Minimal left ideals in some endomorphism semirings of semilattices

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Abstract. Minimal left ideals in some endomorphism semirings of semilattices are investigated.
Keywords: semiring, semimodule, semilattice, ideal, endomorphism.

1. Preliminaries

1.1 Semirings. A semiring is an algebraic structure possesing two associative binary operations (addition and multiplication), where the addition is commutative and the multiplication distributes over the addition.

Let S be a semiring. A non-empty subset I of S is called

- a *left* (*right*) *ideal* if $(I + I) \cup SI \subseteq I$ $((I + I) \cup IS \subseteq I)$;
- an *ideal* if $(I + I) \cup SI \cup IS \subseteq I$;
- a *bi-ideal* if $(S + I) \cup SI \cup IS \subseteq I$.

A (left, right, bi-) ideal I is called *minimal* if $|I| \geq 2$ and $K = I$ whenever K is a (left, right, bi-) ideal such that $K \subseteq I$ and $|K| \geq 2$. The semiring S is called

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(*left-, right-, bi-*) *ideal-simple* if it is non-trivial and has no proper non-trivial (left, right, bi-) ideal.

1.2 Semilattices. A *semilattice* M $(= M(+))$ is a commutative idempotent semigroup. A non-empty subset A of M is an *ideal* if $M + A = A$. Such an ideal is called *prime* if the set $M \setminus A$ is a subsemilattice of M. We have the basic order on M defined by $a \leq b$ iff $a + b = b$.

An element $w \in M$ is *neutral* (*absorbing*) if $x + w = x$ $(x + w = w)$ for every $x \in M$. Such an element is unique (provided that it exists - this fact is denoted by $0_M \in M$ $(o_M \in M)$) and is denoted by $0 = 0_M$ $(o = o_M)$ in the sequel. The fact that no such element exists is denoted by $0_M \notin M$ $(o_M \notin M)$.

1.3 Semimodules. Let S be a semiring. A (*left* $S-$) *semimodule* is a commutative semigroup M $(= M(+))$ together with a scalar multiplication $S \times M \to M$. The semimodule M is called

- *minimal* if $|M| \geq 2$ and $N = M$ whenever N is a non-trivial subsemimodule of M;

- *simple* if it has just two congruence relations;

- *faithful* if for all $a, b \in S$, $a \neq b$, there is at least one $x \in M$ with $ax \neq bx$.

1.4 Endomorphism semirings. Let M $(= M(+))$ be a commutative semigroup. The set $\underline{E} = \underline{E}(M) = \mathrm{End}(M)$ of endomorphisms of M is a semiring via $(f+g)(x) = f(x)+g(x)$ and $(fg)(x) = f(g(x))$ for all $f, g \in \underline{E}$ and $x \in M$. The semiring $\underline{E}$ is unitary and the multiplicatively neutral element is the identity automorphism id_M of M.

2. 0-preserving endomorphisms

Let M be a non-trivial semilattice such that $0_M \in M$. The set $\underline{E}_0 = \{\, f \in \underline{E} \mid f(0_M) = 0_M \,\}$ is a unitary subsemiring of $\underline{E}$ and the constant endomorphism ξ, $\xi(M) = \{0_M\}$, is the zero element of the semiring $\underline{E}_0$. For all $u, v \in M$, we have $q_{u,v} \in E_0$, where $q_{u,v}(x) = 0_M$ for $x \leq u$ and $q_{u,v}(y) = v$ for $y \nleq u$. Thus $q_{u,v}(M) = \{0_M, v\}$ for $u \neq o_M$, $q_{o_M,v} = \xi$ (if $o_M \in M$) and $q_{u,0_M} = \xi$.

Now, let E be a subsemiring of $\underline{E}_0$ such that $q_{u,v} \in E$ for all $u, v \in M$. For every $u \in N = M \setminus \{o_M\}$, put $T_u = \{\, q_{u,v} \mid v \in M \,\}$. The following observations are quite easy.

Lemma 2.1. (i) $q_{u,v_1} + q_{u,v_2} = q_{u,v_1+v_2}$.
(ii) $fq_{u,v} = q_{u,f(v)}$ *for every* $f \in \underline{E}_0$.
(iii) $q_{u,v_1}q_Pu, v_2 = \xi$ *for* $v_2 \leq u$.
$q_{u,v_1}q_{u,v_2} = q_{u,v_1}$ *for* $v_2 \nleq u$.
(v) $q_{u,v}q_{u,v} = \xi$ *for* $v \leq u$.
(vi) $q_{u,v}q_{u,v} = q_{u,v}$ *for* $v \nleq u$. $\square$

Proposition 2.2. *The set T_u is a minimal left ideal of the semiring E for every $u \in N$. The mapping $v \mapsto q_{u,v}$ is an isomorphism of the semimodule ${}_EM$ onto the semimodule ${}_ET_u$.*

Proof. It follows immediately from 2.1 that T_u is a left ideal of E and that the map $v \mapsto q_{u,v}$ is an isomorphism of the semimodules. Furthermore, if $v \neq 0_M$ then $q_{0_M,v_1} q_{u,v} = q_{u,v_1}$ for every $v_1 \in M$. Now, it is clear that T_u is a minimal left ideal of E. □

Proposition 2.3. (i) *$T_u q_{u,v} = \{\xi\}$ for $v \leq u$.*
(ii) *$T_u q_{u,v} = T_u$ for $v \nleq u$.*

Proof. Use 2.1(iii),(iv). □

Proposition 2.4. (i) *The set $\{\, q_{u,v} \mid v \leq u \,\}$ is an ideal of the semiring T_u and it is the greatest proper ideal of that semiring.*
(ii) *The $\{\, q_{u,v} \mid v \nleq u \,\} \cup \{\xi\}$ is just the set of the multiplicative idempotents of T_u ($\xi = q_{u,0_M}$).*
(iii) *The semiring T_u is multiplicatively idempotent iff $u = 0_M$.*
(iv) *The semiring T_u is (left-) ideal-simple iff $u = 0_M$.*

