# A note on relative $\varphi$-order and relative $\varphi$-lower order of entire functions of several complex variables

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**Abstract.** In this paper we introduce the notions of relative $\varphi$-order and relative $\varphi$-lower order of entire functions of several complex variables. We study then some growth properties in connection with relative $\varphi$-order and relative $\varphi$-lower order of entire functions of several complex variables.

**Keywords:** Entire functions, relative $\varphi$-order, relative $\varphi$- lower order, several complex variables, Property (R), Property (P), growth.

## 1. Introduction, definitions and notations

Let $f$ be a non-constant entire function of two complex variables holomorphic in the closed polydisc

$$U = \{(z_1, z_2) : \ |z_i| \leq r_i, \ i = 1, 2 \text{ for all } \ r_1 \geq 0, r_2 \geq 0\}$$

and $M_f(r_1, r_2) = \max\{|f(z_1, z_2)| : |z_i| \leq r_i, \ i = 1, 2\}$. Then in view of maximum principle and Hartogs theorem [8, p. 2, p. 51], $M_f(r_1, r_2)$ is an increasing functions of $r_1, r_2$. In this connection the following definition is well known:

**Definition 1** ([1, 8]). *The order ${}_{v_2}\rho(f)$ and the lower order ${}_{v_2}\lambda(f)$ of an entire function $f$ of two complex variables are defined as*

$$\begin{matrix} {}_{v_2}\rho(f) \\ {}_{v_2}\lambda(f) \end{matrix} = \lim_{r_1, r_2 \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \log M_f(r_1, r_2)}{\log(r_1 r_2)}.$$

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The equivalent formula for ${}_{v_2}\rho(f)$ [8, p. 338] is

$${}_{v_2}\rho(f) = \inf \mu > 0 : M_f(r_1, r_2) < \exp\left[(r_1 r_2)^{\mu}\right] \text{ for all } r_1 \geq R(\mu), r_2 \geq R(\mu) \ .$$

Similarly, one can define ${}_{v_2}\lambda(f)$ as

$${}_{v_2}\lambda(f) = \sup \mu > 0 : M_f(r_1, r_2) > \exp\left[(r_1 r_2)^{\mu}\right] \text{ for all } r_1 \geq R(\mu), r_2 \geq R(\mu) \ .$$

In [6], Chyzhykov et al. introduced the definition of $\varphi$-order of a meromorphic function on single variable in the unit disc. For details about $\varphi$-order, one may see [6]. Extending this notion, it is natural for us to give the $\varphi$-order of entire function holomorphic in the closed polydisc $\{(z_1, z_2) : \ |z_i| \leq r_i, \ i = 1, 2$ for all $r_1 \geq 0, r_2 \geq 0\}$ which is as follows:

**Definition 2.** *Let $\varphi_i(r_1, r_2) \mid i = 1, 2 : [0, +\infty) \times [0, +\infty) \to (0, +\infty)$ be a non-decreasing unbounded function of two variables $r_1$ and $r_2$. The $\varphi$-order of an entire function $f$ of two complex variables denoted by ${}_{v_2}\rho(f, \varphi)$ is defined as:*

$$\begin{aligned} {}_{v_2}\rho(f, \varphi) = \inf \{ & \mu > 0 : M_f(r_1, r_2) < \exp\left[(\varphi_1(r_1, r_2)\, \varphi_2(r_1, r_2))^{\mu}\right], \\ & r_1 \geq R(\mu), r_2 \geq R(\mu) \} . \end{aligned}$$

If we consider $\varphi(r_1, r_2) = \varphi_1(r_1, r_2)\, \varphi_2(r_1, r_2) = r_1 r_2$ or $\varphi_1(r_1, r_2) = \varphi_2(r_1, r_2) = (r_1 r_2)^{\frac{1}{2}}$, then Definition 2 coincides with Definition 1.

Analogously, one can define the $\varphi$-lower order of $f$ of two complex variables denoted by ${}_{v_2}\lambda(f, \varphi)$ as follows :

$$\begin{aligned} {}_{v_2}\lambda(f, \varphi) = \sup \{ & \mu > 0 : M_f(r_1, r_2) > \exp\left[(\varphi_1(r_1, r_2)\, \varphi_2(r_1, r_2))^{\mu}\right]; \\ & r_1 \geq R(\mu), r_2 \geq R(\mu) \} , \end{aligned}$$

where $\varphi_i(r_1, r_2) \mid i = 1, 2 : [0, +\infty) \times [0, +\infty) \to (0, +\infty)$ be a non-decreasing unbounded function of two variables $r_1$ and $r_2$.

Now if we consider Definition 1 for single variable, then the definition coincides with the classical definition of order (see [14]) which is as follows:

**Definition 3** ([14])**.** *The order $\rho(f)$ and the lower order $\lambda(f)$ of an entire function $f$ are defined in the following way:*

$$\begin{matrix} \rho(f) \\ \lambda(f) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log\log M_f(r)}{\log r},$$

*where $M_f(r) = \max\{|f(z)| : |z| = r\}$.*

Further if $f$ is non-constant then $M_f(r)$ is strictly increasing and continuous, and its inverse ${M_f}^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and is such that $\lim_{s\to\infty} {M_f}^{-1}(s) = \infty$. Bernal {[2], [3]} introduced the definition of relative order of $f$ with respect to $g$, denoted by $\rho_g(f)$ as follows:

$$\begin{aligned} \rho_g(f) &= \inf\{\mu > 0 : M_f(r) < M_g(r^{\mu}) \text{ for all } r > r_0(\mu) > 0\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1}(M_f(r))}{\log r}. \end{aligned}$$

The definition coincides with the classical one [14] if $g(z) = \exp z$.

During the past decades, several authors (see [5], [9], [10], [11], [12], [13]) made close investigations on the properties of relative order of entire functions of single variable. In the case of relative order, it was then natural for Banerjee and Dutta [4] to define the relative order of entire functions of two complex variables as follows:

**Definition 4** ([4])**.** *The relative order between two entire functions of two complex variables denoted by ${}_{v_2}\rho_g(f)$ is defined as:*

$$ {}_{v_2}\rho_g(f) = \inf\{\mu > 0 : M_f(r_1, r_2) < M_g(r_1^\mu, r_2^\mu); r_1 \geq R(\mu), r_2 \geq R(\mu)\}$$

*where $f$ and $g$ are entire functions holomorphic in the closed polydisc*

$$U = \{(z_1, z_2) : |z_i| \leq r_i,\ i = 1, 2 \text{ for all } r_1 \geq 0, r_2 \geq 0\}$$

*and the definition coincides with Definition 1 {see [4]} if $g(z) = \exp(z_1 z_2)$.*

Extending this notion, Dutta [7] introduced the idea of relative order of entire functions of several complex variables in the following way:

**Definition 5** ([7])**.** *Let $f(z_1, z_2, ..., z_n)$ and $g(z_1, z_2, ..., z_n)$ be any two entire functions of $n$ variables $z_1, z_2, ..., z_n$ with maximum modulus functions $M_f(r_1, r_2, ..., r_n)$ and $M_g(r_1, r_2, ..., r_n)$ respectively then the relative order of $f$ with respect to $g$, denoted by ${}_{v_n}\rho_g(f)$ is defined by*

$$\begin{aligned} {}_{v_n}\rho_g(f) = \inf\{&\mu > 0 : M_f(r_1, r_2, ..., r_n) < M_g(r_1^\mu, r_2^\mu, ..., r_n^\mu); \\ &\text{for } r_i \geq R(\mu), i = 1, 2, ..., n\}\ . \end{aligned}$$

Similarly, one can define the relative lower order of $f$ with respect to $g$ denoted by ${}_{v_n}\lambda_g(f)$ as follows :

$$\begin{aligned} {}_{v_n}\lambda_g(f) = \sup\{&\mu > 0 : M_f(r_1, r_2, ..., r_n) > M_g(r_1^\mu, r_2^\mu, ..., r_n^\mu); \\ &\text{for } r_i \geq R(\mu), i = 1, 2, ..., n\}\ . \end{aligned}$$

