Solvability for continuous classical optimal control associated with triple hyperbolic boundary value problem

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Abstract. This article deals with the solvability for continuous classical optimal control associated with triple boundary value problem of linear hyperbolic type; for given continuous classical control vector the Galerkin method is used to prove the existence theorem of a unique state vector solution for the hyperbolic boundary value problem as well as for its adjoint equations. The existence theorem of a continuous classical optimal control vector dominating with the considered triple hyperbolic equations is proved. The directional derivative for the cost functional is derived. Finally the necessity theorem(conditions) for optimality of the problem is stated and proved.

Keywords: continuous classical optimal control, triple boundary value problem of linear hyperbolic type, Galerkin Method, the necessary conditions for optimality.

1. Introduction

Various applications in the real life are classified as an optimal control problems (OCPs) in a different field of sciences, for example in medicine [1] , robots [2], engineering [3], economic [4], chemistry[5], electromagnetic [6], pharmacy [7] and many others fields. Usually, in the field of applied mathematics OCPs are dominated by ordinary or partial differential equations (ODES or PDES) which are studying by many researchers, for examples and precisely [8],[9], and [10] are studied OCP dominated by PDES of elliptic, parabolic and hyperbolic type respectively, whilst [11],[12] and [13] studied OCP which are dominating by couple of PDES (CPDES) of elliptic, parabolic and hyperbolic type, whilst [14],[15]and [16] investigated boundary OCP dominated by these three types of PDES respectively, beside these studies, [17] and [18] investigated the OCP dominated by triple linear PDEs (TLPDES) of elliptic and parabolic type. All these investigations encourage us to aim at the study for OCP dominating by TLPDEs of hyperbolic type (TLHPDES). This article is concerned with the investigation for the continuous classical optimal control problem (CCOCP), it includes in the state and proof the existence theorem of a unique solution (state vector solution SVS) for the TLHPDES (as well as the solution vector of the triple adjoint equations (TAES) associated the (TLHPDES)) by employing the Galerkin method (GAM) when the continuous classical control vector (CCCV) is fixed, also the existence theorem for a continuous classical optimal control vector (CCOCV) dominating by TLHPDES is stated and proved. The derivation of the Directional derivative (DDV) for the cost functional is obtained; finally, the necessity theorem (necessity conditions) for optimality (NTHO) of this OCP is sated and proved.

2. Problem description

Let $\Omega \subset R^2, x = (x_1, x_2), Q = [0,T] \times \Omega, \tilde{I} = [0,T], \Gamma = \partial\Omega, \sum = \Gamma \times \tilde{I}$, the CCOCP include in the TSE are given by the following TLHPDES:

(1) $$\varphi_{1tt} - \Delta\varphi_1 + \varphi_1 - \varphi_2 - \varphi_3 = p_1 + \alpha_1 \text{ in } Q,$$

(2) $$\varphi_{2tt} - \Delta\varphi_2 + \varphi_2 + \varphi_3 + \varphi_1 = p_2 + \alpha_2 \text{ in } Q,$$

(3) $$\varphi_{3tt} - \Delta\varphi_3 + \varphi_3 + \varphi_1 - \varphi_2 = p_3 + \alpha_3 \text{ in } Q,$$

with the following boundary conditions (BCs) and the initial conditions (ICs)

(4) $$\varphi_1(x,t) = 0, \text{ on } \sum,$$

(5) $$\varphi_2(x,t) = 0, \text{ on } \sum,$$

(6) $$\varphi_3(x,t) = 0, \text{ on } \sum,$$

(7) $$\varphi_1(x,0) = \varphi_1^0(x), \text{ and } \varphi_{1t}(x,0) = \varphi_1^1(x), \text{ on } \Omega,$$

(8) $$\varphi_2(x,0) = \varphi_2^0(x), \text{ and } \varphi_{2t}(x,0) = \varphi_2^1(x), \text{ on } \Omega,$$

(9) $$\varphi_2(x,0) = \varphi_3^0(x), \text{ and } \varphi_{3t}(x,0) = \varphi_3^1(x) \text{ on } \Omega,$$

where $(p_1, p_2, p_3) \in L^2(Q)$ is a given vector function for each $(x_1, x_2) \in \Omega$, $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in L^2(Q)$ is a CCCV and the corresponding SVS is $\vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3) \in H^2(\Omega)$. The set of admissible CCCV is: $\vec{W}_A = \{\vec{\alpha} \in L^2(Q)\vec{\alpha} \in \vec{A} = A_1 \times A_2 \times A_3 \subset R^3 a.e. \text{ in } Q\}, \vec{A}$ is convex and compact.

The cost function for $\beta > 0$ is:

(10)
$$G_0(\vec{\alpha}) = \|\varphi_1 - \varphi_{1d}\|_Q^2 + \|\varphi_2 - \varphi_{2d}\|_Q^2 + \|\varphi_3 - \varphi_{3d}\|_Q^2 + (\beta/2)(\|\alpha_1\|_Q^2 + \|\alpha_2\|_Q^2 + \|\alpha_3\|_Q^2).$$

Let $\vec{W} = W \times W \times W; W = H^1(\Omega)$, and $\vec{W} = \{\vec{w} : \vec{w} = (w_1, w_2, w_3) \in H^1(\Omega), w_1 = w_2 = w_3 = 0 \text{ on } \partial\Omega\}$.

