Prime-valent one-regular graphs of order $16p$

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Abstract. A graph is *one-regular* and *arc-transitive* if its automorphism group acts on its arcs regularly and transitively, respectively. In this paper, we classify one-regular graphs of prime valency and order $16p$ for each prime p. By analyzing the structure of the full automorphism group of such graphs and using the classification of arc-transitive graphs of order $2p$, we prove that there are only two infinite families of such graphs: one is the cycle C_{16p} with valency 2, the other is the $\mathbb{Z}_p-$cover CF_{16p} of Möbius-Kantor graph F_{16} with valency 3 and $6|(p-1)$.

Keywords: one-regular graph, arc-transitive graph, covering graph.

1. Introduction

Throughout this paper graphs are assumed to be finite, simple, connected and undirected. For group-theoretic concepts or graph-theoretic terms not defined here we refer the reader to [15, 17] or [1, 2], respectively. Let G be a permutation group on a set Ω and $v \in \Omega$. Denote by G_v the stabilizer of v in G, that is, the subgroup of G fixing the point v. We say that G is *semiregular* on Ω if $G_v = 1$ for every $v \in \Omega$ and *regular* if G is transitive and semiregular.

For a graph X, denote by $V(X)$, $E(X)$ and $\text{Aut}(X)$ its vertex set, its edge set and its full automorphism group, respectively. A graph X is said to be *G-vertex-transitive* if $G \leq \text{Aut}(X)$ acts transitively on $V(X)$. X is simply called *vertex-transitive* if it is $\text{Aut}(X)$-vertex-transitive. An *s-arc* in a graph is an ordered $(s+1)$-tuple $(v_0, v_1, \cdots, v_{s-1}, v_s)$ of vertices of the graph X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. In particular, a 1-arc is just an arc and a 0-arc is a vertex. For a subgroup $G \leq \text{Aut}(X)$, a graph X is said to be *(G, s)-arc-transitive* or *(G, s)-regular* if G is transitive or regular on the set of s-arcs in X, respectively. A (G, s)-arc-transitive graph is said to be *(G, s)-transitive* if it is not $(G, s+1)$-arc-transitive. In particular, a $(G, 1)$-arc-transitive graph is called *G-symmetric*. A graph X is simply called *s-arc-transitive*, *s-regular* or *s-transitive* if it is $(\text{Aut}(X), s)$-arc-transitive, $(\text{Aut}(X), s)$-regular or $(\text{Aut}(X), s)$-transitive, respectively.

We denote by C_n and K_n the cycle and the complete graph of order n, respectively. Denote by D_{2n} the dihedral group of order $2n$. As we all known that there is only one connected 2-valent graph of order n, that is, the cycle

C_n, which is 1-regular with full automorphism group D_{2n}. Let p be a prime. Classifying s-transitive and s-regular graphs has received considerable attention. The classification of s-transitive graphs of order p and $2p$ was given in [4] and [5], respectively. Wang [16] characterized the prime-valent s-transitive graphs of order $4p$. The classification of cubic, pentavalent and heptavalent s-transitive graphs of order $16p$ was given in [14], [10] and [11], respectively.

For 2-valent case, s-transitivity always means 1-regularity, and for cubic case, s-transitivity always means s-regularity by Miller [8]. However, for the other prime-valent case, this is not true, see for example [9] for pentavalent case and [12] for heptavalent case. Thus, characterization and classification of prime-valent s-regular graphs is very interesting and also reveals the s-regular global and local actions of the permutation groups on s-arcs of such graphs. In particular, 1-regular action is the most simple and typical situation. In this paper, we classify prime-valent one-regular graphs of order $16p$ for each prime p.

2. Preliminary results

Let X be a connected G-symmetric-transitive graph with $G \leq \mathrm{Aut}(X)$, and let N be a normal subgroup of G. The *quotient graph* X_N of X relative to N is defined as the graph with vertices the orbits of N on $V(X)$ and with two orbits adjacent if there is an edge in X between those two orbits. In view of [13, Theorem 9], we have the following:

Proposition 2.1. *Let X be a connected G-symmetric graph with $G \leq \mathrm{Aut}(X)$ and prime valency $q \geq 3$, and let N be a normal subgroup of G. Then one of the following holds:*

(1) *N is transitive on $V(X)$;*

(2) *X is bipartite and N is transitive on each part of the bipartition;*

(3) *N has $r \geq 3$ orbits on $V(X)$, N acts semiregularly on $V(X)$, the quotient graph X_N is a connected q-valent G/N-symmetric graph.*

To extract a classification of connected prime-valent symmetric graphs of order $2p$ for a prime p from Cheng and Oxley [5], we introduce the graphs $G(2p, q)$. Let V and V' be two disjoint copies of $\mathbb{Z}_p$, say $V = \{0, 1, \cdots, p-1\}$ and $V' = \{0', 1', \cdots, (p-1)'\}$. Let q be a positive integer dividing $p-1$ and $H(p, q)$ the unique subgroup of Z_p^* of order q. Define the graph $G(2p, q)$ to have vertex set $V \cup V'$ and edge set $\{xy' \mid x - y \in H(p, q)\}$.