Now in order to make some progress in the study of relative order of entire functions of several complex variables, one may introduce the definition of relative $\varphi$-order between two entire functions of several complex variables in the following way:

**Definition 6.** *Let $\varphi_i(r_1, r_2, ..., r_n) \mid i = 1, 2, .., n : [0, +\infty) \times [0, +\infty) \times ... \times [0, +\infty) \to (0, +\infty)$ be a non-decreasing unbounded function of $n$ variables $r_1, r_2, ..., r_n$. Also let $f$ and $g$ be any two entire functions of $n$ complex variables with maximum modulus functions $M_f(r_1, r_2, ..., r_n)$ and $M_g(r_1, r_2, ..., r_n)$ respectively then the relative $\varphi$-order of $f$ with respect to $g$, denoted by ${}_{v_n}\rho_g(f, \varphi)$ is defined by*

$$\begin{aligned} {}_{v_n}\rho_g(f, \varphi) = \inf\{&\mu > 0 : M_f(r_1, r_2, ..., r_n) < M_g(\varphi_1^\mu, \varphi_2^\mu, ..., \varphi_n^\mu); \\ &\text{for } r_i \geq R(\mu), i = 1, 2, ..., n\}, \end{aligned}$$

*where $\varphi_i \mid i = 1, 2, .., n$ stand for $\varphi_i(r_1, r_2, ..., r_n) \mid i = 1, 2, .., n$.*

If we consider $\varphi_1(r_1, r_2, ..., r_n) = r_1$, $\varphi_2(r_1, r_2, ..., r_n) = r_2$, ..., $\varphi_n(r_1, r_2, ..., r_n) = r_n$, then Definition 6 coincides with Definition 5. Also if we consider $n = 2$ and $g(z_1 z_2) = \exp(z_1 z_2)$, then Definition 6 coincides with Definition 2.

Likewise, one can define the relative $\varphi$-lower order of $f$ with respect to $g$ denoted by ${}_{v_n}\lambda_g(f, \varphi)$ as follows:

$$\begin{aligned} {}_{v_n}\lambda_g(f,\varphi) = \sup\{&\mu > 0 : M_f(r_1, r_2, ..., r_n) > M_g(\varphi_1^{\mu}, \varphi_2^{\mu}, ..., \varphi_n^{\mu}) ; \\ &\text{for } r_i \geq R(\mu), i = 1, 2, ..., n\}, \end{aligned}$$

where $\varphi_i \mid i = 1, 2, .., n : [0, +\infty) \times [0, +\infty) \times ... \times [0, +\infty) \to (0, +\infty)$ be a non-decreasing unbounded function of $n$ variables $r_1, r_2, ..., r_n$.

Further an entire function $f$ of several complex variables for which relative $\varphi$-order and relative $\varphi$-lower order with respect to another entire function $g$ of several complex variables are the same is called a function of regular relative $\varphi$-growth with respect to $g$. Otherwise, $f$ is said to be irregular relative $\varphi$-growth.with respect to $g$.

In this connection just we state the following two definitions which will be needed in the sequel:

**Definition 7** ([7])**.** *The function $f(z_1, z_2, ..., z_n)$ is said to have Property (R) if for any $\sigma > 1$ and for all large $r_1, r_2, ..., r_n$,*

$$[M_f(r_1, r_2, ..., r_n)]^2 < M_f(r_1^{\sigma}, r_2^{\sigma}, ..., r_n^{\sigma}) .$$

*For examples of functions with or without the Property (R), one may see [7].*

**Definition 8.** *A pair of functions $f(z_1, z_2, ..., z_n)$ and $g(z_1, z_2, ..., z_n)$ of $n$ complex variables are mutually said to have Property (P) if for all sufficiently large values of $r_1, r_2, ..., r_n$, both*

$$M_{f \cdot g}(r_1, r_2, ..., r_n) > M_f(r_1, r_2, ..., r_n)$$

*and*

$$M_{f \cdot g}(r_1, r_2, ..., r_n) > M_g(r_1, r_2, ..., r_n)$$

*hold simultaneously.*

One can easily verify that the functions $f(z_1, z_2, ..., z_n) = \exp(z_1 z_2 ... z_n)$ and $g(z_1, z_2, ..., z_n) = \exp(z_1 z_2 ... z_n)^2$ have the Property (P).

Here, in this paper, we aim at investigating some basic properties of relative $\varphi$-order and relative $\varphi$-lower order of entire functions of several complex variables with respect to another one under somewhat different conditions. We do not explain the standard definitions and notations in the theory of entire function of several complex variables as those are available in [8].

## 2. Lemma

In this section we present a lemma which will be needed in the sequel.

**Lemma 1** ([7]). *Suppose that $f$ be a non constant entire function of several complex variables, $\alpha > 1$ and $0 < \beta < \alpha$. Then, $M_f(\alpha r_1, \alpha r_2, ..., \alpha r_n) > \beta M_f(r_1, r_2, ..., r_n)$ for all sufficiently large $r_1, r_2, ..., r_n$.*

## 3. Theorems.

In this section we present the main results of the paper.

**Theorem 1.** *Let us consider $f_1$, $f_2$ and $g_1$ be any three entire functions of several complex variables. Also let at least $f_1$ or $f_2$ be of regular relative $\varphi$-growth with respect to $g_1$. Then*

$$_{v_n}\lambda_{g_1}(f_1 \pm f_2, \varphi) \leq \max\{\ _{v_n}\lambda_{g_1}(f_1, \varphi),\ _{v_n}\lambda_{g_1}(f_2, \varphi)\}\ .$$

*The equality holds when any one of $_{v_n}\lambda_{g_1}(f_i, \varphi) > {}_{v_n}\lambda_{g_1}(f_j, \varphi)$ with at least $f_j$ is of regular relative $\varphi$-growth with respect to $g_1$ where $i, j = 1, 2$ and $i \neq j$.*

**Proof.** If $_{v_n}\lambda_{g_1}(f_1 \pm f_2, \varphi) = 0$ then the result is obvious. So we suppose that $_{v_n}\lambda_{g_1}(f_1 \pm f_2, \varphi) > 0$. We can clearly assume that $_{v_n}\lambda_{g_1}(f_k, \varphi)$ is finite for $k = 1, 2$. Further let $\max\{_{v_n}\lambda_{g_1}(f_1, \varphi),\ _{v_n}\lambda_{g_1}(f_2, \varphi)\} = \Delta$ and $f_2$ be of regular relative $\varphi$-growth with respect to $g_1$.

Now for any arbitrary $\varepsilon > 0$ from the definition of $_{v_n}\lambda_{g_1}(f_1, \varphi)$, we have for a sequence values of $r_1, r_2, ..., r_n$ tending to infinity that

$$M_{f_1}(r_1, r_2, ..., r_n) < M_{g_1}(\varphi_1^{(\ _{v_n}\lambda_{g_1}(f_1,\varphi)+\varepsilon)}, \varphi_2^{(\ _{v_n}\lambda_{g_1}(f_1,\varphi)+\varepsilon)}, ..., \varphi_n^{(\ _{v_n}\lambda_{g_1}(f_1,\varphi)+\varepsilon)})$$

$$\text{i.e.}, M_{f_1}(r_1, r_2, ..., r_n) < M_{g_1}(\varphi_1^{(\Delta+\varepsilon)}, \varphi_2^{(\Delta+\varepsilon)}, ..., \varphi_n^{(\Delta+\varepsilon)})\ . \tag{3.1}$$

Also for any arbitrary $\varepsilon > 0$ from the definition of $_{v_n}\rho_{g_1}(f_2, \varphi)(=_{v_n} \lambda_{g_1}(f_2, \varphi))$, we obtain for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$M_{f_2}(r_1, r_2, ..., r_n) <$$
$$M_{g_1}(\varphi_1^{(\ _{v_n}\lambda_{g_1}(f_2,\varphi)+\varepsilon)}, \varphi_2^{(\ _{v_n}\lambda_{g_1}(f_2,\varphi)+\varepsilon)}, ..., \varphi_n^{(\ _{v_n}\lambda_{g_1}(f_2,\varphi)+\varepsilon)})$$

$$\text{i.e.},\ M_{f_2}(r_1, r_2, ..., r_n) < M_{g_1}(\varphi_1^{(\Delta+\varepsilon)}, \varphi_2^{(\Delta+\varepsilon)}, ..., \varphi_n^{(\Delta+\varepsilon)})\ . \tag{3.2}$$