Proof. It is easy. □

Proposition 2.5. *The semimodule ${}_ET_u$ is simple.*

Proof. Let $\alpha \neq \mathrm{id}$ be a congruence of T_u. Then $(q_{u,v_1}, q_{u,v_2}) \in \alpha$ for some $v_1 < v_2$, and hence $(\xi, q_{u,v_2}) = (q_{v_1,v_2} q_{u,v_1}, q_{v_1,v_2} q_{u,v_2}) \in \alpha$. Consequently, $(\xi, q_{u,v_2}) = (q_{0_M,v_3}\xi, q_{0_M,v_3} q_{u,v_2}) \in \alpha$ for every $v_3 \in M$. Thus $\alpha = T_u \times T_u$. □

Proposition 2.6. *The semimodule T_u is faithful.*

Proof. Use 2.1(ii). □

Lemma 2.7. *If I is a non-trivial ideal of E then $\bigcup_{u \in N} T_u \subseteq I$.*

Proof. We have $IT_u \subseteq I \cap T_u$ and the latter set is a left ideal of E. If $T_u \nsubseteq I$ then $IT_u = I \cap T_u = \{\xi\}$ and it means that $q_{u,f(v)} = f q_{u,v} = \xi$ for every $f \in I$ and $v \in M$. Since $u \neq o_M$, we get $f(v) = 0_M$ and $f = \xi$. Thus $I = \{\xi\}$, a contradiction. □

Corollary 2.8. *The set $\{\, q_{u_1,v_1} f_1 + \cdots + q_{u_n,v_n} f_n \mid n \geq 1, u_i \in N, v_i \in M, f_i \in E \,\}$ is just the smallest non-trivial ideal of the semiring E.* □

2.9 Let K be a minimal left ideal of the semiring E such that $K \neq T_u$ for every $u \in N$.
Lemma 2.9.1. *$K \cap T_u = \{\xi\}$ for every $u \in N$.*

Proof. It is obvious. □

Lemma 2.9.2. *$Ef = K$ for every $f \in K \setminus \{\xi\}$.*

Proof. The set $K_1 = \{ f \in K \mid Ef = \{\xi\} \}$ is a left ideal of E. If $K_1 = K$ then $EK = \{\xi\}$ and $K = \{\xi, f_0\}$ is two-element. Now, $q_{0_M,v} f_0 = \xi$, and hence $f_0 = \xi$, a contradiction. It follows that $K_1 = \{\xi\}$ and $Ef = K$, since Ef is a left ideal of E. □

Lemma 2.9.3. *There are subsemilattices A and B of M such that:*

(i) *$A \cup B = M$ and $A \cap B = \emptyset$.*

(ii) *$A = \{ x \in M \mid f(x) \neq 0_M \}$ and $B = \{ x \in M \mid f(x) = 0_M \}$ for every $f \in K \setminus \{\xi\}$.*

(iii) *$A + M = A$ (i.e., A is a prime ideal of the semilattice M).*

(iv) *$o_B \notin B$ (i.e., the prime ideal A is not principal).*

Proof. If $f, g \in K \setminus \{\xi\}$ then $g = hf$ for some $h \in E$ by 2.9.2 and we see that $g(x) \neq 0_M$ implies $f(x) \neq 0_M$. Clearly, $A + M = A$ and $B + B = B$. We have $0_M \in B$ and $A \neq \emptyset$, since $f \neq \xi$. If $u = o_B \in B$ then $u \neq o_M$ and $q_{0_M,v} f = q_{u,v} \in K$ for every $v \in M$, a contradiction. □

Lemma 2.9.4. *If $f \in K \setminus \{\xi\}$ and $v \in M$ then $q_{0_M,v} f = q_{A,v} \in K$, where $q_{A,v}(B) = \{0_M\}$ and $q_{A,v}(A) = \{v\}$.*

Proof. It is obvious. □

Proposition 2.9.5. (i) *$K = \{ q_{A,v} \mid v \in M \}$.*

(ii) *The mapping $v \mapsto q_{A,v}$ is an isomorphism of the semimodule ${}_E M$ onto the semimodule ${}_E K$.*

(iii) *$K q_{A,v} = \{\xi\}$ for every $v \in B$.*

(iv) *$K q_{A,v} = K$ for every $v \in A$.*

(v) *The set $\{ q_{A,v} \mid v \in B \}$ is an ideal of the semiring K and it is the greatest proper left ideal of that semiring.*

(vi) *The set $\{ q_{A,v} \mid v \in A \} \cup \{\xi\}$ is just the set of the multiplicative idempotents of K ($\xi = q_{A,0_M}$).*

(vii) *The semiring K is neither multiplicatively idempotent nor ideal-simple.*

Proof. According to 2.9.4, $T_A = \{ q_{A,v} \mid v \in M \} \subseteq K$. The rest is clear (see 2.2, 2.4 and 2.4). □

Theorem 2.10. (i) *The set $T_u = \{ q_{u,v} \mid v \in M \}$ is a minimal left ideal of the semiring E for every $u \in N$.*

(ii) *If K is a minimal left ideal of E such that $K \neq T_u$ for every $u \in N$ then there is a non-principal prime ideal A of the semilattice M such that $K = T_A = \{ q_{A,v} \mid v \in M \}$.*

(iii) *If A is a non-principal prime ideal of M such that $Q_{A,v} \in E$ for at least one $v \in M \setminus \{0_M\}$ then $T_A \subseteq E$ and T_A is a minimal left ideal of E such that $T_A \neq T_u$ for every $u \in N$.*

(iv) *If K is a minimal left ideal of E then ${}_EK$ is a faithful, simple and minimal (left E-) semimodule and the semimodules ${}_EK$ and ${}_EM$ are isomorphic.*

(v) *If K is a minimal left ideal of E then the semiring K is multiplicatively idempotent if and only if $K = T_{0_M}$.*

Proof. Combine the foregoing results. □

Corollary 2.11. *Denote by* $\mathbf{A}$ *the set of prime ideals A of M such that $T_A \subseteq E$* (see 2.10(ii),(iii)). *Then:*