The weak form(wf) of (1)- (9), is given almost everywhere on I by

(11a) $$\langle \varphi_{1tt}, w_1 \rangle + (\nabla\varphi_1, \nabla w_1) + (\varphi_1, w_1) - (\varphi_2, w_1) - (\varphi_3, w_1) = (p_1 + \alpha_1, w_1), \forall w_1 \in W_1$$

(11b) $$(\varphi_1^0, w_1) = (\varphi_1(0), w_1), \text{ and } (\varphi_1^1, w_1) = (\varphi_{1t}(0), w_1), \forall w_1 \in W_1$$

$$\langle \varphi_{2tt}, w_2\rangle + (\nabla\varphi_2, \nabla w_2) + (\varphi_2, w_2) + (\varphi_3, w_2) + (\varphi_1, w_2)$$

(12a) $$= (p_2 + \alpha_2, w_2), \forall w_2 \in w_2$$

$$(\varphi_2^0, w_2) = (\varphi_2(0), w_2), \text{ and } (\varphi_2^1, w_2)$$

(12b) $$= (\varphi_{2t}(0), w_2), \forall w_2 \in W_2$$

$$\langle \varphi_{3tt}, w_3\rangle + (\nabla\varphi_3, \nabla w_3) + (\varphi_3, w_3) + (\varphi_1, w_3) - (\varphi_2, w_3)$$

(13a) $$= (p_3 + {}_3, w_3), \forall w_3 \in W_3$$

$$(\varphi_3^0, w_3) = (\varphi_3(0), w_3), \text{ and } (\varphi_3^1, w_3)$$

(13b) $$= (\varphi_{3t}(0), w_3)\forall w_3 \in W_3.$$

The following assumption is important to study the CCOCV problem (CCOCVP).

2.1 Assum. (A):

The function $p_i (\forall i = 1, 2, 3)$ is satisfied the following condition w.r.t. x and t, i.e.

$$|p_i| \le \gamma_i(x, t), \text{ where } (x, t) \in Q, \gamma_i \in L^2(Q, R).$$

3. The solution for the wf:

Theorem 3.1. *Existence of a Unique Solution for the wf: In addition to assume (A), for each given CCCV $\vec{\alpha} \in L^2(Q)$, the wf (11-13) has a unique solution $\vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$ with $\vec{\varphi} \in L^2(I, W)$ and $\vec{\varphi}_t = (\varphi_{1t}, \varphi_{2t}, \varphi_{3t}) \in L^2(I, W^*)$.*

Proof. Let $\vec{W}_n = W_n \times W_n \times W_n \subset \vec{W}$ (for each n) be the set of continuous and piecewise affine function(CPWAFF) in Ω. $\{\vec{W}_n\}_{n=1}^{\infty}$ be a sequence of subspaces of $\vec{W}$, such that $\forall \vec{w} = (w_1, w_2) \in \vec{W}$, there exists a sequence $\{\vec{w}_n\}$ with $\vec{w}_n = (w_{1n}, w_{2n}) \in \vec{W}_n, \forall n$, and $\vec{w}_n \longrightarrow \vec{w}$ strongly in $\vec{W} \Rightarrow \vec{w}_n \longrightarrow \vec{w}$ strongly in $L^2(\Omega)$. $\{\vec{w}_j = (w_{1j}, w_{2j}, w_{3j}) : j = 1, 2, \ldots, M(n)\}$ be a finite basis of $\vec{W}_n$ (where $\vec{w}_j$ is CPWAFF in Ω, with $\vec{v}_j(x) = 0$ on the boundary Γ) and let $\vec{\varphi}_n = (\varphi_{1n}, \varphi_{2n}, \varphi_{3n})$ be the Galerkin approximate solution (method (GAM)) to the exact solution $\vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$ such that

(14) $$\varphi_{in} = \sum_{j=1}^{n} c_{ij}(t) v_{ij}(x),$$

where $c_{ij}(t)$ is unknown function of t, $\forall i = 1, 2, 3, j = 1, 2, \ldots, n$. Now, the GAM is used to approximate the wf (10)-(13) w.r.t. x, substituting $\vec{\varphi}_{int} = z_{in}$, (for $i = 1, 2, 3$) in the obtained approximation wf, yield to

$$\langle \varphi_{1nt}, w_1\rangle + (\nabla\varphi_{1n}, \nabla w_1) + (\varphi_{1n}, w_1) - (\varphi_{2n}, w_1) - (\varphi_{3n}, w_1)$$

(15a) $$= (p_1 + \alpha_1, w_1), \forall w_1 \in W_n,$$

$$(\varphi_{1n}^0, w_1) = (\varphi_1^0, w_1) \text{ and } (\varphi_{1n}^1, w_1)$$