Since $M_{f_1 \pm f_2}(r_1, r_2, ..., r_n) \leq M_{f_1}(r_1, r_2, ..., r_n) + M_{f_2}(r_1, r_2, ..., r_n)$ for sufficiently large $r_1, r_2, ..., r_n$, we obtain from (3.1) and (3.2) for a sequence values of $r_1, r_2, ..., r_n$ tending to infinity that

$$M_{f_1 \pm f_2}(r_1, r_2, ..., r_n) < 2M_{g_1}\left(\varphi_1^{(\Delta+\varepsilon)}, \varphi_2^{(\Delta+\varepsilon)}, ..., \varphi_n^{(\Delta+\varepsilon)}\right)\ . \tag{3.3}$$

Therefore in view of Lemma 1, we obtain from (3.3) for a sequence values of $r_1, r_2, ..., r_n$ tending to infinity that

$$\begin{aligned} M_{f_1 \pm f_2}(r_1, r_2, ..., r_n) &< M_{g_1}\left(3\varphi_1^{(\Delta+\varepsilon)}, 3\varphi_2^{(\Delta+\varepsilon)}, ..., 3\varphi_n^{(\Delta+\varepsilon)}\right) \\ i.e.,\ M_{f_1 \pm f_2}(r_1, r_2, ..., r_n) &< M_{g_1}\left(\varphi_1^{(\Delta+3\varepsilon)}, \varphi_2^{(\Delta+3\varepsilon)}, ..., \varphi_n^{(\Delta+3\varepsilon)}\right) . \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get from above that

$$_{v_n}\lambda_{g_1}(f_1 \pm f_2, \varphi) \le \Delta = \max\left\{_{v_n}\lambda_{g_1}(f_1, \varphi),\ _{v_n}\lambda_{g_1}(f_2, \varphi)\right\} .$$

Similarly, if we consider that $f_1$ is of regular relative $\varphi$-growth with respect to $g_1$ or both $f_1$ and $f_2$ are of regular relative $\varphi$-growth with respect to $g_1$, then one can easily verify that

$$(3.4) \qquad _{v_n}\lambda_{g_1}(f_1 \pm f_2, \varphi) \le \Delta = \max\left\{_{v_n}\lambda_{g_1}(f_1, \varphi),\ _{v_n}\lambda_{g_1}(f_2, \varphi)\right\}.$$

Now let $_{v_n}\lambda_{g_1}(f_1, \varphi) > {}_{v_n}\lambda_{g_1}(f_2, \varphi)$ and at least $f_2$ be of regular relative $\varphi$-growth with respect to $g_1$. Also let $f = f_1 \pm f_2$. Then in view of (3.4) we get that $_{v_n}\lambda_{g_1}(f, \varphi) \le {}_{v_n}\lambda_{g_1}(f_1, \varphi)$. As, $f_1 = (f \pm f_2)$ and in this case we obtain that $_{v_n}\lambda_{g_1}(f_1, \varphi) \le \max\left\{_{v_n}\lambda_{g_1}(f, \varphi),\ _{v_n}\lambda_{g_1}(f_2, \varphi)\right\}$. As we assume that $_{v_n}\lambda_{g_1}(f_2, \varphi) < {}_{v_n}\lambda_{g_1}(f_1, \varphi)$, therefore we have $_{v_n}\lambda_{g_1}(f_1, \varphi) \le {}_{v_n}\lambda_{g_1}(f, \varphi)$ and hence

$$_{v_n}\lambda_{g_1}(f_1 \pm f_2, \varphi) \ge {}_{v_n}\lambda_{g_1}(f_1, \varphi) = \max\left\{_{v_n}\lambda_{g_1}(f_1, \varphi),\ _{v_n}\lambda_{g_1}(f_2, \varphi)\right\}.$$

Further if we consider $_{v_n}\lambda_{g_1}(f_1, \varphi) < {}_{v_n}\lambda_{g_1}(f_2, \varphi)$ and at least $f_1$ be of regular relative $\varphi$-growth with respect to $g_1$, then one can also verify that

$$(3.5) \qquad _{v_n}\lambda_{g_1}(f_1 \pm f_2, \varphi) \ge \Delta = \max\left\{_{v_n}\lambda_{g_1}(f_1, \varphi),\ _{v_n}\lambda_{g_1}(f_2, \varphi)\right\}.$$

So the conclusion of the second part of the theorem follows from (3.4) and (3.5). □

**Theorem 2.** *Let us consider $f_1$, $f_2$ be any two entire functions of several complex variables with relative $\varphi$-order $_{v_n}\rho_{g_1}(f_1, \varphi)$ and $_{v_n}\rho_{g_1}(f_2, \varphi)$ with respect to another entire function $g_1$ of several complex variables. Then*

$$_{v_n}\rho_{g_1}(f_1 \pm f_2, \varphi) \le \max\left\{ _{v_n}\rho_{g_1}(f_1, \varphi),\ _{v_n}\rho_{g_1}(f_2, \varphi)\right\}.$$

*The equality holds when $_{v_n}\rho_{g_1}(f_1, \varphi) \ne {}_{v_n}\rho_{g_1}(f_2, \varphi)$.*

We omit the proof of Theorem 2 as it can easily be carried out in the line of Theorem 1.

**Theorem 3.** *Let $f_1$, $g_1$ and $g_2$ be any three entire functions of several complex variables such that $_{v_n}\lambda_{g_1}(f_1, \varphi)$ and $_{v_n}\lambda_{g_2}(f_1, \varphi)$ exit. Then*

$$_{v_n}\lambda_{g_1 \pm g_2}(f_1, \varphi) \ge \min\left\{ _{v_n}\lambda_{g_1}(f_1, \varphi),\ _{v_n}\lambda_{g_2}(f_1, \varphi)\right\}.$$

*The equality holds when $_{v_n}\lambda_{g_1}(f_1, \varphi) \ne {}_{v_n}\lambda_{g_2}(f_1, \varphi)$.*

**Proof.** If ${}_{v_n}\lambda_{g_1 \pm g_2}(f_1, \varphi) = \infty$ then the result is obvious. So we suppose that ${}_{v_n}\lambda_{g_1 \pm g_2}(f_1, \varphi) < \infty$. We can clearly assume that ${}_{v_n}\lambda_{g_k}(f_1, \varphi)$ is finite for $k = 1, 2$. Further let $\Psi = \min\left\{{}_{v_n}\lambda_{g_1}(f_1, \varphi),\ {}_{v_n}\lambda_{g_2}(f_1, \varphi)\right\}$. Now for any arbitrary $\varepsilon > 0$ from the definition of ${}_{v_n}\lambda_{g_k}(f_1, \varphi)$, we have for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$M_{g_k}(\varphi_1^{({}_{v_n}\lambda_{g_k}(f_1,\varphi)-\varepsilon)}, \varphi_2^{({}_{v_n}\lambda_{g_k}(f_1,\varphi)-\varepsilon)}, ..., \varphi_n^{({}_{v_n}\lambda_{g_k}(f_1,\varphi)-\varepsilon)}) < M_{f_1}(r_1, r_2, ..., r_n)$$

where $k = 1, 2$.