(i) $\{ T_A \mid a \in \mathbf{A} \}$ *is just the set of (pairwise distinct) minimal left ideals of the semiring E.*

(ii) $A_u = \{ x \in M \mid x \not\leq u \} \in \mathbf{A}$ *for every $u \in N$.*

(iii) *The set $\sum T_A$, $A \in \mathbf{A}$, is just the smallest non-trivial ideal of the semiring E.* □

3. 1-preserving endomorphisms (a)

Let M be a non-trivial semilattice such that $o_M \in M$. The set $\underline{E}_1 = \underline{E}_1(M) = \{ f \in \underline{E} \mid f(o_M) = o_M \}$ is a unitary subsemiring of $\underline{E}$ and the constant endomorphism $\zeta = \zeta_M$, $\zeta(M) = \{o_M\}$, is the bi-absorbing element of the semiring $\underline{E}_1$. For all $u, v \in M$, $u \neq o_M$, we have $p_{u,v} \in \underline{E}_1$, where $p_{u,v}(x) = v$ for $x \leq u$ and $p_{u,v}(y) = o_M$ for $y \not\leq u$. Thus $p_{u,v}(M) = \{v, o_M\}$ and $p_{u,o_M} = \zeta$.

Now, let F be a subsemiring of $\underline{E}_1$ such that $p_{u,v} \in F$ for all $u, v \in M$, $u \neq o_M$. For every $u \in N = M \setminus \{o_M\}$, put $V_u = \{ p_{u,v} \mid v \in M \}$.

The following observations are quite easy.

Lemma 3.1. (i) $p_{u,v_1} + p_{u,v_2} = p_{u,v_1+v_2}$.

(ii) $fp_{u,v} = p_{u,f(v)}$ *for every* $f \in \underline{E}_1$.

(iii) $p_{u,v_1}p_{u,v_2} = \zeta$ *for* $v_2 \not\leq u$.

(iv) $p_{u,v_1}p_{u,v_2} = p_{u,v_1}$ *for* $v_2 \leq u$.

(v) $p_{u,v}p_{u,v} = \zeta$ *for* $v \not\leq u$.

(vi) $p_{u,v}p_{uv} = p_{u,v}$ *for* $v \leq u$. □

Proposition 3.2. *The set V_u is a minimal left ideal of the semiring F for every $u \in N$. The mapping $v \mapsto p_{u,v}$ is an isomorphism of the semimodule ${}_FM$ onto the semimodule ${}_FV_u$.*

Proof. It follows immediately from 3.1 that V_u is a left ideal of F and the map $v \mapsto p_{u,v}$ is an isomorphism of the semimodules. Furthermore, if $v \neq o_M$ then $p_{v,v_1}p_{u,v} = p_{u,v_1}$ for every $v_1 \in M$. Now, it is clear that V_u is a minimal left ideal of F. □

Proposition 3.3.(i) $V_up_{u,v} = \{\zeta\}$ *for* $v \not\leq u$.

(ii) $V_up_{u,v} = V_u$ *for* $v \leq u$.

Proof. Use 3.1 (iii), (iv). □

Proposition 3.4. (i) *The set* $\{ p_{u,v} \,|\, v \not\leq u \}$ *is a bi-ideal of the semiring* V_u *and it is the greatest proper left ideal of that semiring.*

(ii) *The set* $\{ p_{u,v} \,|\, v \leq u \} \cup \{\zeta\}$ *is just the set of multiplive idempotents of* V_u $(\zeta = p_{u,o_M})$.

(iii) *The semiring* V_u *is multiplicatively idempotent iff* $u = o_N \in N$.

(iv) *The semiring* V_u *is (left-) ideal-simple iff* $u = o_N \in N$.

Proof. It is easy. □

Proposition 3.5. *The semimodule* ${}_FV_u$ *is simple.*

Proof. Let $\alpha \neq \mathrm{id}$ be a congruence of ${}_FV_u$. Then $(p_{u,v_1}, p_{u,v_2}) \in \alpha$ for some $v_1 < v_2$, and hence $(p_{u,v_2}, \zeta) = (p_{v_1,v_2}p_{u,v_1}, p_{v_1,v_2}p_{u,v_2}) \in \alpha$. Consequently, $(p_{u,v_1}, \zeta) \in \alpha$ and, finally, $(p_{u,v_3}, \zeta) = (p_{v_1,v_3}p_{u,v_1}, p_{v_1,v_3}\zeta) \in \alpha$ for every $v_3 \in M$. Thus $\alpha = V_u \times V_u$. □

Proposition 3.6. *The semimodule* ${}_FV_u$ *is faithful.*

Proof. Use 3.1(ii). □

Lemma 3.7. *If* I *is a non-trivial ideal of* F *then* $\bigcup_{u\in N} V_u \subseteq I$.

Proof. We can proceed similarly as in the proof of 2.7. □

Corollary 3.8. *The set* $\{ p_{u_1,v_1}f_1 + \dots p_{u_n,v_n}f_n \,|\, n \geq 1, u_i \in N, v_i \in M, f_i \in F \}$ *is just the smallest non-trivial ideal of the semiring* F. □

3.9 Let K be a minimal left ideal of the semiring F such that $K \neq V_u$ for every $u \in N$.

Lemma 3.9.1. $K \cap V_u = \{\zeta\}$ *for every* $u \in N$.

Proof. It is obvious. □

Lemma 3.9.2. $Ff = K$ *for every* $f \in K \setminus \{\zeta\}$.

Proof. If $f \in F \setminus \{\zeta\}$ then $f(v) \neq o_M$ for at least one $v \in N$ and $p_{f(v),v} \neq \zeta$. Now, we can proceed similarly as in the proof of 2.9.2. □

Lemma 3.9.3. *There is a proper ideal* A *of the semilattice* M *such that* $A = \{ x \in M \,|\, f(x) = o_M \}$ *for every* $f \in K \setminus \{\zeta\}$ *and we put* $B = M \setminus A$.

