New independent paracompact spaces

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Abstract. The purpose of the present paper is to introduce a new type of paracompactness which is called ω_δ- paracompact and to obtain some results of paracompact spaces, one of which is the image of ω_δ- paracompact is paracompact under ω_δ- continuous surjection which maps ω_δ- open sets onto open set. We give an example shows that this type of paracompact is independent with standard paracompact space.

Keywords: ω_δ-open set, ω_δ-paracompact Space,ω_δ-paracompact subset.

1. Introduction

As defined by J. Dieudonne [1], a space X is said to be paracompact if each open covering has locally finite open refinement. C. H. Dowker [2] generalize this concept and introduced the class of countably paracompact spaces. A space X is said to be countably paracompact if each countable open cover of X has a locally finite open refinement. Generalizing the concept of paracompact spaces, K. Y. Al Zoubi [7] and M. K. Singal and Shashi Prabha Arya [8]. A space X is said to be S- Paracompact if each open cover has a locally finite semi open refinement and a space X is said to be R- Paracompact if each open covering of X of cardinality R has a locally finite open refinement. Since, then a lot of work has been done of S- Paracompact spaces and many interesting results have been obtained [10, 11]. This type of paracompact space is on the set that is different of open set and other types of sets like as ω_p- open sets [4]. The

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object of the present paper is to present some results of new independent type of paracompact spaces.

2. Preliminaries

Throughout this paper a space will always mean a topological space in which no separation axioms is assumed unless explicitly stated. A subset G of a space X is called δ - open [8], if for each $x \in G$, there exists an open set U containing x such that $IntClU \subseteq G$. For a subset A of a space X, the $Int_\delta(A)$, $Cl_\delta(A)$ will be denoted the δ- interior and δ- closure of A respectively. A space X is said to be locally-countable [5] if each point of X has a countable open neighborhood. Let (X, τ) be a space, a subset A of X is said to be ω_δ- open set [3] if for each $x \in U$, there exists an open set G containing x such that $G - Int_\delta U$ is countable. The complement of ω_δ- open set is called ω_δ- closed set. If A is a subset of a space X, then the ω_δ- Interior $(\omega_\delta Int(A))$ of A is a union of all ω_δ- open sets of X which contained in A and the ω_δ- Closure $(\omega_\delta Cl(A))$ of A is the intersection of all ω_δ- closed sets which containing A.

3. ω_δ-paracompact spaces

The main purpose of this section is to define ω_δ- Paracompact spaces, and obtain some characterizations, properties, and relationships.

A family $\{A_\lambda : \lambda \in \Lambda\}$ of subsets of a space (X, τ) is called ω_δ- locally finite [4], if for each $x \in X$, there exist an ω_δ- open set G containing x such that $\{\lambda \in \Lambda : G \cap A_\lambda \neq \emptyset\}$ is finite.

Definition 3.1. *A space X is called an ω_δ- Paracompact space, if each ω_δ- open covering of X has an ω_δ- locally finite ω_δ- open refinement.*

Proposition 3.2. *A topological space (X, τ) is ω_δ- Paracompact if and only if the topological space $(X, \tau_{\omega_\delta})$ is paracompact.*

From [Proposition 3.8, 3], we get the following result:

Proposition 3.3. *If a topological space (X, τ) is locally countable, then $(X, \tau_{\omega_\delta})$ is paracompact.*

Lemma 3.4. *If a covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of a space X has ω_δ- locally finite ω_δ- open refinement, then there exist an ω_δ- locally finite ω_δ- open covering $\{G_\lambda\}_{\lambda\in\Lambda}$ of X such that $G_\lambda \subseteq U_\lambda$, for all $\lambda \in \Lambda$.*