(15b) $$= (\varphi_1^1, w_1), \forall w_1 \in V_n,$$

$$\begin{aligned} &\langle \varphi_{2nt}, w_2\rangle + (\nabla\varphi_{2n}, \nabla w_2) + (\varphi_{2n}, w_2) + (\varphi_{3n}, w_2) + (\varphi_{1n}, w_2) \\ &= (p_2 + \alpha_2, w_2), w_2 \in W_n, \end{aligned} \tag{16a}$$

$$\begin{aligned} &(\varphi_{2n}^0, w_2) = (\varphi_2^0, w_2),\ \text{and}\ (\varphi_{2n}^1, w_2) \\ &= (\varphi_2^1, w_2), \forall w_2 \in W_n, \end{aligned} \tag{16b}$$

$$\begin{aligned} &\langle y_3nt, w_3\rangle + (\nabla\varphi_{3n}, \nabla w_3) + (\varphi_{3n}, w_3) + (\varphi_{1n}, w_3) - (\varphi_{2n}, w_3) \\ &= (p_3 + \alpha_3, w_3), \forall w_3 \in W_n, \end{aligned} \tag{17a}$$

$$\begin{aligned} &(\varphi_{3n}^0, w_3) = (\varphi_3^0, w_3),\ \text{and}\ (\varphi_3^1, w_3) \\ &= (\varphi_3^1, w_3)\forall w_3 \in W_n, \end{aligned} \tag{17b}$$

where $\varphi_{in}^0 = \varphi_{in}^0(x) = \varphi_{in}(x,0) \in W_n$ (respectively $z_{in}^0 = \varphi_{in}^1 = \varphi_{in}^1(x) = \varphi_{int}(x,0) \in L^2(\Omega)$ be the projection of φ_i^0 onto W (be the projection of $\varphi_i^1 = \varphi_{it}$ onto $L^2(\Omega)$, $\forall i = 1,2,3$, i.e.

$$\varphi_{in}^0 \longrightarrow \varphi_i^0 \text{ strongly in } W, \text{ with } \|\vec{\varphi}_n^0\|_1 \le b_0 \text{ and } \quad \|\varphi_n^0\|_0 \le b_0 \tag{18}$$

$$\varphi_{in}^1 \longrightarrow \varphi_i^1 \text{ strongly in } L^2(\Omega) \text{ and } \quad \|\vec{\varphi}_n^1\|_0 \le b_1. \tag{19}$$

Substituting (14) for $i = 1,2,3$ in (17)-(19) respectively and then setting $w_i = w_{il}, \forall l = 1,2,\ldots,n$, the obtained equations are equivalent to the following linear system (LS) of 1^{st} order ODEs with ICs (which has a unique solution), i.e.

$$A_1\acute{C}_1(t) + B_1C_1(t) - EC_2(t) - FC_3(t) = b_1, \tag{20a}$$

$$A_1C_1(0) = b_1^0,\ \text{and}\ A_1\bar{C}_1(0) = b_1^1, \tag{20b}$$

$$A_2\acute{C}_2(t) + B_2C_2(t) + GC_3(t) + HC_1(t) = b_2, \tag{21a}$$

$$A_2C_2(0) = b_2^0,\ \text{and}\ A_2\bar{C}_2(0) = b_2^1, \tag{21b}$$

$$A_3\acute{C}_3(t) + B_3C_3(t) + RC_1(t) - WC_2(t) = b_3, \tag{22a}$$

$$A_3C_3(0) = b_3^0,\ \text{and}\ A_3\bar{C}_3(0) = b_3^1. \tag{22b}$$

Where $A_i = (a_{ilj})_{(n\times n)}, a_{ilj} = (w_{ij}, w_{il}), B_i = (b_{ilj})_{(n\times n)}, b_{ilj} = (\nabla w_{ij}, \nabla w_{il}) + (w_{ij}, w_{il}), E = (e_{lj})_{(n\times n)}, e_{lj} = (w_{2j}, w_{1l}), F = (f_{lj})_{(n\times n)}, f_{lj} = (w_{3j}, w_{1l}), G = (g_{lj})_{(n\times n)}, g_{lj} = (w_{3j}, w_{2l}), H = (h_{lj})_{(n\times n)}, h_{lj} = (w_{1j}, w_{2l}), R = (r_{lj})_{(n\times n)}, r_{lj} = (w_{1j}, w_{3l}), W = (w_{lj})_{(n\times n)}, w_{lj} = (w_{2j}, w_{3l}), b_{il}^0 = (y_i^0, w_{il}), b_i^0 = (b_{il}^0), b_i = (b_{il})_{(n\times 1)}, b_{il} = (p_i + \alpha_i, v_{il}), \acute{C}_i(t) = (\acute{c_{ij}}(t))_{(n\times 1)}, C_i(t) = (c_{ij}(t))_{(n\times 1)}, \bar{C}_i(0) = (\bar{c}_{ij}(0))_{(n\times 1)}C_i(0) = (c_{ij}(0)_{(n\times 1)}, \forall l = 1,2,3,\ldots,n, i = 1,2,3$

Then corresponding to the sequence $\{\vec{W}_n\}$, the following approximation problems are hold, i.e. for each $\vec{w}_n = (w_{1n}, w_{2n}, w_{3n}) \subset W_n,$ and $n = 1,2,\ldots$

$$\begin{aligned} &\langle \varphi_{1ntt}, w_{1n}\rangle + (\nabla\varphi_{1n}, \nabla w_{1n}) + (\varphi_{1n}, w_{1n}) - (\varphi_{2n}, w_{1n}) \\ &- (\varphi_{3n}, w_{1n}) = (p_1 + \alpha_1, w_{1n}), \end{aligned} \tag{23a}$$

$$(\varphi_{1n}^0, w_1n) = (\varphi_1^0, w_{1n}), (\varphi_{1n}^1, w_1) = (\varphi_1^1, w_{1n}) \tag{23b}$$

$$\langle\varphi_{2ntt}, w_{2n}\rangle + (\nabla\varphi_{2n}, \nabla w_{2n}) + (\varphi_{2n}, w_{2n}) + (\varphi_{1n}, w_{2n})$$