Proof. Use 3.9.2. □

Lemma 3.9.4. *Let* $f \in K \setminus \zeta$, $v \in M$, *and let* $o_m \neq w \in f(M)$. *Then* $p_{w,v}f = p_{A,v} \in K$, *where* $p_{A,v}(B) = \{v\}$ *and* $p_{A,v}(A) = \{o_M\}$.

Proof. We have $g = p_{w,v}f \in K$ and, by 3.9.3, $A = \{ x \in M \,|\, g(x) = o_M \} = \{ x \in M \,|\, f(x) \not\leq w \}$. Thus $B = \{ y \in M \,|\, g(y) \neq o_M \} = \{ y \in M \,|\, f(y) \leq w \}$ and $g(B) = \{v\}$. □

Proposition 3.9.5. (i) *A is a prime ideal of M, $o_B \notin B$ and $K = \{\, p_{A,v} \mid v \in M \,\} = V_A$.*

(ii) *The mapping $v \mapsto p_{A,v}$ is an isomorphism of the semimodule ${}_F M$ onto the semimodule ${}_F K$.*

(iii) *$Kp_{A,v} = \{\zeta\}$ for $v \in A$.*

(iv) *$Kp_{A,v} = K$ for $v \in B$.*

(v) *The set $\{\, p_{A,v} \mid v \in A \,\}$ is a bi-ideal of the semiring K and it is the greatest proper left ideal of that semiring.*

(vi) *The set $\{\, p_{A,v} \mid v \in B \,\} \cup \{\zeta\}$ is just the set of multiplicative idempotents of K ($\zeta = p_{A,o_M}$).*

(vii) *The semiring K is neither multiplicatively idempotent nor bi-ideal-simple.*

Proof. According to 3.9.4, $V_A = \{\, p_{A,v} \mid v \in M \,\} \subseteq K$. The rest is clear (see 3.2, 3.3 and 3.4) $\square$

Theorem 3.10. (i) *The set $V_u = \{\, p_{u,v} \mid v \in M \,\}$ is a minimal left ideal of the semiring F for every $u \in N$.*

(ii) *If K is a minimal left ideal of F such that $K \neq V_u$ for every $u \in N$ then there is a non-principal prime ideal A of the semilattice M such that $K = V_A = \{\, p_{A,v} \mid v \in M \,\}$.*

(iii) *If A is a non-principal prime ideal of M such that $p_{A,v} \in F$ for at least one $v \in M \setminus \{o_M\}$ then $V_A \subseteq F$ and V_A is a minimal left ideal of F such that $V_A \neq V_u$ for every $u \in N$.*

(iv) *If K is a minimal left ideal of F then ${}_F K$ is a faithful, simple and minimal F-semimodule and the semimodules ${}_F K$ and ${}_F M$ are isomorphic.*

(v) *If K is a minimal left ideal of F then the semiring K is multiplicatively idempotent if and only if $o_N \in N$ and $K = V_{o_N}$.*

Proof. See the foregoing results. $\square$

Corollary 3.11. *Denote by $\mathbf{B}$ the set of prime ideals A of M such that $V_A \subseteq F$* (see 3.10(ii),(iii)). *Then:*

(i) *$\{\, V_A \mid A \in \mathbf{B} \,\}$ is just the set of (pair-wise distinct) minimal left ideals of the semiring F.*

(ii) *$A_u = \{\, x \in M \mid x \not\leq u \,\} \in \mathbf{B}$ for every $u \in N$.*

(iii) *The set $\bigcup V_A$, $A \in \mathbf{B}$, is just the smallest non-trivial ideal of the semiring F.* $\square$

4. 1-preserving endomorphisms (b)

Let M be an *antichain*, i.e., a semilattice containing at least three elements such that $o_M \in M$ and $x + y = o_M$ for all $x, y \in M$, $x \neq y$, and put $N = M \setminus \{o_M\}$. Now, a transformation f of M belongs to $\underline{E}_1$ iff $f(o_M) = o_M$ and $f(x) \neq f(y)$ for all $x, y \in M$ such that $x \neq y$ and $f(x) \neq o_M$.

Proposition 4.1. *Let E be a subsemiring of of $\underline{E}_1$ such that $\zeta \in E$. The following conditions are equivalent:*

(i) *The semimodule ${}_EM$ is minimal.*

(ii) $N \subseteq E(x) = \{\, f(x) \,|\, f \in E \,\}$ *for every* $x \in N$.

(iii) $E(x) = M$ *for every* $x \in N$.

(iv) *The semimodule ${}_EM$ is simple.*

Proof. Clearly, the first three conditions are equivalent. Now, let α be a congruence of ${}_EM$ and $P = \{\, x \in M \,|\, (x, o_M) \in \alpha \,\}$. Then P is a subsemimodule of M and if M is minimal then $P = \{o_M\}$. On the other hand, if $(x, y) \in \alpha$, $x \neq y$, then $(x, o_M) = (x + x, x + y) \in \alpha$ and $x \in P$. Thus (i) implies (iv). Conversely, for every $x \in M$, the relation $\alpha_x = (E(x) \times E(x)) \cup \mathrm{id}_M$ is a congruence of ${}_EM$. If $\alpha = M \times M$ then $E(x) = M$. If $\alpha_x = \mathrm{id}_M$ then $E(x) = \{o_M\}$ and $x \in Q = \{\, y \in M \,|\, F(y) = \{o_M\} \,\}$. Of course, Q is a subsemimodule of ${}_EM$ and $\beta = (Q \times Q) \cup \mathrm{id}_M$ is a congruence of ${}_EM$. If $\beta = \mathrm{id}_M$ then $x = o_M$. If $\beta = M \times M$ then $Q = M$, $E(M) = \{o_M\}$ and ${}_EM$ is not simple. Thus (iv) implies (iii). $\square$

Lemma 4.2. *Let $f, g \in \underline{E}_1$. Then $f + g = \zeta$ iff $f(x) = o_M$ whenever $x \in M$ is such that $f(x) = g(x)$.*

Proof. It is obvious. $\square$

In the remaining part of this section, let E be a non-trivial subsemiring of $\underline{E}_1$ satisfying the equivalent conditions of 4.1 and let K be a left ideal of E.