Therefore from above we get for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$M_{g_k}\left(\varphi_1^{(\Psi-\varepsilon)}, \varphi_2^{(\Psi-\varepsilon)}, ..., \varphi_n^{(\Psi-\varepsilon)}\right) < M_{f_1}(r_1, r_2, ..., r_n) \quad \text{where } k = 1, 2. \tag{3.6}$$

Since $M_{g_1 \pm g_2}(r_1, r_2, ..., r_n) \leq M_{g_1}(r_1, r_2, ..., r_n) + M_{g_2}(r_1, r_2, ..., r_n)$ for sufficiently large $r_1, r_2, ..., r_n$, we obtain from above and Lemma 1 for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$\begin{aligned}
&M_{g_1 \pm g_2}\left(\varphi_1^{(\Psi-\varepsilon)}, \varphi_2^{(\Psi-\varepsilon)}, ..., \varphi_n^{(\Psi-\varepsilon)}\right) < M_{g_1}\left[\varphi_1^{(\Psi-\varepsilon)}, \varphi_2^{(\Psi-\varepsilon)}, ..., \varphi_n^{(\Psi-\varepsilon)}\right] \\
&+ M_{g_2}\left[\varphi_1^{(\Psi-\varepsilon)}, \varphi_2^{(\Psi-\varepsilon)}, ..., \varphi_n^{(\Psi-\varepsilon)}\right] \\
&i.e., M_{g_1 \pm g_2}\left(\varphi_1^{(\Psi-\varepsilon)}, \varphi_2^{(\Psi-\varepsilon)}, ..., \varphi_n^{(\Psi-\varepsilon)}\right) < 2M_{f_1}(r_1, r_2, ..., r_n) \\
&i.e., M_{g_1 \pm g_2}\left(\left(\frac{1}{3}\right)\varphi_1^{(\Psi-\varepsilon)}, \left(\frac{1}{3}\right)\varphi_2^{(\Psi-\varepsilon)}, ..., \left(\frac{1}{3}\right)\varphi_n^{(\Psi-\varepsilon)}\right) < M_{f_1}(r_1, r_2, ..., r_n) \\
&i.e., M_{g_1 \pm g_2}\left(\varphi_1^{(\Psi-3\varepsilon)}, \varphi_2^{(\Psi-3\varepsilon)}, ..., \varphi_n^{(\Psi-3\varepsilon)}\right) < M_{f_1}(r_1, r_2, ..., r_n).
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get from above that

$${}_{v_n}\lambda_{g_1 \pm g_2}(f_1, \varphi) \geq \Psi = \min\left\{ {}_{v_n}\lambda_{g_1}(f_1, \varphi),\ {}_{v_n}\lambda_{g_2}(f_1, \varphi)\right\}. \tag{3.7}$$

Now let ${}_{v_n}\lambda_{g_1}(f_1, \varphi) < {}_{v_n}\lambda_{g_2}(f_1, \varphi)$ and $g = g_1 \pm g_2$. Then in view of (3.7) we get that ${}_{v_n}\lambda_g(f_1, \varphi) \geq {}_{v_n}\lambda_{g_1}(f_1, \varphi)$. Further, $g_1 = (g \pm g_2)$ and in this case we obtain that ${}_{v_n}\lambda_{g_1}(f_1, \varphi) \geq \min\left\{{}_{v_n}\lambda_g(f_1, \varphi),\ {}_{v_n}\lambda_{g_2}(f_1, \varphi)\right\}$. As we assume that ${}_{v_n}\lambda_{g_1}(f_1, \varphi) < {}_{v_n}\lambda_{g_2}(f_1, \varphi)$, therefore we have ${}_{v_n}\lambda_{g_1}(f_1, \varphi) \geq {}_{v_n}\lambda_g(f_1, \varphi)$ and hence

$${}_{v_n}\lambda_{g_1 \pm g_2}(f_1, \varphi) \leq {}_{v_n}\lambda_{g_1}(f_1, \varphi) = \min\left\{ {}_{v_n}\lambda_{g_1}(f_1, \varphi),\ {}_{v_n}\lambda_{g_2}(f_1, \varphi)\right\}.$$

Similarly, if we consider ${}_{v_n}\lambda_{g_1}(f_1, \varphi) > {}_{v_n}\lambda_{g_2}(f_1, \varphi)$, then one can also derive that

$${}_{v_n}\lambda_{g_1 \pm g_2}(f_1, \varphi) \leq \Psi = \min\left\{ {}_{v_n}\lambda_{g_1}(f_1, \varphi),\ {}_{v_n}\lambda_{g_2}(f_1, \varphi)\right\}. \tag{3.8}$$

So the conclusion of the second part of the theorem follows from (3.7) and (3.8). $\square$

**Theorem 4.** *Let $f_1$, $g_1$ and $g_2$ be any three entire functions of several complex variables such that ${}_{v_n}\rho_{g_1}(f_1,\varphi)$ and ${}_{v_n}\rho_{g_2}(f_1,\varphi)$ exit. Also let $f_1$ be of regular relative $\varphi$-growth with respect to at least any one of $g_1$ or $g_2$. Then*

$$ {}_{v_n}\rho_{g_1\pm g_2}(f_1,\varphi) \geq \min\left\{ {}_{v_n}\rho_{g_1}(f_1,\varphi),\ {}_{v_n}\rho_{g_2}(f_1,\varphi)\right\} . $$

*The equality holds when any one of ${}_{v_n}\rho_{g_i}(f_1,\varphi) < {}_{v_n}\rho_{g_j}(f_1,\varphi)$ with at least $f_1$ is of regular relative $\varphi$-growth with respect to $g_j$ where $i,j=1,2$ and $i\neq j$.*

We omit the proof of Theorem 4 as it can easily be carried out in the line of Theorem 3.

**Theorem 5.** *Let $f_1$, $f_2$, $g_1$ and $g_2$ be any four entire functions of several complex variables. Then*

$$ \begin{aligned} {}_{v_n}\rho_{g_1\pm g_2}(f_1\pm f_2,\varphi) \leq \max[&\min\{{}_{v_n}\rho_{g_1}(f_1,\varphi),\ {}_{v_n}\rho_{g_2}(f_1,\varphi)\},\\ &\min\{{}_{v_n}\rho_{g_1}(f_2,\varphi),\ {}_{v_n}\rho_{g_2}(f_2,\varphi)\}] \end{aligned} $$

*when the following two conditions hold:*

*$(i)$ ${}_{v_n}\rho_{g_i}(f_1,\varphi) < {}_{v_n}\rho_{g_j}(f_1,\varphi)$ with at least $f_1$ is of regular relative $\varphi$-growth with respect to $g_j$ for $i=1,\ 2,\ j=1,2$ and $i\neq j$; and*

*$(ii)$ ${}_{v_n}\rho_{g_i}(f_2,\varphi) < {}_{v_n}\rho_{g_j}(f_2,\varphi)$ with at least $f_2$ is of regular relative $\varphi$-growth with respect to $g_j$ for $i=1,\ 2,\ j=1,2$ and $i\neq j$.*

*The equality holds when*

$$ {}_{v_n}\rho_{g_1}(f_i,\varphi) <_{v_n} \rho_{g_1}(f_j,\varphi) \ \ \textit{and} \ \ {}_{v_n}\rho_{g_2}(f_i,\varphi) <_{v_n} \rho_{g_2}(f_j,\varphi) $$

*hold simultaneously for $i=1,2$; $j=1,2$ and $i\neq j$.*

**Proof.** Let the conditions $(i)$ and $(ii)$ of the theorem hold. Therefore in view of Theorem 2 and Theorem 4 we get that

$$ \begin{aligned} &\max[\min\{ {}_{v_n}\rho_{g_1}(f_1,\varphi),\ {}_{v_n}\rho_{g_2}(f_1,\varphi)\},\min\{ {}_{v_n}\rho_{g_1}(f_2,\varphi),\ {}_{v_n}\rho_{g_2}(f_2,\varphi)\}]\\ &= \max[ {}_{v_n}\rho_{g_1\pm g_2}(f_1,\varphi),\ {}_{v_n}\rho_{g_1\pm g_2}(f_2,\varphi)]\\ &\geq {}_{v_n}\rho_{g_1\pm g_2}(f_1\pm f_2,\varphi). \end{aligned} \tag{3.9} $$

Since ${}_{v_n}\rho_{g_1}(f_i,\varphi) < {}_{v_n}\rho_{g_1}(f_j,\varphi)$ and ${}_{v_n}\rho_{g_2}(f_i,\varphi) < {}_{v_n}\rho_{g_2}(f_j,\varphi)$ hold simultaneously for $i=1,2$; $j=1,2$ and $i\neq j$, we obtain that

either $\min\{ {}_{v_n}\rho_{g_1}(f_1,\varphi),\ {}_{v_n}\rho_{g_2}(f_1,\varphi)\} > \min\{ {}_{v_n}\rho_{g_1}(f_2,\varphi),\ {}_{v_n}\rho_{g_2}(f_2,\varphi)\}$ or

$\min\{ {}_{v_n}\rho_{g_1}(f_2,\varphi),\ {}_{v_n}\rho_{g_2}(f_2,\varphi)\} > \min\{ {}_{v_n}\rho_{g_1}(f_1,\varphi),\ {}_{v_n}\rho_{g_2}(f_1,\varphi)\}$ holds.

