green sequence.

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