Lemma 4.3. *Let $w \in M$. The mapping $\tau_w : f \mapsto f(w)$ is a homomorphism of the semimodule ${}_EK$ into the semimodule ${}_EM$. If $K(w) \neq \{o_M\}$ then $\tau(K) = M$.*

Proof. It is easy. $\square$

Proposition 4.4. *If the semimodule ${}_EK$ is simple then K is a minimal left ideal of E and the semimodules ${}_EK$ and ${}_EM$ are isomorphic.*

Proof. Since ${}_EK$ is simple, the left ideal K is non-trivial and, taking $f \in K \setminus \{o_M\}$, we find $w \in M$ with $f(w) \neq o_M$. Then $K(w) = M$ and τ_w is an isomorphism of the semimodules (use 4.3). $\square$

Lemma 4.5. *For every $w \in M$, the set $L_w = \{\, f \in E \,|\, f(w) = o_M \,\}$ is a left ideal of the semiring E.*

Proof. It is obvious. $\square$

Lemma 4.6. *Let $w \in M$ be such that $K \cap L_w = \{\zeta\}$ (e.g., if K is minimal and $K \nsubseteq L_w$). If $f, g \in K$ are such that $f(w) \neq g(w)$ then $f + g = \zeta$.*

Proof. We have $f+g \in K \cap L_w$. □

Lemma 4.7. *If K is minimal then $f+g=\zeta$ for all $f,g \in K$, $f \neq g$.*

Proof. Since $f \neq g$, there is $w \in M$ with $f(w) \neq g(w)$. Clearly, $K \nsubseteq L_w$ and 4.6 applies. □

Theorem 4.8. *If K is a minimal left ideal of the semiring E then the semimodules ${}_EK$ and ${}_EM$ are isomorphic.*

Proof. Since K is non-trivial, there is $w \in M$ with $K(w) \neq \{o_M\}$. Then $K \cap L_w = \{\zeta\}$ and, in view of 4.3, we have to show that $\ker \tau_w = \mathrm{id}_K$. If $f,g \in K$ are such that $f(w)=g(w)$ then $f(w)=(f+g)(w)=g(w)$ and either $f(w)=o_M=g(w)$ and $f=\zeta=g$, or $(f+g)(w) \neq o_M$, $f+g \neq \zeta$ and $f=g$ by 4.7. □

Remark 4.9. Assume that K is a minimal left ideal and let $w \in M$ be such that $K(w)=M$ (equivalently, $K \nsubseteq L_w$). If $f \in K$ then $Kf=\{\zeta\}$ iff $K \subseteq L_{f(w)}$. On the other hand, if $Kf \neq \{\zeta\}$ then $Kf=K$, $K(f(w))=M$ and $K \cap L_{f(w)}=\{\zeta\}$.

Lemma 4.10. *If $f \in \underline{E}_1$ is such that $f^2=f$ then $f(x) \in \{x, o_M\}$ for every $x \in M$.*

Proof. It is obvious. □

Lemma 4.11. *Let K be a minimal left ideal, $A=\{\, w \in N \mid K \subseteq L_w \,\}$ and let $f \in K$ be such that $f^2=f \neq \zeta$. Then:*

(i) *$f(x)=x$ for every $x \in M \setminus A$.*

(ii) *f is right multiplicatively neutral in K.*

(iii) *If $A=\emptyset$ then $f=\mathrm{id}_M$ and $K=E$.*

(iv) *If $A \neq \emptyset$ then $K=\bigcap L_w$, $w \in A$.*

Proof. (i) This follows from 4.10.

(ii) If $g \in K$ then $gf(x)=g(x)$ for every $x \in M \setminus A$ and $gf(y)=o_M=g(y)$ for every $y \in A$. Thus $gf=g$.

(iii) By (i), $f=\mathrm{id}_M$.

(iv) If $g \in \bigcap L_w$ then $gf=g$. □

Corollary 4.12. *Let K be a minimal left ideal of E. Then:*

(i) *K contains at most two multiplicatively idempotent elements.*

(ii) *K is not multiplicatively idempotent.* □

5. 1-preserving endomorphisms (c)

Let K and L be semilattices containing at least three elements and such that $o_K \in K$ and $o_L \in L$. The set $Q = (\{o_K\} \times L) \cup (K \times \{o_L\})$ is an ideal of the cartesian product $K \times L$ and we put $M = (K \times L)/Q = \{o_M\} \cup (K' \times L')$, $K' = K \setminus \{o_K\}$, $L' = L \setminus \{o_L\}$. Clearly, $|M| \geq 5$.

If $f \in \underline{E}_1(K)$ and $g \in \underline{E}_1(L)$ then $(f \times g)(Q) \subseteq Q$ and we put $f * g = (f \times g)/Q \in \underline{E}_1(M)$. Thus $(f_1 * g_1)+(f_2 * g_2) = (f_1+f_2)*(g_1+g_2)$, $(f_1 * g_1)(f_2 * g_2) = f_1 f_2 * g_1 g_2$, and hence $\underline{E}_1(K) * \underline{E}_1(L)$ is a subsemiring of $\underline{E}_1(M)$. In fact, if E is a subsemiring of $\underline{E}_1(K)$ and F is a subsemiring of $\underline{E}_1(L)$ then $E * F$ is a subsemiring of $\underline{E}_1(M)$.

In the remaining part of this section, assume that L is an antichain and take a non-trivial subgroup G of $\mathrm{Aut}(L)$ such that $f(x) \neq g(x)$ for all $x \in L'$, $f, g \in G$, $f \neq g$. Let S be a subsemiring of $\underline{E}_1(K) * (G \cup \{\zeta_L\})$ such that $p_{u,v} * g \in S$ for all $u \in K'$, $v \in K$ and $g \in G$ (see the preceding two sections). One checks easily that $o_S = \zeta_M = p_{u,o_K} * g \in S$ is the bi-absorbing element of S and $AA \subseteq A$, where $A = \{\, f \in \underline{E}_1(K) \,|\, f * g \in S, g \in G \,\}$,. Furthermore, $S \subseteq \underline{E}_1(K) * G$ and $B = \{\, f \in \underline{E}_1(K) \,|\, f * \mathrm{id}_M \in S \,\}$ is a subsemiring of $\underline{E}_1(K)$ such that $p_{u,v} \in B$ for all $u \in K'$ and $v \in K$.