Proof. Let $\{V_\gamma\}_{\gamma\in\Gamma}$ be the ω_δ- locally finite ω_δ- open refinement of $\{U_\lambda\}_{\lambda\in\Lambda}$. Therefore, there exists a function $\beta \to \Lambda$ such that $V_\gamma \subseteq U_{\beta\gamma=\lambda}$, for each $\gamma \in \Gamma$. Let $G_\lambda = \bigcup_{\gamma\in\Gamma,\beta(\gamma)=\lambda} V_\gamma$, then the family $\{G_\lambda\}_{\lambda\in\Lambda}$ is an ω_δ- open covering of X with the property taht $G_\lambda \subseteq U_\lambda$, for each $\lambda \in \Lambda$. Also, $\{U_\lambda\}_{\lambda\in\Lambda}$ is ω_δ- locally finite. If $x \in X$, then there is an ω_δ- open set W containing x such that the set $\Gamma_0 = \{\gamma \in \Gamma : W \cap V_\lambda \neq \emptyset\}$ is finite, since $W \cap G_\lambda \neq \emptyset$ if and only if $\lambda = \beta(\gamma)$, for some $\gamma \in \Gamma_0$, so the set $\{\lambda \in \Lambda : W \cap G_\lambda \neq \emptyset\}$ is finite. Hence, the proof is complete. □

Corollary 3.5. *A space X is ω_δ- paracompact if and only if for every ω_δ- open covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of X, there exists an ω_δ- locally finite ω_δ- open covering $\{V_\lambda\}_{\lambda\in\Lambda}$ of X such that $V_\lambda \subseteq U_\lambda$, for each $\lambda \in \Lambda$.*

The ω_δ- boundary of a subset A of a space $X(\omega_\delta b(A))$ is the difference between $\omega_\delta\ Cl(A)$ and $\omega_\delta Int(A)$.

Corollary 3.6. *If $\{A_\lambda : \lambda \in \Lambda\}$ is an ω_δ- locally finite family of subsets of X, then:*

1) *$\{\omega_\delta Cl(A_\lambda) : \lambda \in \Lambda\}$ is also ω_δ- locally finite and $\omega_\delta Cl(\cup\{A_\lambda : \lambda \in \Lambda\}) = \cup\{\omega_\delta Cl(A_\lambda) : \lambda \in \Lambda\}$.*

2) *$\omega_\delta b(\bigcup_{\lambda\in\Lambda} A_\lambda) \subseteq \bigcup_{\lambda\in\Lambda} \omega_\delta b(A_\lambda)$.*

Proof. 1) Let $x \in X$. Since, $\{A_\lambda : \lambda \in \Lambda\}$ is ω_δ- locally finite, so there exists an ω_δ- open set G containing x such that the set $\{\lambda : \lambda \in \Lambda : G \cap A_\lambda \neq \emptyset\}$ is finite. Since, $G \cap A_\lambda = \emptyset$ if and only if $G \cap \omega_\delta Cl(A_\lambda) = \emptyset$, so $\{\lambda : \lambda \in \Lambda : G \cap \omega_\delta Cl(A_\lambda) \neq \emptyset\}$ is finite. Hence, $\{\omega_\delta Cl(A_\lambda) : \lambda \in \Lambda\}$ is ω_δ- locally finite. Since, $\cup\{\omega_\delta Cl(A_\lambda) : \lambda \in \Lambda\} \subseteq \omega_\delta Cl(\cup\{A_\lambda : \lambda \in \Lambda\})$. To prove $\omega_\delta Cl(\cup\{A_\lambda : \lambda \in \Lambda\}) \subseteq \cup\{\omega_\delta Cl(A_\lambda) : \lambda \in \Lambda\}$. Let $x \notin \cup\{\omega_\delta Cl(A_\lambda) : \lambda \in \Lambda\}$. Since by what we have proved above $\{\omega_\delta Cl(A_\lambda) : \lambda \in \Lambda\}$ is ω_δ- locally finite, so there exist an ω_δ- open set U containing x such that $\Lambda_0 = \{\lambda \in \Lambda : U \cup \omega_\delta Cl(A_\lambda) \neq \emptyset\}$ is finite. Set $V = U \cup (\cap\{X - \omega_\delta Cl(A_\lambda) : \lambda \in \Lambda_0\})$ is ω_δ- open subsets of X containing x such that $V \cap (\cup\{A_\lambda : \lambda \in \lambda\}) = \cup\{V \cap A_\lambda : \lambda \in \Lambda\} = \emptyset$. Thus $x \notin \omega_\delta Cl(\cup\{A_\lambda \in \Lambda\})$, hence $\omega_\delta Cl(\cup\{A_\lambda : \lambda \in \Lambda\}) \subseteq \cup\{\omega_\delta Cl(A_\lambda) : \lambda \in \Lambda\}$. Therefore, $\omega_\delta Cl(\cup\{A_\lambda : \lambda \in \Lambda\}) = \cup\{\omega_\delta Cl(A_\lambda) : \lambda \in \Lambda\}$.