Now in view of the conditions $(i)$ and $(ii)$ of the theorem, it follows from above that

either ${}_{v_n}\rho_{g_1\pm g_2}(f_1,\varphi) > {}_{v_n}\rho_{g_1\pm g_2}(f_2,\varphi)$ or ${}_{v_n}\rho_{g_1\pm g_2}(f_2,\varphi) > {}_{v_n}\rho_{g_1\pm g_2}(f_1,\varphi)$

which is the condition for holding equality in (3.9).

Hence the theorem follows. □

**Theorem 6.** *Let $f_1, f_2, g_1$ and $g_2$ be any four entire functions of several complex variables. Then*

$$
\begin{aligned}
{}_{v_n}\lambda_{g_1 \pm g_2}(f_1 \pm f_2, \varphi) &\geq \min[\max\{ {}_{v_n}\lambda_{g_1}(f_1, \varphi),\ {}_{v_n}\lambda_{g_1}(f_2, \varphi)\},\\
&\max\{ {}_{v_n}\lambda_{g_2}(f_1, \varphi),\ {}_{v_n}\lambda_{g_2}(f_2, \varphi)\}]
\end{aligned}
$$

*when the following two conditions hold:*

*(i) ${}_{v_n}\lambda_{g_1}(f_i, \varphi) > {}_{v_n}\lambda_{g_1}(f_j, \varphi)$ with at least $f_j$ is of regular relative $\varphi$-growth with respect to $g_1$ for $i = 1,\ 2,\ j = 1, 2$ and $i \neq j$; and*

*(ii) ${}_{v_n}\lambda_{g_2}(f_i, \varphi) > {}_{v_n}\lambda_{g_2}(f_j, \varphi)$ with at least $f_j$ is of regular relative $\varphi$-growth with respect to $g_2$ for $i = 1,\ 2,\ j = 1, 2$ and $i \neq j$.*

*The equality holds when ${}_{v_n}\lambda_{g_i}(f_1, \varphi) < {}_{v_n}\lambda_{g_j}(f_1, \varphi)$ and ${}_{v_n}\lambda_{g_i}(f_2, \varphi) < {}_{v_n}\lambda_{g_j}(f_2, \varphi)$ hold simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$.*

**Proof.** Suppose that the conditions $(i)$ and $(ii)$ of the theorem hold. Therefore in view of Theorem 1 and Theorem 3, we obtain that

$$
\begin{aligned}
&\min[\max\{ {}_{v_n}\lambda_{g_1}(f_1, \varphi),\ {}_{v_n}\lambda_{g_1}(f_2, \varphi)\},\\
&\max\{ {}_{v_n}\lambda_{g_2}(f_1, \varphi),\ {}_{v_n}\lambda_{g_2}(f_2, \varphi)\}]\\
&= \min[ {}_{v_n}\lambda_{g_1}(f_1 \pm f_2, \varphi), {}_{v_n}\lambda_{g_2}(f_1 \pm f_2, \varphi)] \geq {}_{v_n}\lambda_{g_1 \pm g_2}(f_1 \pm f_2, \varphi).
\end{aligned}
\tag{3.10}
$$

Since ${}_{v_n}\lambda_{g_i}(f_1, \varphi) < {}_{v_n}\lambda_{g_j}(f_1, \varphi)$ and ${}_{v_n}\lambda_{g_i}(f_2, \varphi) < {}_{v_n}\lambda_{g_j}(f_2, \varphi)$ hold simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$, we get that

either $\max\{ {}_{v_n}\lambda_{g_1}(f_1, \varphi),\ {}_{v_n}\lambda_{g_1}(f_2, \varphi)\} < \max\{ {}_{v_n}\lambda_{g_2}(f_1, \varphi),\ {}_{v_n}\lambda_{g_2}(f_2, \varphi)\}$ or

$\max\{ {}_{v_n}\lambda_{g_2}(f_1, \varphi),\ {}_{v_n}\lambda_{g_2}(f_2, \varphi)\} < \max\{ {}_{v_n}\lambda_{g_1}(f_1, \varphi),\ {}_{v_n}\lambda_{g_1}(f_2, \varphi)\}$ holds.

Since condition $(i)$ and $(ii)$ of the theorem hold, it follows from above that

either ${}_{v_n}\lambda_{g_1}(f_1 \pm f_2, \varphi) < {}_{v_n}\lambda_{g_2}(f_1 \pm f_2, \varphi)$ or ${}_{v_n}\lambda_{g_2}(f_1 \pm f_2, \varphi) < {}_{v_n}\lambda_{g_1}(f_1 \pm f_2, \varphi)$

which is the condition for holding equality in (3.10).

Hence the theorem follows. □

**Theorem 7.** *Let $f_1$, $f_2$ and $g_1$ be any three entire functions of several complex variables. Also let at least $f_1$ or $f_2$ be of regular relative $\varphi$-growth with respect to $g_1$. Then*

$$
{}_{v_n}\lambda_{g_1}(f_1 \cdot f_2, \varphi) \leq \max\{ {}_{v_n}\lambda_{g_1}(f_1, \varphi),\ {}_{v_n}\lambda_{g_1}(f_2, \varphi)\}
$$

*provided $g_1$ has the Property (R). The equality holds when $f_1$ and $f_2$ satisfy Property (P).*

**Proof.** Suppose that ${}_{v_n}\lambda_{g_1}(f_1 \cdot f_2, \varphi) > 0$. Otherwise if ${}_{v_n}\lambda_{g_1}(f_1 \cdot f_2, \varphi) = 0$ then the result is obvious. Let us consider that $f_2$ is of regular relative $\varphi$-growth with respect to $g_1$. Also suppose that $\max\{{}_{v_n}\lambda_{g_1}(f_1, \varphi), {}_{v_n}\lambda_{g_1}(f_2, \varphi)\} = \Delta$. We can clearly assume that ${}_{v_n}\lambda_{g_1}(f_k, \varphi)$ is finite for $k = 1, 2$.