(24a) $$+ (\varphi_{3n}, w_{2n}) = (p_2 + \alpha_2, w_{2n}),$$

(24b) $$(\varphi^0_{2n}, w_{2n}) = (\varphi^0_2, w_{2n}), (\varphi^1_{2n}, w_2) = (\varphi^1_2, w_{2n})$$

$$\langle\varphi_{3ntt}, w_3\rangle + (\nabla\varphi_{3n}, \nabla w_{3n}) + (\varphi_{3n}, w_{3n}) + (\varphi_{1n}, w_{3n}) - (\varphi_{2n}, w_{3n})$$

(25a) $$= (p_3 + \alpha_3, w_{3n}),$$

(25b) $$(\varphi^0_{3n}, w_{3n}) = (\varphi^0_3, w_{3n}), (\varphi^1_3, w_{3n}) = (\varphi^1_3, w_{3n}),$$

which has a sequence of unique solution $\{\vec{\varphi}_n\}$.

Substituting $w_{in} = \varphi_{int}$, for $i = 1, 2, 3$ in ((23a),(24a),(25a)) respectively, then adding the three obtained equations, using Lemma 1.2 in [20] for the 1st term of the L.H.S., once get

$$28(d/dt)[\|\vec{\varphi}_{nt}\|^2_0 + \|\varphi_n\|^2_1] = 2((\varphi_{2n}, \varphi_{1nt}) + (\varphi_{3n}, \varphi_{1nt}) - (\varphi_{1n}, \varphi_{2nt})$$
$$- (\varphi_{3n}, \varphi_{2nt}) - (\varphi_{1n}, \varphi_{3nt}) + (\varphi_{2n}, \varphi_{3nt}) + (p_1 + \alpha_1, \varphi_{1nt})$$

(26) $$+ (p_2 + \alpha_2, \varphi_{2nt}) + (p_3 + \alpha_3, \varphi_{3nt})),$$

$$(d/dt)[\|\vec{\varphi}_{nt}\|^2_0 + \|\vec{\varphi}_n\|^2_1] \leq 2(|(\varphi_2 n, \varphi_{1nt})|$$
$$+ |(\varphi_{3n}, \varphi_{1nt})| + |(\varphi_{1n}, \varphi_{2nt})| + |(\varphi_{3n}, \varphi_{2nt})|$$
$$+ |(\varphi_{1n}, \varphi_{3nt})| + |(\varphi_{1n}, \varphi_{3nt})| + |(\varphi_{2n}, \varphi_{3nt})| + |(p_1 + \alpha_1, \varphi_{1nt})|$$

(27) $$+ |(p_2 + \alpha_2, \varphi_{2nt})| + |(p_3 + \alpha_2, \varphi_{3nt})|).$$

Using the C-S inequality for the R.H.S. of (27), integrating both sides on $[0, t]$, using

$$\|\varphi_{in}\|_0 \leq \|\varphi_{in}\|_1 \leq \|\vec{\varphi}_n\|_1, \|\varphi_{int}\|_0 \leq \|\vec{\varphi}_{nt}\|_0,$$

and assum (A), to get

$$\int_0^t (d/dt)[\|\vec{\varphi}_{nt}(t)\|^2_0 + \|\vec{\varphi}_n\|^2_1]dt \leq$$

(28) $$2\int_0^t (\|\vec{\varphi}_{nt}\|^2_0 + \|\vec{\varphi}_n\|^2_1)dt + \int_0^t \sum_{i=1}^3 (\|\gamma_i\|^2_0 + \|u_{ii}\|^2_0)dt + \int_0^t \|\vec{\varphi}_{nt}\|^2_0 dt +$$

$$\sum_{i=1}^3 (\|\gamma_i\|^2_Q + \|\check{b}_i\|^2_Q) + \lambda_1 \int_0^t (\|\vec{\varphi}_{nt}\|^2_0 + \|\vec{\varphi}_(n)\|^2_1)dt \leq$$

$$\lambda_2 + \lambda_1 \int_0^t (\|\vec{\varphi}_{nt}\|^2_0 + \|\vec{\varphi}_n\|^2_1)dt,$$

where, $\lambda_2 = \sum_{i=1}^3 (\bar{b}_1 + \check{b}_i), \lambda_1 \geq 3$, with $\|\gamma_i\|^2_Q \leq \bar{b}_i, \|\alpha_i\|_Q \leq \check{b}_i$, for each $i = 1, 2, 3$. Since $\|\vec{\varphi}^0_n\|_1 \leq b_1$, and $\|\vec{\varphi}^1_n\|_0 \leq b_0$, with $\lambda_3 = b_0 + b_1 + \lambda_2$, inequality (28) becomes $\|\vec{\varphi}_{nt}(t)\|^2_0 + \|\vec{\varphi}_n(t)\|^2_1 \leq \lambda_3 + \lambda_1 \int_0^t (\|\vec{\varphi}_n\|^2_0 + \|\vec{\varphi}_{nt}\|^2_1)dt$.