Lemma 5.1. *Let $u \in K'$, $v_1, v_2 \in K$ and $g_1, g_2 \in G$. Then:*
(i) *If $g_1 = g_2$ then $(p_{u,v_1} * g_1) + (p_{u,v_2} * g_2) = p_{u,v_1+v_2} * g_1$.*
(ii) *If $g_1 \neq g_2$ then $(p_{u,v_1} * g_1) + (p_{u,v_2} * g_2) = \zeta_M$.*

Proof. We have $(p_{u,v_1} * g_1) + (p_{u,v_2} * g_2) = (p_{u,v_1+v_2} * (g_1 + g_2))$. □

Lemma 5.2. *Let $u \in K'$, $v \in K$, $g \in G$ and let $f \in \underline{E}_1(K)$ and $h \in G$ be such that $f * h \in S$. Then $(f * h)(p_{u,v} * g) = p_{u,f(v))} * (hg)$.*

Proof. It is obvious. □

Proposition 5.3. *For every $u \in K'$, the set $W_u = \{\, p_{u,v} * g \,|\, v \in K, g \in G \,\}$ is a minimal left ideal of the semiring S.*

Proof. According to 5.1 and 5.2, the set W_u is a left ideal of S. If $v \in K'$, $w \in K$ and $g, h \in S$ then $p_{v.w} * hg^{-1})(p_{u,v} * g) = p_{u,w} * h$. Thus $S(p_{u,v} * g) = W_u$. □

Lemma 5.4. *Let $f_1, f_2 \in \underline{E}_1(K)$ and $g_1, g_2 \in G$ be such that $f_1 * g_1 = f_2 * g_2$. Then $f_1 = f_2$ and if $f_1 \neq \zeta_K$ then $g_1 = g_2$.*

Proof. Let $x_0 \in K$ be such that $f_1(x_0) \neq f_2(x_0)$. Then $x_0 \in K'$, $(f_1(x_0), g_1(y)) \in Q$ and $(f_2(x_0), g_2(y)) \in Q$ for every $y \in L'$. If $f_1(x_0) \neq o_K$ then $g_1(y) = o_L$, a contradiction. Thus $f_1(x_0) = o_K$ and, similarly, $f_2(x_0) = o_K$. It follows that $f_1 = f_2$. The rest is clear. □

Lemma 5.5. *Let $u \in K'$, $v_1, v_2 \in K$ and $g_1, g_2 \in G$. Then $p_{u,v_1} * g_1 = p_{u,v_2} * g_2$ iff $v_1 = v_2$ and either $g_1 = g_2$ or $v_1 = o_K$.*

Proof. This follows from 5.4. □

Proposition 5.6. *Let $u \in K'$ and $z \in L'$. Define a mapping $\nu_{u,z} : W_u \to M$ by $\nu_{u,z}(p_{u,v} * g) = (v, g(z))$ for $v \in K'$ and $\nu_{u,z}(o_S) = o_M$* (see 5.5)*. Then $\nu_{u,z}$ is an injective homomorphism of the semimodule ${}_SW_u$ into the semimodule ${}_SM$. The homomorphism $\nu_{u,z}$ is an isomorphism of the semimodules iff $G(z) = L'$ (i.e., the group G operates transitively on L').*

Proof. Use 5.1,…,5.5. □

Proposition 5.7. *Let $u \in K'$. The semimodule ${}_SW_u$ is simple, minimal and faithful.*

Proof. The semimodule is minimal due to 5.3. By 5.2, $(f * h)(p_{u,v} * \mathrm{id}_L) = p_{u,f(v)} * h$ and, using 5.5, we conclude easily the our semimodule is faithful as well.

It remains to show that ${}_SW_u$ is simple. For, let $\alpha \neq \mathrm{id}$ be a congruence of the semimodule. Then $(p_{u,v_1} * g_1, p_{u,v_2} * g_2) \in \alpha$, where $p_{u,v_1} * g_1 \neq p_{u,v_2} * g_2$ and we can assume that $v_1 \in K'$. If $g_1 \neq g_2$ then $(p_{u,v_1} * g_1, \zeta) \in \alpha$ follows from 5.1(ii). But $S(p_{u,v_1} * g_1) = W_u$ by 5.3, and hence $\alpha = W_u \times W_u$. Assume, therefore, that $g_1 = g_2 = g$ and $v_2 \nleq v_1$. Then $(p_{U,v_1} * g, p_{u,v_3} * g) \in \alpha$, where $v_1 < v_3 = v_1 + v_2$. From this, $(p_{u,v_4} * g, \zeta) = ((p_{v_1,v_4} * g)(p_{u,v_1} * g), (p_{v_1,vb_4} * g)(p_{u,v_3} * g)) \in \alpha$ for every $v_4 \in K$. Thus $\alpha = W_u \times W_u$. □

Remark 5.8. S is a subsemiring of $\underline{E}_1(M)$, and so the (left S-) semimodule ${}_SM$ is faithful. In view of 5.6, the semimodule ${}_SM$ is minimal iff G operates transitively on L', and then the semimodules ${}_SM$ and ${}_SW_u$ are isomorphic, so that ${}_SM$ is simple by 5.7. If $y \in L'$ then $R = (K' \times G(y)) \cup \{o_M\}$ is an ideal of the semimodule ${}_SM$ and $(R \times R) \cup \mathrm{id}_M$ is a congruence of ${}_SM$. If $\tau = M \times M$ (e.g., if ${}_SM$ is simple) then $R = M$ and G operates transitively on L'.