Since $M_{f_1 \cdot f_2}(r_1, r_2, ..., r_n) < M_{f_1}(r_1, r_2, ..., r_n) \cdot M_{f_2}(r_1, r_2, ..., r_n)$ for all large $r_1, r_2, ..., r_n$, we have from (3.1), (3.2) for a sequence values of $r_1, r_2, ..., r_n$ tending to infinity that

$$M_{f_1 \cdot f_2}(r_1, r_2, ..., r_n) < \left[M_{g_1}\left(\varphi_1^{(\Delta+\varepsilon)}, \varphi_2^{(\Delta+\varepsilon)}, ..., \varphi_n^{(\Delta+\varepsilon)}\right)\right]^2 .$$

Also, in view of Definition 7, we obtain from above for any $\delta > 1$ and for a sequence values of $r_1, r_2, ..., r_n$ tending to infinity that

$$M_{f_1 \cdot f_2}(r_1, r_2, ..., r_n) < M_{g_1}\left(\varphi_1^{\delta(\Delta+\varepsilon)}, \varphi_2^{\delta(\Delta+\varepsilon)}, ..., \varphi_n^{\delta(\Delta+\varepsilon)}\right),$$

since $g_1$ has the Property (R). Since $\varepsilon > 0$ is arbitrary, now letting $\delta \to 1^+$, we get from above that

$$_{v_n}\lambda_{g_1}(f_1 \cdot f_2, \varphi) \le \Delta = \max\{_{v_n}\lambda_{g_1}(f_1, \varphi),\ _{v_n}\lambda_{g_1}(f_2, \varphi)\}.$$

Similarly, if we consider that $f_1$ is of regular relative $\varphi$-growth with respect to $g_1$ or both $f_1$ and $f_2$ are of regular relative $\varphi$-growth with respect to $g_1$, then also one can easily verify that

$$_{v_n}\lambda_{g_1}(f_1 \cdot f_2, \varphi) \le \Delta = \max\{\ _{v_n}\lambda_{g_1}(f_1, \varphi),\ _{v_n}\lambda_{g_1}(f_2, \varphi)\}. \tag{3.11}$$

Now let $f_1$ and $f_2$ satisfy Property (P), then of course we have $M_{f_1 \cdot f_2}(r_1, r_2, ..., r_n) > M_{f_1}(r_1, r_2, ..., r_n)$ and $M_{f_1 \cdot f_2}(r_1, r_2, ..., r_n) > M_{f_2}(r_1, r_2, ..., r_n)$ for all sufficiently large values of $r_1, r_2, ..., r_n$. Therefore from the definition of relative $\varphi$-lower order, we get for a sequence values of $r_1, r_2, ..., r_n$ tending to infinity that

$$\begin{aligned} &M_{f_1}(r_1, r_2, ..., r_n) < M_{f_1 \cdot f_2}(r_1, r_2, ..., r_n) \\ &\quad \le M_{g_1}\left(\varphi_1^{\left(\ _{v_n}\lambda_{g_1}(f_1 \cdot f_2, \varphi)+\varepsilon\right)}, \varphi_2^{\left(\ _{v_n}\lambda_{g_1}(f_1 \cdot f_2, \varphi)+\varepsilon\right)}, ..., \varphi_n^{\left(\ _{v_n}\lambda_{g_1}(f_1 \cdot f_2, \varphi)+\varepsilon\right)}\right). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get from above that $_{v_n}\lambda_{g_1}(f_1 \cdot f_2, \varphi) \ge\ _{v_n}\lambda_{g_1}(f_1, \varphi)$. Similarly $_{v_n}\lambda_{g_1}(f_1 \cdot f_2, \varphi) \ge\ _{v_n}\lambda_{g_1}(f_2, \varphi)$ and therefore

$$_{v_n}\lambda_{g_1}(f_1 \cdot f_2, \varphi) \ge \Delta = \max\{_{v_n}\lambda_{g_1}(f_1, \varphi),\ _{v_n}\lambda_{g_1}(f_2, \varphi)\}. \tag{3.12}$$

Hence the theorem follows from (3.11) and (3.12). □

Now we state the following theorem which can easily be carried out in the line of Theorem 7 and therefore its proof is omitted.

**Theorem 8.** *Let us consider $f_1$, $f_2$ be any two entire functions of several complex variables with relative $\varphi$-order $_{v_n}\rho_{g_1}(f_1, \varphi)$ and $_{v_n}\rho_{g_1}(f_2, \varphi)$ with respect to another entire function $g_1$ of several complex variables. Then*

$$_{v_n}\rho_{g_1}(f_1 \cdot f_2, \varphi) \le \max\{_{v_n}\rho_{g_1}(f_1, \varphi),\ _{v_n}\rho_{g_1}(f_2, \varphi)\}$$

*provided $g_1$ has the Property (R). The equality holds when $f_1$ and $f_2$ satisfy Property (P).*

**Theorem 9.** *Let $f_1$, $g_1$ and $g_2$ be any three entire functions of several complex variables. Also let $\lambda_{g_1}(f_1)$ and $\lambda_{g_2}(f_1)$ exist. Then*

$$_{v_n}\lambda_{g_1\cdot g_2}(f_1,\varphi) \geq \min\left\{_{v_n}\lambda_{g_1}(f_1,\varphi),\ _{v_n}\lambda_{g_2}(f_1,\varphi)\right\}$$

*provided $g_1 \cdot g_2$ has the Property (R). The equality holds when $g_1$ and $g_2$ satisfy Property (P).*

**Proof.** Suppose that $_{v_n}\lambda_{g_1\cdot g_2}(f_1,\varphi) < \infty$. Otherwise if $_{v_n}\lambda_{g_1\cdot g_2}(f_1,\varphi) = \infty$ then the result is obvious. Also suppose that $\min\left\{_{v_n}\lambda_{g_1}(f_1,\varphi), _{v_n}\lambda_{g_2}(f_1,\varphi)\right\} = \Psi$. We can clearly assume that $_{v_n}\lambda_{g_k}(f_1,\varphi)$ is finite for $k = 1, 2$.

As $M_{g_1\cdot g_2}(r_1, r_2, ..., r_n) < M_{g_1}(r_1, r_2, ..., r_n) \cdot M_{g_2}(r_1, r_2, ..., r_n)$ for all large $r_1, r_2, ..., r_n$, we get in view of (3.6) for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$M_{g_1\cdot g_2}\left(\varphi_1^{(\Psi-\varepsilon)}, \varphi_2^{(\Psi-\varepsilon)}, ..., \varphi_n^{(\Psi-\varepsilon)}\right) < \left[M_{f_1}(r_1, r_2, ..., r_n)\right]^2$$

$$i.e., \left[M_{g_1\cdot g_2}\left(\varphi_1^{(\Psi-\varepsilon)}, \varphi_2^{(\Psi-\varepsilon)}, ..., \varphi_n^{(\Psi-\varepsilon)}\right)\right]^{\frac{1}{2}} < M_{f_1}(r_1, r_2, ..., r_n).$$

Now in view of Definition 7 we obtain from above for any $\delta > 1$ and for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$M_{g_1\cdot g_2}\left(\varphi_1^{\frac{(\Psi-\varepsilon)}{\delta}}, \varphi_2^{\frac{(\Psi-\varepsilon)}{\delta}}, ..., \varphi_n^{\frac{(\Psi-\varepsilon)}{\delta}}\right) < M_{f_1}(r_1, r_2, ..., r_n)$$

since $g_1 \cdot g_2$ has the Property (R). Since $\varepsilon > 0$ is arbitrary, now letting $\delta \to 1^+$, we obtain from above that

$$_{v_n}\lambda_{g_1\cdot g_2}(f_1,\varphi) \geq \Psi = \min\left\{_{v_n}\lambda_{g_1}(f_1,\varphi), _{v_n}\lambda_{g_2}(f_1,\varphi)\right\}. \tag{3.13}$$

Now let $g_1$ and $g_2$ satisfy Property (P), then of course we have $M_{g_1\cdot g_2}(r_1, r_2, ..., r_n) > M_{g_1}(r_1, r_2, ..., r_n)$ and $M_{g_1\cdot g_2}(r_1, r_2, ..., r_n) > M_{g_2}(r_1, r_2, ..., r_n)$ for all sufficiently large values of $r_1, r_2, ..., r_n$. Therefore from the definition of relative $\varphi$-lower order, we get for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$\begin{aligned}
&M_{g_1}(\varphi_1^{(_{v_n}\lambda_{g_1\cdot g_2}(f_1,\varphi)-\varepsilon)}, \varphi_2^{(_{v_n}\lambda_{g_1\cdot g_2}(f_1,\varphi)-\varepsilon)}, ..., \varphi_n^{(_{v_n}\lambda_{g_1\cdot g_2}(f_1,\varphi)-\varepsilon)})\\
&< M_{g_1\cdot g_2}(\varphi_1^{(_{v_n}\lambda_{g_1\cdot g_2}(f_1,\varphi)-\varepsilon)}, \varphi_2^{(_{v_n}\lambda_{g_1\cdot g_2}(f_1,\varphi)-\varepsilon)}, ..., \varphi_n^{(_{v_n}\lambda_{g_1\cdot g_2}(f_1,\varphi)-\varepsilon)})\\
&\leq M_{f_1}(r_1, r_2, ..., r_n).
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get from above that $_{v_n}\lambda_{g_1}(f_1,\varphi) \geq {_{v_n}\lambda_{g_1\cdot g_2}(f_1,\varphi)}$. Similarly $_{v_n}\lambda_{g_2}(f_1,\varphi) \geq {_{v_n}\lambda_{g_1\cdot g_2}(f_1,\varphi)}$ and therefore

$$_{v_n}\lambda_{g_1\cdot g_2}(f_1,\varphi) \leq \Psi = \min\left\{_{v_n}\lambda_{g_1}(f_1,\varphi), _{v_n}\lambda_{g_2}(f_1,\varphi)\right\}. \tag{3.14}$$