2) Since $\omega_\delta b(\bigcup_{\lambda\in\Lambda} A_\lambda) = \omega_\delta Cl(\bigcup_{\lambda\in\Lambda} A_\lambda) \cap \omega_\delta Cl(X - \bigcup_{\lambda\in\Lambda} A_\lambda)$
$= \bigcup_{\lambda\in\Lambda} \omega_\delta Cl(A_\lambda) \cap \omega_\delta Cl(X - \bigcup_{\lambda\in\Lambda} A_\lambda) \subseteq \bigcup_{\lambda\in\Lambda}(\omega_\delta Cl(A_\lambda) \cap \omega_\delta Cl(X - A_\lambda)) = \bigcup_{\lambda\in\Lambda} \omega_\delta b(A_\lambda)$.

□

Corollary 3.7. *Let X be an ω_δ- paracompact space, and let H and F be two subsets in which F is an ω_δ- closed subset of X which is disjoint from H. If for every $x \in F$, there exist disjoint ω_δ- open sets U_x and V_x containing x and H, respectively. Then, there are disjoint ω_δ- open sets U and V containing F and H, respectively.*

Proof. Consider the ω_δ- open covering $\{U_x\}_{x\in F} \cup \{X - F\}$ of an ω_δ- paracompact space X. Then by Corollary 3.5, there exists an ω_δ- locally finite ω_δ- open covering $\{G_x\}_{x\in F} \cup \{G\}$ of X such that $G \subseteq X - F$ and $G_x \subseteq U_x$, for each $x \in F$. Since $U_x \cap V_x = \emptyset$, then $G_x \cap V_x = \emptyset$, so $\omega_\delta Cl(G_x) \cap V_x = \emptyset$, for each $x \in F$.

Then by Proposition 3.6, the sets $U = \bigcup_{x\in F} G_x$ and $V = X - \bigcup_{x\in F} \omega_\delta Cl(G_x)$ are the required ω_δ- open sets of X. Thus, completes the proof. □

Definition 3.8 ([4]). *A space X is said to be:*

1) *$\omega_\delta - T_2$ space, if for each distinct points x and y of X, there exists disjoint ω_δ- open sets U and V containing x and y, respectively.*
2) *$\omega_\delta - T_1$ space, if for each pair of distinct points of X, there exist ω_δ- open sets U and V such that $x \in U$, $y \notin U$ and $y \in V, y \notin V$.*
3) *ω_δ- regular space, if each ω_δ- closed subset H of X and a point x in X such that $x \notin H$, there exist disjoint ω_δ- open sets U and V containing x and H, respectively.*
4) *ω_δ- normal space, if for each pair of disjoint ω_δ- closed sets A and B in X, there exist disjoint ω_δ- open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*

Proposition 3.9. *Each ω_δ- paracompact $\omega_\delta - T_2$ (ω_δ- regular) space is an ω_δ- normal space.*

Proof. Let X be an ω_δ- paracompact $\omega_\delta - T_2$ space and let x_0 be any point in X, which is not in arbitrary ω_δ- closed subset F of X. Therefore, for each $x \in F$, there are disjoint ω_δ- open sets U_x and V_x containing x and $\{x_0\}$, so by Proposition 3.7, there exist disjoint ω_δ- open sets U and V containing F and x_0. This shows that X is ω_δ- regular. Thus, X is ω_δ- paracompact ω_δ- regular space. Let F and H be any two disjoint ω_δ- closed subsets of X. Since, H is ω_δ- closed, so by ω_δ- regularity of X, for each $x \in F$, there exist disjoint ω_δ- open sets U_x and V_x containing x and H. Therefore, by Proposition 3.7, there exist disjoint ω_δ- open sets U and V containing F and H. Thus, X is an ω_δ- normal space. □

Example 3.10. Since the usual space on $\mathbb{R}$ is an $\omega_\delta - T_2$ but not ω_δ- normal space, so by **Proposition** 3.9, it is not ω_δ- paracompact.