Using the Belman-Gronwall (B-G) inequality, to get $\forall t \in [0, T]$, $\|\vec{\varphi}_{nt}(t)\|^2_0 + \|\vec{\varphi}_n(t)\|^2_1 \leq \lambda_3 e^{\lambda_1} = b^2(c)\|\vec{\varphi}_{nt}(t)\|^2_0 \leq b^2(c)$ and $\|\vec{\varphi}_n(t)\|^2_1 \leq b^2(c), \forall t \in [0, T]$ Easily once can obtained that $\|\vec{\varphi}_{nt}(t)\|_Q \leq b_1(c)$ and $\|\vec{\varphi}_n(t)\|_{L^2(I,V)} \leq b(c)$.

Then the Alaoglu's theorem (ATh) can be applied here, thus there exists a subsequence of $\{\vec{\varphi}_n\}_{n\in N}$, for simplicity let be $\{\vec{\varphi}_n\}_{n\in N}$, s.t. $\vec{\varphi}_{nt} \longrightarrow \vec{\varphi}$ weakly in $L^2(Q)$ and $\vec{\varphi}_n \longrightarrow \vec{\varphi}$ weakly in $L^2(I,V)$, and since Now, multiplying both sides of (23a),(24a),(25a) by

$$y_i(t) \in C^2[0,T], \text{ s.t. } y_i(T) = y_i(T) = 0, y_i(0) \neq 0, y_i'(0) \neq 0, \forall i = 1,2,3 \tag{29}$$

integrating on $[0,T]$, finally integrating by parts twice the 1st term of each one of the obtained three equations, yield to

$$\begin{aligned}
&-\int_0^T d/dt(\varphi_{1n}, w_{1n})(w_1')(t)dt + \int_0^T [(\nabla\varphi_{1n}, \nabla w_{1n}) + (\varphi_{1n}, w_{1n})\\
&-(\varphi_{2n}, w_{1n}) - (\varphi_{3n}, w_{1n})]y_1(t)dt\\
&= \int_0^T (p_1 + \alpha_1, w_{1n})y_1(t)dt + (\varphi_{1n}^1, w_{1n})y_1(0),
\end{aligned} \tag{30}$$

$$\begin{aligned}
&\int_0^T (\varphi_{1n}, w_{1n})(y_1'')(t)dt + \int_0^T [(\nabla\varphi_{1n}, \nabla w_{1n}) + (\varphi_{1n}, w_{1n})\\
&-(\varphi_{2n}, w_{1n}) - (\varphi_{3n}, w_{1n})]y_1(t)dt\\
&= \int_0^T (p_1 + \alpha_1, w_{1n})y_1(t)dt + (\varphi_{1n}^1, w_{1n})y_1(0) + (\varphi_{1n}^0, w_{1n})(y_1')(0),
\end{aligned} \tag{31}$$

$$\begin{aligned}
&-\int_0^T d/dt(\varphi_{2n}, w_{2n})(y_2')(t)dt + \int_0^T [(\nabla\varphi_{2n}, \nabla w_{2n})\\
&+(\varphi_{2n}, w_{2n}) + (\varphi_{1n}, w_{2n}) + (\varphi_{3n}, w_{2n})]y_2(t)dt\\
&= \int_0^T (p_2 + \alpha_2, w_{2n})y_2(t)dt + (\varphi_{2n}^0, w_{2n})y_2(0),
\end{aligned} \tag{32}$$

$$\begin{aligned}
&\int_0^T (\varphi_{2n}, w_{2n})(y_2'')(t)dt + \int_0^T [(\nabla\varphi_2 n, \nabla w_2 n) + (\varphi_2 n, w_2 n)\\
&+(\varphi_1 n, w_{2n}) + (\varphi_{3n}, w_{2n})]y_2(t)dt\\
&= \int_0^T (p_2 + \alpha_2, w_{2n})\varphi_2(t)dt + (\varphi_{2n}^0, w_2 n)\varphi_2(0) + (\varphi_{2n}^0, w_2 n)(\varphi_2')(0),
\end{aligned} \tag{33}$$

$$\begin{aligned}
&-\int_0^T d/dt(\varphi_{3n}, w_{3n})(y_3')(t)dt + \int_0^T [(\nabla\varphi_{3n}, \nabla w_{3n}) + (\varphi_{3n}, w_{3n})\\
&+(\varphi_{1n}, w_{3n}) - (\varphi_{2n}, w_{3n})]y_3(t)dt\\
&= \int_0^T (p_3 + \alpha_3, w_{3n})y_3(t)dt + (\varphi_{3n}^1, w_{3n})y_3(0),
\end{aligned} \tag{34}$$

$$\begin{aligned}
&\int_0^T (\varphi_{3n}, w_{3n})(y_3'')(t)dt + \int_0^T [(\nabla\varphi_{3n}, \nabla w_{3n}) + (\varphi_{3n}, w_{3n})\\
&+(\varphi_{1n}, w_{3n}) - (\varphi_{2n}, w_{3n})]y_3(t)dt\\
&= \int_0^T (p_3 + \alpha_3, w_{3n})\varphi_1(t)dt + (\varphi_{3n}^1, w_{3n})y_3(0) + (\varphi_{3n}^0, w_{3n})(y_3')(0).
\end{aligned} \tag{35}$$