Proposition 5.9. (i) *$W_u(p_{u,v} * g) = \{\zeta\}$ for $v \nleq u$.*

(ii) *$W_u(p_{u,v} * g) = W_u$ for $v \leq u$.*

Proof. It is easy. □

Proposition 5.10. (i) *The set $\{ p_{u,v} * g \mid v \nleq u \}$ is a bi-ideal of the semiring W_u and it is the greatest proper left ideal of that semiring.*

(ii) *The set $\{ p_{u,v} * \mathrm{id}_L \mid v \leq u \} \cup \{\zeta\}$ is just the set of multiplicative idempotents of W_u.*

(iii) *The semiring W_u is not multiplicatively idempotent.*

(iv) *The semiring W_u is not bi-ideal-simple.*

Proof. Easy. □

5.11 Let R be a minimal left ideal of the semiring S such that $R \neq W_u$ for every $u \in N$.

Lemma 5.11.1. $R \cap W_u = \{\zeta\}$.

Proof. It is obvious. □

Lemma 5.11.2. $S(f * g) = R$ for every $f * g \in R \setminus \{\zeta\}$.

Proof. We have $f \in \underline{E}_1(K) \setminus \{\zeta_K\}$, and hence $f(v) \neq o_K$ for at least one $v \in N$ and $(p_{f(v),v} * \mathrm{id}_M)(f * g) = (p_{f(v),v} f * g) \neq \zeta$. Now, we can proceed similarly as in the proof of 2.9.2. □

Lemma 5.11.3. *There is a proper ideal A of the semilattice K such that $A = \{\, x \in K \mid f(x) \neq o_K \,\}$ for every $f * g \in R \setminus \{\zeta\}$.*

Proof. Use 5.11.2. □

Lemma 5.11.4. *Let $f * g \in R \setminus \{\zeta\}$, $v \in M$ and let $o_K \neq w \in f(K)$. Then $p_{w,v} f * g = p_{A,v} * g \in K$, where $B = K \setminus A$, $p_{A,v}(B) = \{v\}$ and $p_{A,v}(A) = \{o_K\}$.*

Proof. We have $p_{w,v} f * g = (p_{w,v} * \mathrm{id}_L)(f * g) \in R$ and, by 5.11.3, $A = \{\, x \in K \mid p_{w,v} f(x) = o_K \,\} = \{\, x \in K \mid f(x) \not\leq w \,\}$. Thus $B = \{\, y \in K \mid f(x) \neq o_K \,\} = \{\, y \in K \mid f(y) \leq w \,\}$ and $p_{w,v} f(B) = \{v\}$. □

Proposition 5.11.5. (i) *A is a prime ideal of the semilattice K, $o_B \notin B$ and $W_A = \{\, p_{A,v} * g \mid v \in K, g \in G \,\} = R$.*

(ii) *The mapping $\nu_{A,z} : R \to M$, where $z \in L'$ and $\nu_{A,z}(p_{A,v} * g) = (v, g(z))/Q$ is an injective homomorphism of the semimodule ${}_S R$ into the semimodule ${}_S M$. This homomorphism is an isomorphism of the semimodules iff G operates transitively on L'.*

(iii) *$R(p_{A,v} * g) = \{\zeta\}$ for $v \in A$.*

(iv) *$R(p_{A,v} * g) = R$ for $v \in B$.*

(v) *The set $\{\, p_{A,v} * g \mid v \in A, g \in G \,\}$ is a bi-ideal of the semiring R and it is the greatest proper left ideal of that semiring.*

(vi) *The set $\{\, p_{A,v} * \mathrm{id}_L \mid v \in B \,\} \cup \{\zeta\}$ is just the set of multiplicative idempotents of R.*

(vii) *The semiring R is neither multiplicatively idempotent nor bi-ideal-simple.*

Proof. (i) First, take $f * g \in R \setminus \{\zeta\}$. By 5.11, we get $p_{A,v} * g \in R$ for every $v \in K$. If $v \in K'$ and $h \in G$ then $p_{A,v} * h = (p_{A,v} * hg^{-1})(p_{A,v} * g) \in R$. Thus $W_A \subseteq R$. On the other hand, W_A is a non-trivial left ideal of S and, R being minimal, we find that $R = W_A$.

(ii) We can proceed similarly as in the proof of 5.6.

The remaining assertions are easy to check. □

Theorem 5.12. (i) *The set* $W_u = \{ p_{u,v} * g \mid v \in K, g \in G \}$ *is a minimal left ideal of the semiring* S *for every* $u \in K'$.

(ii) *if* R *is a minimal left ideal of* S *such that* $R \neq W_u$ *for every* $u \in K'$ *then there is a non-principal prime ideal* A *of the semilattice* K *such that* $R = W_A = \{ p_{A<v} * g \mid v \in K, g \in G \}$, *where* $p_{A,v}(A) = \{o_K\}$ *and* $p_{A,v}(K \setminus A) = \{v\}$.

(iii) *If* A *is a non-principal prime ideal of* K *such that* $p_{A,v} * g \in S$ *for at least one* $v \in K'$ *and at least one* $g \in G$ *then* $W_A \subseteq S$ *and* W_A *is aminimal left ideal of* S *such that* $W_A \neq W_u$ *for every* $u \in K'$.

(iv) *If* P *is a minimal left ideal of* S *then* ${}_SP$ *is a faithful, simple and minimal (left* S*-) semimodule. Besides, if* $z \in L'$ *then* ${}_SP$ *is isomorphic to the subsemimodule* ${}_S(K \times G(z))/Q$ *of* ${}_SM$.

(v) *If* P_1 *and* P_2 *are minimal left ideals of* S *then the semimodules* ${}_SP_1$ *and* ${}_SP_2$ *are isomorphic.*

(vi) *If* P *is a minimal left ideal of* S *then the semiring* P *is neither multiplicatively idempotent nor bi-ideal-simple.*

Proof. Combine the foregoing results. □

Corollary 5.13. *Denote by* $\mathbf{C}$ *the set of prime ideals* A *of* K *such that* $W_A \subseteq S$. *Then:*

(i) *The set* $\{ W_A \mid A \in \mathbf{C} \}$ *is just the set of (pair-wise distinct) minimal left ideals of the semiring* S.

(ii) $A_u = \{ x \in K \mid x \nleq u \} \in \mathbf{C}$ *for every* $u \in K'$.