Proposition 2.2. *Let X be a connected q-valent symmetric graph of order $2p$ with p, q primes. Then X is isomorphic to K_{2p} with $q = 2p-1$, $K_{p,p}$ or $G(2p, q)$ with $q|(p-1)$. Furthermore, if $(p, q) \neq (11, 5)$ then* $\mathrm{Aut}(G(2p, q)) = (\mathbb{Z}_p \rtimes \mathbb{Z}_q) \rtimes \mathbb{Z}_2$*; if $(p, q) = (11, 5)$ then* $\mathrm{Aut}(G(2p, q)) = \mathrm{PGL}(2, 11)$.

From [7, pp.12-14], we can deduce the non-abelian simple groups whose orders have at most three different prime divisors.

Proposition 2.3. *Let G be a non-abelian simple group. Suppose that the order $|G|$ has at most three different prime divisors. Then G is called K_3 simple group and isomorphic to one of the following groups.*

Table 1: **Non-abelian simple $\{2, 3, p\}$-groups**

Group	Order	Group	Order
A_5	$2^2 \cdot 3 \cdot 5$	$PSL(2, 17)$	$2^4 \cdot 3^2 \cdot 17$
A_6	$2^3 \cdot 3^2 \cdot 5$	$PSL(3, 3)$	$2^4 \cdot 3^3 \cdot 13$
$PSL(2, 7)$	$2^3 \cdot 3 \cdot 7$	$PSU(3, 3)$	$2^5 \cdot 3^3 \cdot 7$
$PSL(2, 8)$	$2^3 \cdot 3^2 \cdot 7$	$PSU(4, 2)$	$2^6 \cdot 3^4 \cdot 5$

3. Classification

This section is devoted to classifying prime-valent one-regular graphs of order $16p$ for each prime p. Let q be a prime. In what follows, we always denote by X the q-valent one-regular graph of order $16p$. Set $A = \mathrm{Aut}(X)$, $v \in V(X)$. Then the vertex stabilizer $A_v \cong \mathbb{Z}_q$ and hence $|A| = 16pq$. Clearly, if $q = 2$, then $X \cong C_{16p}$ with $A \cong D_{32p}$.

Let $q = 3$. Then by [14, Theorem 2.7], we can have the classification of cubic one-regular graphs of order $16p$. For convenience, we use the same notation CF_{16p} as in [14] to denote the cyclic $\mathbb{Z}_p$-covering graph of Möbius-Kantor graph F_{16}. Denote by $D_8 \circ \mathbb{Z}_4$ the central product of D_8 and $\mathbb{Z}_4$.

Lemma 3.1. *If $q = 3$, then $X \cong CF_{16p}$ and $A \cong \mathbb{Z}_p \rtimes ((D_8 \circ \mathbb{Z}_4) \rtimes \mathbb{Z}_3)$ with $6|(p-1)$.*

For $q = 5$ or 7, by [10, Theorem 1.1] and [11, Theorem 1.1], it is easy to see that there is no new graph. Thus, we treat with the case $p \geq 3$ and $q > 7$ by proving the following lemma.

Lemma 3.2. *Let $p \geq 3$ and $q > 7$. Then there is no new graph.*

Proof. Recall that $|A| = 16pq$ and $A_v \cong \mathbb{Z}_q$. If A is non-solvable, then A must contain a non-solvable composition factor H, which is isomorphic to a non-abelian simple group. It forces that $|H|\big|16pq$ and H is a K_3 simple group. However, since $q > 7$ and $p \geq 3$, this is impossible be Proposition 2.3. Thus, A is solvable. We divide the proof into the following two cases: $p = q$ and $p \neq q$.

Case 1: Suppose that $p = q$. Then $|A| = 16p^2$ and $A_v \cong \mathbb{Z}_p$.

Let P be a Sylow p-subgroup of A. Then $|P| = p^2$. Note that $p = q > 7$. Thus, by Sylow Theorem, the number of Sylow p-subgroups of A is $kp+1 = |A :$

$N_A(P)|$ for some integer k. Since $|A| = 16p^2$, we have that $(kp+1)|16$. Suppose that P is not normal in A. Then $kp + 1 > 1$ and hence $k \geq 1$. However, $p > 7$ implies that $kp + 1$ cannot be a divisor of 16 for each $k \geq 1$, a contradiction. Thus, P is normal in A. This means that P is the only Sylow p-subgroup of A. Since $A_v \cong \mathbb{Z}_p$, we have that $A_v \leq P$, that is, $A_v = P_v \neq 1$. By Proposition 2.1, P is transitive or has two orbits on $V(X)$. Clearly, both are impossible because $|P| = p^2$ and $|V(X)| = 16p$.

Case 2: Suppose that $p \neq q$. Then $|A| = 16pq$ and $A_v \cong \mathbb{Z}_q$.