Proposition 3.11. *Let $\{A_\lambda : \lambda \in \Lambda\}$ be a family of subsets of a space X and $\{B_\gamma : \gamma \in \Gamma\}$ be an ω_δ- locally finite ω_δ- closed cover of X such that for each $\gamma \in \Gamma$, the set $\{\lambda \in \Lambda : B_\gamma \cap A_\lambda \neq \emptyset\}$ is finite. Then there exists an ω_δ- locally finite family $\{G_\lambda : \lambda \in \Lambda\}$ of ω_δ- open sets of X such that $A_\lambda \subseteq G_\lambda$, for each $\lambda \in \Lambda$.*

Proof. For each λ, let $G_\lambda = X - (\cup\{B_\gamma : B_\gamma \cap A_\lambda = \emptyset\})$. Then $A_\lambda \subseteq G_\lambda$, and since $\{B_\gamma : \gamma \in \Gamma\}$ is ω_δ- closed sets, so $\omega_\delta Cl(B_\gamma) = B_\gamma$, for all $\gamma \in \Gamma$. Since, $\{B_\gamma : \gamma \in \Gamma\}$ is ω_δ- locally finite, so by Proposition 3.6, $\omega_\delta Cl(\bigcup_{\gamma\in\Gamma} B_\gamma) = \bigcup_{\gamma\in\Gamma} \omega_\delta Cl(B_\gamma) = \bigcup_{\gamma\in\Gamma} B_\gamma$, this implies that G_λ is an ω_δ- open sets, for each $\lambda \in \Gamma$. Let $x \in X$. Since, $\{B_\gamma : \gamma \in \Gamma\}$ is ω_δ- locally finite, so there is an ω_δ- open set U containing x such that the set $\gamma_0 = \{\gamma \in \Gamma : U \cup B_\gamma \neq \emptyset\}$ is finite.

Thus, $U \cap B_\gamma = \emptyset$, for each $\gamma \notin \Gamma_0$. Therefore, $U \subseteq \cup\{B_\gamma : \gamma \in \Gamma_0\}$. Also, since for each $\gamma \in \Gamma_0, G_\lambda \cap B_\gamma = \emptyset$ if and only if $A_\lambda \cap B_\gamma = \emptyset$. The finiteness of $\{\lambda \in \Lambda : B_\gamma \cap A_\lambda \neq \emptyset\}$ implies the finiteness of $\{\lambda \in \Lambda : U \cap G_\lambda \neq \emptyset\}$. Thus, $\{G_\lambda : \lambda \in \Lambda\}$ is ω_δ- locally finite of ω_δ- open subsets of X. □

Theorem 3.12. *A space X is ω_δ- paracompact ω_δ- normal if and only if every ω_δ- open covering of X has an ω_δ- locally finite ω_δ- closed refinement.*

Proof. Necessity. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an ω_δ- open covering of an ω_δ- paracompact ω_δ- normal space X. So by Corollary 3.5, there exists an ω_δ- locally finite ω_δ- open covering $\{V_\lambda\}_{\lambda\in\Lambda}$ of X such that $V_\lambda \subseteq U_\lambda$, for all $\lambda \in \Lambda$. Since, X is an ω_δ- normal space, then by [Theorem 5.27, 4], there exists an ω_δ- locally finite ω_δ- closed refinement of $\{V_\lambda\}_{\lambda\in\Lambda}$ which also covers X.