(iii) *The set* $\bigcup W_A$, $A \in \mathbf{C}$, *is just the smallest non-trivial ideal of the semiring* S. □

Proposition 5.14. *The following conditions are equivalent:*

(i) *There is a minimal left ideal* R *of* S *such that* $R(+)$ *is an antichain.*

(ii) *For every minimal left ideal* P *of the semiring* S, *the semklattice* $P(+)$ *is an antichain.*

(iii) *The semilattice* K *is an antichain.*

(iv) *The semilattice* M *is an antichain.*

Proof. It is easy. □

Lemma 5.15. *The following conditions are equivalent for a minimal left ideal* P *of* S:

(i) *The set* $P \setminus \{\zeta\}$ *is a subsemigroup of the multiplicative semigroup of* P.

(ii) $K' + K' = K'$ *and* $P = W_A$, *where* $A = \{o_K\}$.

Proof. It is easy. □

6. 0,1-preserving endomorphisms

Let M be a non-trivial semilattice such that $0_M, o_M \in M$. The set $\underline{E}_{0,1} = \{ f \in \underline{E} \mid f(0_M) = 0_M, f(o_M) = o_M \}$ is a unitary subsemiring of $\underline{E}$. For every $u \in N \setminus \{o_M\}$, we have $r_u \in \underline{E}_{0,1}$, where $r_u(x) = 0_M$ for $x \leq u$ and $r_u(y) = o_M$ for $y \not\leq u$. Clearly, r_{0_M} is additively absorbing.

Now, let E be a subsemiring of $\underline{E}_{0,1}$ such that $r_u \in E$ for every $u \in N$. For every pair $(u, v) \in N \times N$, $u < v$, the set $X_{(u,v)} = \{r_u, r_v\}$ is a minimal left ideal of E.

6.1 Let K be a minimal left ideal of E such that $K \neq X_{(u,v)}$ for all $u, v \in N$, $u < v$.

Lemma 6.1.1. *Let $u \in N$, $f \in K$ and $A = A_{f,u} = \{ x \in M \mid f(x) \not\leq u \}$. Then:*
(i) *A is a prime ideal of M.*
(ii) *$r_u f = r_A \in K$, where $r_A(A) = \{o_M\}$ and $r_A(B) = \{0_M\}$, $B = M \setminus A$.*
(iii) *If $u = o_B \in B$ then $r_A = r_U$.*

Proof. It is easy. □

Lemma 6.1.2. *Let A_1, A_2 be prime ideals of M such that $r_{A_1}, r_{A_2} \in K$. Then either $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$.*

Proof. Let, on the contrary, $A_1 \nsubseteq A_2 \nsubseteq A_1$. The set $A_3 = A_1 \cup A_2$ is a prime ideal and $r_{A_3} = r_{A_1} + r_{A_2} \in K$. Now, the sets $K_1 = \{r_{A_1}, r_{A_3}\}$ and $K_2 = \{r_{A_2}, r_{A_3}\}$ are two-element left ideals contained in K. Consequently, $K_1 = K_2 = K$, $r_{A_1} = r_{A_2}$ and $A_1 = A_2$, a contradiction. □

Corollary 6.1.3. *Let A_1, A_2 be two different prime ideals such that $r_{A_1}, r_{A_2} \in K$. Then $K = \{r_{A_1}, r_{A_2}\}$ and either $A_1 \subset A_2$ or $A_2 \subset A_1$.* □

Now, assume that $r_A \in K$ for exactly one prime ideal A and put $B = M \setminus A$ (see 6.1.1).

Lemma 6.1.4. *$f(A) = \{o_M\}$ for every $f \in K$.*

Proof. We have $A = \{ x \in M \mid f(x) \not\leq u \}$ for every $u \in N$ (use 6.1.1). Thus $f(A) = \{o_M\}$. □

Lemma 6.1.5. *$f(B) = \{0_M\}$ for every $f \in K$.*

Proof. We have $B = M \setminus A = \{ y \in M \mid f(y) \leq 0_M \}$. □

Corollary 6.1.6. *$K = \{r_A\}$.* □

Theorem 6.2. (i) *For all $u, v \in N$, $u < v$, the set $\{r_u, r_v\}$ is a minimal left ideal of E.*

(ii) *If K is a minimal left ideal of E then there are prime ideals A_1, A_2 of M such that $A_1 \subset A_2$ and $K = \{r_{A_1}, r_{A_2}\}$.*

Proof. See 6.1. □

Theorem 6.3. *Denote by* $\mathbf{D}$ *the set of prime ideals of* M *such that* $r_A \in E$ (see 6.1). *Then:*

(i) *Minimal left ideals of the semiring* E *are just the two-element sets* $\{r_{A_1}, r_{A_2}\}$, $A_1, A_2 \in \mathbf{D}$, $A_1 \subset A_2$.

(ii) $A_u = \{\, x \in M \mid x \not\leq u \,\} \in \mathbf{D}$ *for every* $u \in N$.

(iii) $P = M \setminus \{0_M\} \in \mathbf{D}$ *and* $r_P = r_{0_M}$.

(iv) *For every* $A \in \mathbf{D}$, $A \neq P$, *the two-element set* $\{r_A, r_{0_M}\}$ *is a minimal left ideal of* E.

(v) $\{\, r_A \mid A \in \mathbf{D} \,\}$ *is an ideal and it is the smallest right ideal of* E.

Proof. It is easy (use 6.2). □

7. Non-preserving endomorphisms

Let M be a non-trivial semilattice and let E be a subsemiring of $\underline{E}(M)$ containing all constant endomorphisms μ_u, $\mu_u(M) = \{u\}$, $u \in M$. Now, the set $Q = \{\, \mu_u \mid u \in M \,\}$ is an ideal and, in fact, it is the smallest left ideal of the semiring E. The two-element sets $\{\mu_u, \mu_v\}$, where $u < v$, are minimal right ideals and there are no more. The semimodules ${}_EM$ and ${}_EQ$ are isomorphic via $u \mapsto \mu_u$.

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