Hence the theorem follows from (3.13) and (3.14). □

**Theorem 10.** *Let $f_1$, $g_1$ and $g_2$ be any three entire functions of several complex variables. Also let $f_1$ be of regular relative $\varphi$-growth with respect to at least any one of $g_1$ or $g_2$. Then*

$$ {}_{v_n}\rho_{g_1\cdot g_2}(f_1,\varphi) \geq \min\{ {}_{v_n}\rho_{g_1}(f_1,\varphi), {}_{v_n}\rho_{g_2}(f_1,\varphi)\} $$

*provided $g_1 \cdot g_2$ has the Property (R). The equality holds when $g_1$ and $g_2$ satisfy Property (P).*

We omit the proof of Theorem 10 as it can easily be carried out in the line of Theorem 9.

Now we state the following two theorems without their proofs as those can easily be carried out with the help of Theorem 8, Theorem 7, Theorem 9 and Theorem 10 and

in the line of Theorem 5 and Theorem 6 respectively.

**Theorem 11.** *Let $f_1$, $f_2$, $g_1$ and $g_2$ be any four entire functions of several complex variables. Also let $g_1 \cdot g_2$ satisfy the Property (R). Then,*

$$ {}_{v_n}\rho_{g_1\cdot g_2}(f_1\cdot f_2,\varphi) = $$
$$ \max\left[\min\{ {}_{v_n}\rho_{g_1}(f_1,\varphi), {}_{v_n}\rho_{g_2}(f_1,\varphi)\}, \min\{ {}_{v_n}\rho_{g_1}(f_2,\varphi), {}_{v_n}\rho_{g_2}(f_2,\varphi)\}\right], $$

*when the following four conditions hold:*

($i$) *$f_1$ is of regular relative $\varphi$-growth with respect to at least any one of $g_1$ or $g_2$;*

($ii$) *$f_2$ is of regular relative $\varphi$-growth with respect to at least any one of $g_1$ or $g_2$;*

($iii$) *$f_1$ and $f_2$ satisfy Property (P); and*

($iv$) *$g_1$ and $g_2$ satisfy Property (P).*

**Theorem 12.** *Let $f_1$, $f_2$, $g_1$ and $g_2$ be any four entire functions of several complex variables. Also let $g_1 \cdot g_2$, $g_1$ and $g_2$ satisfy the Property (R). Then,*

$$ {}_{v_n}\lambda_{g_1\cdot g_2}(f_1\cdot f_2,\varphi) = \min[\max\{ {}_{v_n}\lambda_{g_1}(f_1,\varphi), {}_{v_n}\lambda_{g_1}(f_2,\varphi)\}, $$
$$ \max\{ {}_{v_n}\lambda_{g_2}(f_1,\varphi), {}_{v_n}\lambda_{g_2}(f_2,\varphi)\}], $$

*when the following four conditions hold:*

($i$) *At least $f_1$ or $f_2$ is of regular relative $\varphi$-growth with respect to $g_1$;*

($ii$) *At least $f_1$ or $f_2$ is of regular relative $\varphi$-growth with respect to $g_2$;*

($iii$) *$f_1$ and $f_2$ satisfy Property (P);*

($iv$) *$g_1$ and $g_2$ satisfy Property (P).*

**Theorem 13.** *Let $f$, $g$ and $h$ be any three entire functions of several complex variables such that $0 < {}_{v_n}\lambda_h(f,\varphi) \leq {}_{v_n}\rho_h(f,\varphi) < \infty$ and $0 < {}_{v_n}\lambda_h(g) \leq {}_{v_n}\rho_h(g) < \infty$. Then*

$$ \frac{{}_{v_n}\lambda_h(f,\varphi)}{{}_{v_n}\rho_h(g)} \leq {}_{v_n}\lambda_g(f,\varphi) \leq \min\left\{\frac{{}_{v_n}\lambda_h(f,\varphi)}{{}_{v_n}\lambda_h(g)}, \frac{{}_{v_n}\rho_h(f,\varphi)}{{}_{v_n}\rho_h(g)}\right\} $$
$$ \leq \max\left\{\frac{{}_{v_n}\lambda_h(f,\varphi)}{{}_{v_n}\lambda_h(g)}, \frac{{}_{v_n}\rho_h(f,\varphi)}{{}_{v_n}\rho_h(g)}\right\} \leq {}_{v_n}\rho_g(f,\varphi) \leq \frac{{}_{v_n}\rho_h(f,\varphi)}{{}_{v_n}\lambda_h(g)}. $$

**Proof.** From the definitions of ${}_{v_n}\rho_g(f,\varphi)$ it follows for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$M_f(r_1, r_2, ..., r_n) < M_g(\varphi_1^{({}_{v_n}\rho_g(f,\varphi)+\varepsilon)}, \varphi_2^{({}_{v_n}\rho_g(f,\varphi)+\varepsilon)}, ..., \varphi_n^{({}_{v_n}\rho_g(f,\varphi)+\varepsilon)}) \tag{3.15}$$

and for a sequence values of $r_1, r_2, ..., r_n$ tending to infinity we obtain that

$$M_f(r_1, r_2, ..., r_n) > M_g(\varphi_1^{({}_{v_n}\rho_g(f,\varphi)-\varepsilon)}, \varphi_2^{({}_{v_n}\rho_g(f,\varphi)-\varepsilon)}, ..., \varphi_n^{({}_{v_n}\rho_g(f,\varphi)-\varepsilon)}). \tag{3.16}$$

Similarly from the definition of ${}_{v_n}\lambda_g(f,\varphi)$, we have for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$M_f(r_1, r_2, ..., r_n) > M_g(\varphi_1^{({}_{v_n}\lambda_g(f,\varphi)-\varepsilon)}, \varphi_2^{({}_{v_n}\lambda_g(f,\varphi)-\varepsilon)}, ..., \varphi_n^{({}_{v_n}\lambda_g(f,\varphi)-\varepsilon)}) \tag{3.17}$$

and also for a sequence values of $r_1, r_2, ..., r_n$ tending to infinity that

$$M_f(r_1, r_2, ..., r_n) < M_g(\varphi_1^{({}_{v_n}\lambda_g(f,\varphi)+\varepsilon)}, \varphi_2^{({}_{v_n}\lambda_g(f,\varphi)+\varepsilon)}, ..., \varphi_n^{({}_{v_n}\lambda_g(f,\varphi)+\varepsilon)}). \tag{3.18}$$

Further from the definitions of ${}_{v_n}\rho_h(g)$ it follows for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$M_g(\varphi_1, \varphi_2, ..., \varphi_n) < M_h(\varphi_1^{({}_{v_n}\rho_h(g)+\varepsilon)}, \varphi_2^{({}_{v_n}\rho_h(g)+\varepsilon)}, ..., \varphi_n^{({}_{v_n}\rho_h(g)+\varepsilon)}) \tag{3.19}$$

and for a sequence values of $r_1, r_2, ..., r_n$ tending to infinity we obtain that

$$M_g(\varphi_1, \varphi_2, ..., \varphi_n) > M_h(\varphi_1^{({}_{v_n}\rho_h(g)-\varepsilon)}, \varphi_2^{({}_{v_n}\rho_h(g)-\varepsilon)}, ..., \varphi_n^{({}_{v_n}\rho_h(g)-\varepsilon)}). \tag{3.20}$$