Since $|A| = 16pq$ and $A_v \cong \mathbb{Z}_q$, we have that A_v is a Sylow q-subgroup of A. It forces that the Sylow q-subgroups of A cannot be normal in A. Recall that A is solvable. Thus, all normal subgroups of A are solvable. It follows that A has a maximal normal r-subgroup with $r = 2$ or p.

Assume that A has a maximal normal p-subgroup M. Then $M \cong \mathbb{Z}_p$ because $|A| = 16pq$. Clearly, M acting on $V(X)$ has 16 orbits. By Proposition 2.1, X_M is a q-valent symmetric graph of order 16. By [6], X_M is isomorphic to Möbius-Kantor graph F_{16} with $q = 3$, Q_4^d with $q = 5$, or $K_{8,8} - 8K_2$ with $q = 7$, where Q_4^d denotes the graph obtained by connecting all long diagonal of 4-cube Q_4. This contradicts our hypothesis that $q > 7$. Thus, the Sylow p-subgroup of A cannot be normal in A.

Assume that A has a maximal normal 2-subgroup N. Since N is a 2-subgroup, we have that N acting on $V(X)$ has at least p orbits. By Proposition 2.1, X_N is a q-valent symmetric graph of order $16p/|N|$. Recall that $q > 7$ is an odd prime and there is no graph of odd order and odd valency. Thus, $|N| \neq 16$ and $|N| = 2$, 4 or 8.

Let $|N| = 2$. Then X_N is a q-valent symmetric graph of order $8p$ and $|A/N| = 8pq$. Note that A is solvable. Thus, A/N is also solvable. Since $A_v \cong \mathbb{Z}_q$, we have that A/N has no normal q-subgroup. If A/N has a normal p-subgroup $K/N \cong \mathbb{Z}_p$. Then $|K| = 2p$. Since $p \geq 3$, we have that K has a normal Sylow p-subgroup M by Sylow Theorem. It forces that M is characteristic in K and hence normal in A. By the above argument, this is impossible. This implies that A/N has a non-trivial normal 2-subgroup, this contradicts the maximality of N.

Let $|N| = 4$. Then X_N is a q-valent symmetric graph of order $4p$ and $|A/N| = 4pq$. If A/N has a normal Sylow q-subgroup $H/N \cong \mathbb{Z}_q$, then H is normal in A. Clearly, $|H| = 4q$. Note that $q > 7$. Thus, Sylow Theorem implies that H has a normal Sylow q-subgroup $Q \cong \mathbb{Z}_q$. It forces that Q is characteristic in H and hence normal in A. This is impossible because a Sylow q-subgroup of A cannot be normal in A. If A/N has a normal Sylow p-subgroup $K/N \cong \mathbb{Z}_p$, then $|K| = 4p$ and K is normal in A. Recall $p \geq 3$. Suppose that $p = 3$. Then X_N is a q-valent symmetric graph of order 12. Since $q > 7$, by [6], the only possible is $X_N \cong K_{12}$. In this case, $|A/N| = 12{\cdot}11$ and $A/N \lesssim S_{12}$. However, by Magma [3], S_{12} has no arc-transitive subgroup of order $12{\cdot}11$, a contradiction. Suppose that $p \geq 5$. Then by Sylow Theorem K must have a normal Sylow p-subgroup P. It forces that P is characteristic in K and hence normal in A. This is also impossible because by the above argument A has no normal Sylow p-subgroup.

Let $|N| = 8$. Then X_N is a q-valent symmetric graph of order $2p$ and $|A/N| = 2pq$. Recall that $q \neq p$ and $q \neq 3$. By Proposition 2.2, $X_N \cong K_{2p}$ with $q = 2p-1$ or $G(2p, q)$ with $q|(p-1)$ with $A \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_q) \rtimes \mathbb{Z}_2$. For the former, $q = 2p-1$ is a prime. Thus, $A/N \lesssim S_{2p}$ and A/N is 2-transitive on $V(X)$. By Burnside's Theorem, any 2-transitive permutation group is almost simple or affine. Since A/N is solvable, we have that A/N is affine. It forces that A/N must have a normal subgroup isomorphic to $\mathbb{Z}_p$. By the above argument, this is impossible. For the later, $A/N \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_q) \rtimes \mathbb{Z}_2$. It is easy to see that A/N has a normal Sylow p-subgroup. Since $q > 7$ and $q|(p-1)$, we have that $p \geq 23$. With an easy calculation by using Sylow Theorem, we can deduce that A has a normal Sylow p-subgroup, a contradiction. □

Combining the above arguments with the cases $q = 2, 5, 7$, and by Lemmas 3.1-3.2, we have the following result.

Theorem 3.3. *Let p, q be two primes and X a connected q-valent one-regular graph of order $16p$. Then $X \cong C_{16p}$ with valency 2 and $\mathrm{Aut}(X) \cong D_{32p}$ or $X \cong CF_{16p}$ with valency 3, $\mathrm{Aut}(X) \cong \mathbb{Z}_p \rtimes ((D_8 \circ \mathbb{Z}_4) \times \mathbb{Z}_3)$ and $6|(p-1)$.*

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