First, since

$$
\begin{array}{lll}
w_{in} \longrightarrow w_i \quad \text{strongly in } W & \Longrightarrow & \left\{ \begin{array}{ll} \left.\begin{array}{l} w_{in}y_i(t) \longrightarrow w_iy_i(t) \\ w_{in}y'_i(t) \longrightarrow w_iy'_i(t) \end{array}\right\} & \text{strongly in } L^2(I,W) \\ w_{in}y_i(t) \longrightarrow w_iy_i(0) & \text{strongly in } L^2(\Omega) \\ \left.\begin{array}{l} w_{in}y'_i(t) \longrightarrow y'_i(t) \\ w_{in}y''_i(t) \longrightarrow w_iy''_i(t) \end{array}\right\} & \text{strongly in } L^2(I,Q) \\ w_{in}y'_i(t) \longrightarrow w'_iy_i(0) & \text{strongly in } L^2(\Omega) \end{array}\right. \\
w_{in} \longrightarrow w_i \quad \text{strongly in } L^2(\Omega) & \Longrightarrow &
\end{array}
$$

Second, $\varphi_{int} \longrightarrow \varphi_{it}$ weakly in $L^2(Q)$ and $\varphi_{in} \longrightarrow \varphi_i$ weakly in $L^2(I,W)$ and strongly in $L^2(Q)$.

Third, since $w_{in}y_i \longrightarrow w_iy_i$ weakly in $L^2(I,W)$, then

$$\int_0^T (p_i + \alpha_i, w_{in})y_i(t)d \to \int_0^T (p_i + \alpha_i, w_i)y_i(t)dt, \forall i = 1, 2, 3.$$

From these convergences, (18) and (19), we can passaged the limits in (30)-(34), to get

$$
\begin{aligned}
&-\int_0^T (\varphi_{1t}, w_1)(y_1)'(t)dt + \int_0^T [(\nabla\varphi_1, \nabla w_1) + (\varphi_1, w_1) \\
&- (\varphi_2, w_1) - (\varphi_3, w_1)]y_1(t)dt \\
&= \int_0^T (p_1 + \alpha_1, w_1)y_1(t)dt + (\varphi_1^1, w_1)y_1(0)
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
&\int_0^T (\varphi_1, w_1)(y_1)''(t)dt + \int_0^T [(\nabla\varphi_1, \nabla w_1) + (\varphi_1, w_1) \\
&- (\varphi_2, w_1) - (\varphi_3, w_1)]y_1(t)dt \\
&= \int_0^T (p_1 + \alpha_1, w_1)y_1(t)dt + (\varphi_1^1, w_1)y_1(0) + (\varphi_1^0, w_1)y'_1(0)
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
&-\int_0^T (\varphi_{2t}, w_2)(\varphi_2)'(t)dt + \int_0^T [(\nabla\varphi_2, \nabla w_2) + (\varphi_2, w_2) \\
&+ (\varphi_1, w_2) + (\varphi_3, w_2)]y_2(t)dt \\
&= \int_0^T (p_2 + \alpha_2, w_2)y_2(t)dt + (\varphi_2^0, w_2)y_2(0)
\end{aligned} \tag{38}
$$

$$
\begin{aligned}
&\int_0^T (y_2, w_2)(y_2)''(t)dt + \int_0^T [(\nabla y_2, \nabla w_2) + (y_2, w_2) \\
&+ (y_1, w_2) + (y_3, w_2)]y_2(t)dt \\
&= \int_0^T (p_2 + \alpha_2, w_2)y_2(t)dt + (y_2^0, w_2)y_2(0) + (y_2^0, w_2)y'_2(0)
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
&-\int_0^T (\varphi_{3t}, w_3)(y_3)'(t)dt + \int_0^T [(\nabla\varphi_3, \nabla w_3) + (\varphi_3, w_3) \\
&+ (\varphi_1, w_3) - (\varphi_2, w_3)]y_3(t)dt
\end{aligned}
$$

$$= \int_0^T (p_3 + \alpha_3, w_3) y_3(t) dt + (\varphi_3^1, w_3) y_3(0) \tag{40}$$

$$\int_0^T (\varphi_3, w_3)(y_3)''(t) dt + \int_0^T [(\nabla\varphi_3, \nabla w_3) + (\varphi_3, w_3)$$
$$+ (\varphi_1, w_3) - (\varphi_2, w_3)] y_3(t) dt$$

$$= \int_0^T (p_3 + \alpha_3, w_3) y_3(t) dt + (\varphi_3^1, w_3) y_3(0) + (\varphi_3^0, w_3) y_3'(0) \tag{41}$$