Likewise from the definitions of ${}_{v_n}\lambda_h(g)$ it follows for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$M_g(\varphi_1, \varphi_2, ..., \varphi_n) > M_h(\varphi_1^{({}_{v_n}\lambda_h(g)-\varepsilon)}, \varphi_2^{({}_{v_n}\lambda_h(g)-\varepsilon)}, ..., \varphi_n^{({}_{v_n}\lambda_h(g)-\varepsilon)}) \tag{3.21}$$

and for a sequence values of $r_1, r_2, ..., r_n$ tending to infinity we obtain that

$$M_g(\varphi_1, \varphi_2, ..., \varphi_n) < M_h(\varphi_1^{({}_{v_n}\lambda_h(g)+\varepsilon)}, \varphi_2^{({}_{v_n}\lambda_h(g)+\varepsilon)}, ..., \varphi_n^{({}_{v_n}\lambda_h(g)+\varepsilon)}). \tag{3.22}$$

Now, from (3.18) and in view of (3.19), for a sequence values of $r_1, r_2, ..., r_n$ tending to infinity, we get that

$$\begin{aligned}&M_f(r_1, r_2, ..., r_n) < M_h(\varphi_1^{({}_{v_n}\lambda_g(f,\varphi)+\varepsilon)({}_{v_n}\rho_h(g)+\varepsilon)},\\&\varphi_2^{({}_{v_n}\lambda_g(f,\varphi)+\varepsilon)({}_{v_n}\rho_h(g)+\varepsilon)}, ..., \varphi_n^{({}_{v_n}\lambda_g(f,\varphi)+\varepsilon)({}_{v_n}\rho_h(g)+\varepsilon)}).\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get from above that

$$\begin{aligned}&{}_{v_n}\lambda_h(f,\varphi) \leq {}_{v_n}\lambda_g(f,\varphi) \cdot {}_{v_n}\rho_h(g)\\&i.e., \frac{{}_{v_n}\lambda_h(f,\varphi)}{{}_{v_n}\rho_h(g)} \leq {}_{v_n}\lambda_g(f,\varphi)\ .\end{aligned} \tag{3.23}$$

Analogously from (3.17) and in view of (3.20) it follows for a sequence values of $r_1, r_2, ..., r_n$ tending to infinity that

$$M_f(r_1, r_2, ..., r_n) > M_h(\varphi_1^{({}_{v_n}\lambda_g(f,\varphi)-\varepsilon)({}_{v_n}\rho_h(g)-\varepsilon)},$$
$$\varphi_2^{({}_{v_n}\lambda_g(f,\varphi)-\varepsilon)({}_{v_n}\rho_h(g)-\varepsilon)}, ..., \varphi_n^{({}_{v_n}\lambda_g(f,\varphi)-\varepsilon)({}_{v_n}\rho_h(g)-\varepsilon)}).$$

As $\varepsilon(> 0)$ is arbitrary, so we obtain from above that

$${}_{v_n}\rho_h(f, \varphi) \geq {}_{v_n}\lambda_g(f, \varphi) \cdot {}_{v_n}\rho_h(g),$$
$$i.e., \ {}_{v_n}\lambda_g(f, \varphi) \leq \frac{{}_{v_n}\rho_h(f, \varphi)}{{}_{v_n}\rho_h(g)} . \tag{3.24}$$

Similarly from (3.16) and in view of (3.21) we have for a sequence values of $r_1, r_2, ..., r_n$ tending to infinity that

$$M_f(r_1, r_2, ..., r_n) > M_h(\varphi_1^{({}_{v_n}\rho_g(f,\varphi)-\varepsilon)({}_{v_n}\lambda_h(g)-\varepsilon)},$$
$$\varphi_2^{({}_{v_n}\rho_g(f,\varphi)-\varepsilon)({}_{v_n}\lambda_h(g)-\varepsilon)}, ..., \varphi_n^{({}_{v_n}\rho_g(f,\varphi)-\varepsilon)({}_{v_n}\lambda_h(g)-\varepsilon)}).$$

Since $\varepsilon > 0$ is arbitrary, it follows from above that

$${}_{v_n}\rho_h(f, \varphi) \geq {}_{v_n}\rho_g(f, \varphi) \cdot {}_{v_n}\lambda_h(g),$$
$$i.e., \ {}_{v_n}\rho_g(f, \varphi) \leq \frac{{}_{v_n}\rho_h(f, \varphi)}{{}_{v_n}\lambda_h(g)} . \tag{3.25}$$

Further from (3.15) and in view of (3.19) we get for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$M_f(r_1, r_2, ..., r_n) < M_h(\varphi_1^{({}_{v_n}\rho_g(f,\varphi)+\varepsilon)({}_{v_n}\rho_h(g)+\varepsilon)},$$
$$\varphi_2^{({}_{v_n}\rho_g(f,\varphi)+\varepsilon)({}_{v_n}\rho_h(g)+\varepsilon)}, ..., \varphi_n^{({}_{v_n}\rho_g(f,\varphi)+\varepsilon)({}_{v_n}\rho_h(g)+\varepsilon)}).$$

As $\varepsilon > 0$ is arbitrary, we obtain from above that

$${}_{v_n}\rho_h(f, \varphi) \leq {}_{v_n}\rho_g(f, \varphi) \cdot {}_{v_n}\rho_h(g),$$
$$i.e., \ {}_{v_n}\rho_g(f, \varphi) \geq \frac{{}_{v_n}\rho_h(f, \varphi)}{{}_{v_n}\rho_h(g)} . \tag{3.26}$$

Moreover, from (3.15) and in view of (3.22) we have for a sequence values of $r_1, r_2, ..., r_n$ tending to infinity that

$$M_f(r_1, r_2, ..., r_n) < M_h(\varphi_1^{({}_{v_n}\rho_g(f,\varphi)+\varepsilon)({}_{v_n}\lambda_h(g)+\varepsilon)},$$
$$\varphi_2^{({}_{v_n}\rho_g(f,\varphi)+\varepsilon)({}_{v_n}\lambda_h(g)+\varepsilon)}, ..., \varphi_n^{({}_{v_n}\rho_g(f,\varphi)+\varepsilon)({}_{v_n}\lambda_h(g)+\varepsilon)}).$$

Since $\varepsilon > 0$ is arbitrary, it follows from above that

$${}_{v_n}\lambda_h(f, \varphi) \leq {}_{v_n}\rho_g(f, \varphi) \cdot {}_{v_n}\lambda_h(g)$$

$$i.e., \ {}_{v_n}\rho_g(f,\varphi) \geq \frac{{}_{v_n}\lambda_h(f,\varphi)}{{}_{v_n}\lambda_h(g)} \ . \tag{3.27}$$

Likewise from (3.17) and (3.21), it follows for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$M_f(r_1, r_2, ..., r_n) > M_h(\varphi_1^{( {}_{v_n}\lambda_g(f,\varphi)-\varepsilon)( {}_{v_n}\lambda_h(g)-\varepsilon)},$$
$$\varphi_2^{( {}_{v_n}\lambda_g(f,\varphi)-\varepsilon)( {}_{v_n}\lambda_h(g)-\varepsilon)}, ..., \varphi_n^{( {}_{v_n}\lambda_g(f,\varphi)-\varepsilon)( {}_{v_n}\lambda_h(g)-\varepsilon)}).$$

As $\varepsilon > 0$ is arbitrary,we obtain from above that

$${}_{v_n}\lambda_h(f,\varphi) \geq {}_{v_n}\lambda_g(f,\varphi) \cdot {}_{v_n}\lambda_h(g),$$
$$i.e., \ {}_{v_n}\lambda_g(f,\varphi) \leq \frac{{}_{v_n}\lambda_h(f,\varphi)}{{}_{v_n}\lambda_h(g)} \ . \tag{3.28}$$

Hence the theorem follows from (3.23), (3.24), (3.25), (3.26), (3.27) and (3.28). □

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Accepted: 13.06.2019