Sufficiency. Let X be a space with the property that every ω_δ- open covering of X it has an ω_δ- locally finite ω_δ- closed refinement. Thus, by [Theorem 5.27, 4], X is ω_δ- normal space. It remains only to show that X is ω_δ- paracompact. Let $\{W_\lambda\}_{\lambda\in\Lambda}$ be an ω_δ- open covering of X and $\{F_\gamma\}_{\gamma\in\Gamma}$ be an ω_δ- locally finite ω_δ- closed refinement of $\{W_\lambda\}_{\lambda\in\Lambda}$. Therefore, for each $x \in X$, there exists an ω_δ- open set U_x containing x such that the set $\{\gamma \in \Gamma : U_x \cap F_\gamma \neq \emptyset\}$ is finite. Consider $\{E_\nu\}_{\nu\in\vartheta}$ is an ω_δ- locally finite ω_δ- closed refinement of the ω_δ- open covering $\{U_x\}_{x\in X}$ of X, then for each $\nu \in \vartheta$, the set $\{\gamma \in \Gamma : F_\gamma \cap E_\nu \neq \emptyset\}$ is finite. So by Proposition 3.11, there exists an ω_δ- locally finite family $\{G_\gamma\}_{\gamma\in\Gamma}$ of ω_δ- open sets of X such that $F_\gamma \subseteq G_\gamma$, for each $\gamma \in \Gamma$, which is also covers X. Since, $\{F_\gamma\}_{\gamma\in\Gamma}$ is refinement of $\{W_\lambda\}_{\lambda\in\Lambda}$, so for each $\gamma \in \Gamma$, there is $\lambda(\gamma) \in \Gamma$ such that $F_\gamma \subseteq W_{\lambda(\gamma)}$. Therefore, $\{G_\gamma \cap W_\lambda(\gamma)\}_{\gamma\in\Gamma}$ is an ω_δ- locally finite ω_δ- open refinement of $\{W_\lambda\}_{\lambda\in\Lambda}$. Hence, X is ω_δ- paracompact space. □

4. ω_δ-paracompact subset

In this section, we study some results of paracompactness on a subspace.

Proposition 4.1. *Every ω_δ- paracompact, δ- open subset of a space X is an ω_δ- paracompact subspace.*

Proof. Let A be an ω_δ- paracompact, δ- open subset of a space X and let $\{U_\lambda\}_{\lambda\in\Lambda}$ is a covering of A by ω_δ- open subsets of A. Then by [Theorem 3.4,4], $\{U_\lambda\}_{\lambda\in\Lambda}$ is a covering of A by ω_δ- open subsets of X. By hypothesis, there exists an ω_δ- locally finite ω_δ- open refinement $\{W_\gamma\}_{\gamma\in\Gamma}$ of the family $\{V_\lambda\}_{\lambda\in\Lambda}$ which covers A also. Therefore, by [Proposition 3.6, 4], $\{W_\gamma \cap A\}_{\gamma\in\Gamma}$ is an ω_δ- locally finite ω_δ- open refinement of $\{U_\lambda\}_{\lambda\in\Lambda}$ in A. Thus, A is an ω_δ- paracompact subspace of X. □

Proposition 4.2. *An ω_δ- closed subset of an ω_δ- paracompact space is an ω_δ- paracompact subset.*

Proof. Let F be an ω_δ- closed subset of an ω_δ- paracompact space X and let $\{U_\lambda\}_{\lambda\in\Lambda}$ is a covering of F by ω_δ- open sets of X. Then $\{U_\lambda\}_{\lambda\in\Lambda} \cup \{X - F\}$ is an ω_δ- open covering of X. By hypothesis and by Corollary 3.5, there exists an ω_δ- locally finite ω_δ- open covering $\{W_\lambda\}_{\lambda\in\Lambda} \cup \{W\}$ of X such that $W \subseteq X - F$ and $W_\lambda \subseteq U_\lambda$, for each $\lambda \in \Lambda$. Therefore, $\{W_\lambda\}_{\lambda\in\Lambda}$ is an ω_δ- locally finite ω_δ- open refinement of $\{U_\lambda\}_{\lambda\in\Lambda}$ which covers F. This shows that F is an ω_δ- paracompact relative to X. □

Proposition 4.3. *If a space X is $\omega_\delta - T_2$ space and has a subset F which is ω_δ- paracompact relative to X, then for each $x \in X - F$, there exist two disjoint ω_δ- open sets of X containing x and F.*