Case 1. Choose $y_i \in C^2[0,T]$, s.t. $y_i(0) = (y_i)'(0) = y_i(T) = (y_i)'(T) = 0, \forall i = 1,2,3.$ in (35), (37), (39), integration by parts twice the 1^{st} terms in the L.H.S. of each one of the obtained equation, yield to

$$\int_0^T \langle \varphi_{1tt}, w_1 \rangle y_1(t) dt + \int_0^T [(\nabla\varphi_1, \nabla w_1)$$
$$+ (\varphi_1, w_1) - (\varphi_2, w_1) - (\varphi_3, w_1)] y_1(t) dt = \int_0^T (p_1 + \alpha_1, w_1) y_1(t) dt \tag{42}$$

$$\int_0^T \langle \varphi_{2tt}, w_2 \rangle y_2(t) dt + \int_0^T [(\nabla\varphi_2, \nabla w_2)$$
$$+ (\varphi_2, w_2) + (\varphi_1, w_2) + (\varphi_3, w_2)] y_2(t) dt = \int_0^T (p_2 + \alpha_2, w_2) y_2(t) dt \tag{43}$$

$$\int_0^T \langle \varphi_{3tt}, w_3 \rangle y_3(t) dt + \int_0^T [(\nabla\varphi_3, \nabla w_3) + (\varphi_3, w_3)$$
$$+ (\varphi_1, w_3) - (\varphi_2, w_3)] y_3(t) dt = \int_0^T (p_3 + \alpha_3, w_3) y_3(t) dt \tag{44}$$

Which give that $(\varphi_1, \varphi_2, \varphi_3)$ is a solution of ((11a) , (12a) , (13a)) a.e. on I.

Case 2. Choose $y_i \in C^2[0,T]$, s.t. $y_i(T) \neq 0 \& y_i(0) \neq 0, \forall i = 1,2,3.$ Multiplying both sides of (11a), (12a), and (13a) by $y_1(t), y_2(t)$ and $y_3(t)$ respectively, integrating for $0 \leq t \leq T$, integrating by parts the 1^{st} term in the L.H.S. of each one of the three equations, then subtracting each one of these obtained equations from those in (34) , (36) & (38) respectively, one has $(y_i^1, w_i) y_i(0) = (y_i t(0), w_i) y_i(0)$.

From the last two cases easily once get the initial conditions (11b), (12b) and (13b). To prove that $\vec{\varphi}_n \longrightarrow \vec{\varphi}$ strongly in $L^2(I, W)$, we start by integrating (26) for $0 \leq t \leq T$, to get

$$\|\vec{\varphi}_{nt}(T)\|_Q^2 - \|\varphi_{nt}(0)\|_Q^2 + 2\int_0^T \|\vec{\varphi}_n(t)\|_1^2 dt = (43a) + (34b) \tag{45}$$

$$(43a) = 2((\varphi_{2n}, \varphi_{1nt}) + (\varphi_{3n}, \varphi_{1nt}) - (\varphi_{1n}, \varphi_{2nt})$$
$$- (\varphi_{3n,}\varphi_{2nt}) - (\varphi_{1n}, \varphi_{3nt}) + (\varphi_{2n}, \varphi_{3nt}))$$
$$(43b) = 2((p_1 + \alpha_1, \varphi_{1nt}) + (p_2 + \alpha_2, \varphi_{2nt}) + (p_3 + \alpha_3, \varphi_{3nt})).$$

Same steps which is applied to get (26)&(45), can be also applied here when we have $\vec{\varphi}$ and $\vec{\varphi}_t$, i.e.

$$\|\vec{\varphi}_t(T)\|_Q^2 - \|\vec{\varphi}_t(0)\|_Q^2 + 2\int_0^T \|\vec{\varphi}(t)\|_1^2 dt = (44a) + (44b) \tag{46}$$

$$(44a) = 2((\varphi_2, \varphi_{1t}) + (\varphi_3, \varphi_{1t}) - (\varphi_1, \varphi_{2t}) - (\varphi_3, \varphi_{2t}) - (\varphi_1, \varphi_{3t}) + (\varphi_2, \varphi_{3t}))$$
$$(44b) = 2((p_1 + \alpha_1, \varphi_{1t}) + (p_2 + \alpha_2, \varphi_{2t}) + (p_3 + \alpha_3, \varphi_{3t})).$$

Since

$$\|\vec{\varphi}_{nt}(T) - \vec{\varphi}_t(T)\|_Q^2 - \|\vec{\varphi}_{nt}(0) - \vec{\varphi}\|_t(0)\|_Q^2 + 2\int_0^T \|\vec{\varphi}_n(t) - \vec{\varphi}(t)\|_1^2 dt$$
$$= (45a) - (45b) - (45c) \tag{47}$$

$$(45a) = \|\vec{\varphi}_{nt}(T)\|_Q^2 - \|\vec{\varphi}_{nt}(0)\|_Q^2 + 2\int_0^T \|\vec{\varphi}_n(t)\|_1^2 dt$$
$$(45b) = (\vec{\varphi}_{nt}(T), \vec{\varphi}_t(T)) - (\vec{\varphi}_{nt}(0), \vec{\varphi}_t(0)) + 2\int_0^T (\vec{\varphi}_n(t), \vec{\varphi}(t))_1 dt$$
$$(45c) = (\vec{\varphi}_t(T), \vec{\varphi}_{nt}(T) - \vec{\varphi}_t(T)) - (\vec{\varphi}_t(0), \vec{\varphi}_{nt}(0) - \vec{\varphi}_t(0))$$
$$+ 2\int_0^T (\vec{\varphi}(t), \vec{\varphi}_n(t) - \vec{\varphi}(t))_1 dt$$