Proof. Let F be an ω_δ- paracompact subset of an $\omega_\delta - T_2$ space X and let x be any point of $X - F$. Then for each $y \in F$, there exist ω_δ- open sets U_y and V_y such that $y \in U_y$, $x \in V_y$ and $U_y \cap V_y = \emptyset$. This implies that $\omega_\delta Cl(U_y) \cap V_y = \emptyset$. Hence, $x \notin \omega_\delta Cl(U_y)$, for each $y \in F$. Now, $\{U_y\}_{y\in F}$ is cover of F by ω_δ- open subsets of X. Thus, by hypothesis and by Corollary 3.5, there exists an ω_δ- locally finite covering $\{W_y\}_{y\in F}$ of F by ω_δ- open subsets of X such that for each $y \in F$, $W_y \subseteq U_y$. Therefore, $x \notin \omega_\delta Cl(W_y)$, for each $y \in F$. Hence, by Proposition 3.6, $U = \bigcup_{y\in F} W_y$ and $V = X - \bigcup_{y\in F} \omega_\delta Cl(W_y)$, which are ω_δ- open sets which containing F and x respectively. □

From Proposition 4.3, we get the following result:

Corollary 4.4. *Every ω_δ- paracompact subset of an $\omega_\delta - T_2$ space is an ω_δ- closed.*

Corollary 4.5. *Every ω_δ- regular, $\omega_\delta - T_1$ space is an $\omega_\delta - T_2$ space.*

Proposition 4.6. *If X is an ω_δ- regular $\omega_\delta - T_1$ space and F is a subset of X which is ω_δ- paracompact relative to X, then for each ω_δ- open set U in X containing F in X, there exists an ω_δ- closed set H in X containing F and it is contained in U.*

Proof. Since X is ω_δ- regular $\omega_\delta - T_1$ space, so by Corollary 4.5, and Corollary 4.4, F is ω_δ- closed subset of X. Therefore, by [Theorem 5.4, 4], for each $x \in F$, there is ω_δ- open set U_x such that $x \in U_x \subseteq \omega_\delta Cl(U_x) \subseteq U$. Since F is ω_δ- paracompact relative to X, so there exists an ω_δ- locally finite family $\{V_\gamma\}_{\gamma\in\Gamma}$ of F by ω_δ- open sets of X which refines $\{U_x\}_{x\in F}$ and covers F. Therefore, by Proposition 3.6, $H = \bigcup_{\gamma\in\Gamma} \omega_\delta Cl(V_\gamma)$ is required ω_δ- closed set. □

Definition 4.7. *A topological space (X, τ) is called ω_δ- connected space, if X is not a union of two nonempty disjoint ω_δ- open sets, otherwise it is ω_δ- disconnected. Obviously from Definition 4.7, we get the following result:*

Theorem 4.8. *A space X is ω_δ- disconnected if and only if there exists a nonempty proper subset of X which is both ω_δ- open and ω_δ- closed in X.*

Corollary 4.9. *A space X is ω_δ- connected if and only if the only nonempty subset of X which is both ω_δ- open and ω_δ- closed is X itself.*

Theorem 4.10. *Let X be ω_δ- disconnected space, then the following statements are equivalent:*

1) *X is an ω_δ- paracompact space.*
2) *Every proper ω_δ- closed subset of X is ω_δ- paracompact relative to X.*
3) *Every proper δ- open,ω_δ- closed subset of X is ω_δ- paracompact subspace.*
4) *Every proper δ- open, ω_δ - clopen subset of X is ω_δ- paracompact subspace.*
5) *There exists a proper δ- open, ω_δ- clopen subset F of X such that both F and $X - F$ are ω_δ- paracompact subspace of X.*

Proof. $(1 \to 2)$ and $(2 \to 3)$, Follows from Proposition 4.2 and Proposition 4.1, respectively.

$(3 \to 4)$ and $(4 \to 5)$ are obvious. We prove if (5), we get (1). Let X be a space that contains a proper δ- open,ω_δ- clopen subset F in which both F and $X - F$ are ω_δ- paracompact, and let $\{G_\lambda\}_{\lambda\in\Lambda}$ be any ω_δ- open cover of X. Then $\{F \cap G_\lambda\}_{\lambda\in\Lambda}$ and $\{(X - F) \cap G_\lambda\}_{\lambda\in\Lambda}$ are covers F and $X - F$ by ω_δ- open subset of F and $X - F$, respectively. Therefore, there exist ω_δ- locally finite refinements $\{V_\gamma\}_{\gamma\in\Gamma}$ and $\{V_\nu\}_{\nu\in\vartheta}$ of $\{F \cap G_\lambda\}_{\lambda\in\Lambda}$ and $\{(X-F)\cap G_\lambda\}_{\lambda\in\Lambda}$ covering F and $X - F$, respectively such that V_γ is an ω_δ- open in F, for each $\gamma \in \Gamma$ and V_ν is an ω_δ- open in $X - F$, for each $\nu \in \vartheta$. By [Theorem 3.4, 4], both V_γ and V_ν are ω_δ- open sets in X, for each $\gamma \in \Gamma$ and $\nu \in \vartheta$. Therefore, $\{V_\mu\}_{\mu\in\gamma\cup\vartheta}$ is an ω_δ- locally finite ω_δ- open refinement of $\{G_\lambda\}_{\lambda\in\Lambda}$ which covers X. Hence, X is ω_δ- paracompact space. □

Remark 4.11. From Theorem 4.10, we notice that, if X is ω_δ- connected, then by Corollary 4.9, the only ω_δ- clopen subsets of X are empty set and X itself. So the condition that X is ω_δ- disconnected is essential.

A function $f : X \to Y$ is said to be an ω_δ- continuous [3], if the inverse image of each open subset of Y is an ω_δ- open subset in X.

Proposition 4.12. *Let $f : X \to Y$ be an ω_δ- continuous surjection which maps ω_δ- open sets onto open sets. If K is ω_δ- paracompact relative to X, then $f(K)$ is paracompact relative to Y.*

Proof. Let $\{G_\lambda\}_{\lambda\in\Lambda}$ be any covering of $f(K)$ by open sets of Y. Since, f is ω_δ- continuous surjection function, then $\{f^{-1}(G_\lambda)\}$ is a covering of $f(K)$ by ω_δ- open subsets of X. But, K is ω_δ- paracompact relative to X, thus, there exists an ω_δ- locally finite ω_δ- open family $\{V_\gamma\}_{\gamma\in\Gamma}$ of subsets of X which refines $\{f^{-1}(G_\lambda)\}_{\lambda\in\Lambda}$ and covers K, so by hypothesis, $\{f(V_\gamma)\}_{\gamma\in\Gamma}$ is a locally finite family of open subsets of Y which refines $\{G_\lambda\}_{\lambda\in\Lambda}$ and covers $f(K)$. Therefore, $f(K)$ is paracompact relative to Y. □

The ω_δ- paracompactness and paracompactness are independent, as shown in the following examples:

Example 4.13. Consider the co-countable topology $(\mathbb{R}, \tau_{COC})$, $(\tau_{COC})_{\omega_\delta} = \{\emptyset, \mathbb{R}\}$, so $\mathbb{R}$ is ω_δ- paracompact, but it is not paracompact.

Example 4.14. Consider the closed unit interval I of the usual topology $(\mathbb{R}, \mathfrak{U})$. Since, I is compact subset of $\mathbb{R}$, so $(I, \mathfrak{U}_I)$ is compact space and hence it is paracompact. Since $(I, \mathfrak{U}_I)$ is $\omega_\delta - T_2$ but not ω_δ- normal, so by Proposition 3.9, it is not ω_δ- paracompact.

Theorem 4.15. *The union of ω_δ- locally finite family of ω_δ- open ω_δ- paracompact subsets of a space X is ω_δ- paracompact subset.*

Proof. Let $\{U_\alpha : \alpha \in I\}$ be any ω_δ- locally finite family of ω_δ- open, ω_δ-paracompact sets and take $U = \cup\{U_\alpha : \alpha \in I\}$. Let $\{V_\beta : \beta \in J\}$ be any ω_δ-open covering of U, by ω_δ- open subsets of X. Then, for each α, $\{V_\beta \cap U_\alpha : \beta \in J\}$ is a covering of U_α by ω_δ- open sets. Since U_α is ω_δ- paracompact relative to X, then there exist ω_δ- locally finite family of ω_δ- open sets, $\{D_\lambda : \lambda \in K^\alpha\}$ of X which refines $\{V_\beta \cap U_\alpha : \beta \in J\}$ and covers U_α, where K is infinite cardinal. Consider the family $F = \{D_\lambda : \lambda \in K^\alpha, \alpha \in I\}$. Then F is ω_δ- locally finite ω_δ-open refinement of $\{V_\beta : \beta \in J\}$ and hence, U is ω_δ- paracompact relative to X. □

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