Since $\vec{\varphi}_n \longrightarrow \vec{\varphi}$ strongly in $L^2(Q)$, and $\vec{\varphi}_{nt} \longrightarrow \vec{\varphi}$ weakly in $L^2(Q)$, then from (45) and the assum.(A), we obtain

$$(45a) = (43a) + (43b)(44a) + (44b)$$

by the same way that it used to get (19), we can get also that

$$\vec{\varphi}_{nt}(T) \longrightarrow \vec{\varphi}_t(T) \text{ strongly in } L(\Omega)^2. \tag{48}$$

On the other hand, since $\vec{\varphi}_n \longrightarrow \vec{\varphi}$ weakly in $L^2(I, W)$, then using (19 & 46), we get

$$(45b) \longrightarrow (44a) + (44b).$$

All the terms in $(45c)$ imply to zero, so as the 1st two terms in the L.H.S. of (45), hence (45) gives $\int_0^T \|\vec{\varphi}_n(t) - \vec{\varphi}(t)\|_1^2 dt \longrightarrow 0$ as $n \longrightarrow \infty$, so we get that $\vec{\varphi}_n \longrightarrow \vec{\varphi}$ strongly in $L^2(I, W)$. Uniqueness of the solution: Let $\vec{\varphi} = (\vec{\varphi}_1, \vec{\varphi}_2, \vec{\varphi}_3)$ and $\vec{\bar{\varphi}} = (\bar{\varphi}_1, \bar{\varphi}_2, \bar{\varphi}_3)$ be two solutions of the wf (11-13), i.e. $\vec{\varphi}$ and $\vec{\bar{\varphi}}$ are satisfied the wf (11-13), subtracting each equation from the other and then setting $w_i = \varphi_i - \vec{\bar{\varphi}}_i$, for each $i = 1, 2, 3$, yields to $\langle(\varphi_i - \bar{\varphi}_i)_{tt}, \varphi_i - \bar{\varphi}_i\rangle + \|\varphi_i - \bar{\varphi}_i\|_1^2 = 0$, $((\varphi_i - \bar{\varphi}_i)(0), (\varphi_i - \bar{\varphi}_i)(0)) = 0$ for $w_i = (\varphi_i - \bar{\varphi}_{1i})_t$ we have $((\varphi_i - \bar{\varphi}_i)_t(0), (\varphi_i - \bar{\varphi}_i)_t(0)) = 0$.

Adding the above equalities (for $i = 1, 2, 3$), using Lemma 1.2 in ref. [20] for the 1^{st} in L.H.S. of the obtained equation which will be positive, integrating both sides from 0 to t, using the initial conditions, and finally applying the B

-G inequality, to get $\int_0^t \left[(d/dt)\|(\vec{\varphi}-\vec{\bar{\varphi}})_t(t)\|_Q^2 + 2\|(\vec{\varphi}-\vec{\bar{\varphi}})\|_1^2\right] dt \le 2L\int_0^t \|(\vec{\varphi}-\vec{\bar{\varphi}})\|_1^2 dt \Rrightarrow \|(\vec{\varphi}-\vec{\bar{\varphi}})(t)\|_1^2 = 0, \forall t \in I. \Rrightarrow \|(\vec{\varphi}-\vec{\bar{\varphi}})(t)\|_{L^2(I,W)} = 0 \Rrightarrow$ the solution is unique. □

4. Existence of a CCOC:

Lemma 4.1. *In addition to assum. (A), assume that $\vec{\varphi}$ and $\vec{\varphi}+\delta\vec{\varphi}$ are the SVS corresponding to the CVS $\vec{\alpha}$ and $\vec{\alpha}+\delta\vec{\alpha}$ respectively with $\vec{\alpha}$ and $\delta\vec{\alpha}$ are bounded in $L^2(Q)$ then $\|\delta\vec{\varphi}_\varepsilon\|_{L^\infty(I,L^2(\Omega))} \le a\|\delta\vec{\alpha}\|_Q, \|\delta\vec{\varphi}_\varepsilon\|_{L^2(I,V)} \le a\|\delta\vec{\alpha}\|_Q$ and $\|\delta\vec{\varphi}_\varepsilon\|_Q \le a\|(\delta\vec{\alpha})\|_Q$, with $a \in \mathbb{R}^+$.*

Proof. Let $\overrightarrow{\delta\alpha} = \vec{\bar{\alpha}} - \vec{\alpha}$, where $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3), \vec{\bar{\alpha}} = (\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3) \in L^2(Q)$, then $\vec{\alpha}_\varepsilon = \vec{\alpha} + \varepsilon(\overrightarrow{\delta\alpha}) \in L^2(Q)$ for $\varepsilon > 0$, and then by Theorem 3.1, $\vec{\varphi} = \vec{\varphi}_{\vec{\alpha}} = (\varphi_1, \varphi_2, \varphi_3)$ and $\vec{\varphi}_\varepsilon = \vec{\varphi}_{\vec{\alpha}_\varepsilon} = (\varphi_{1\varepsilon}, \varphi_{2\varepsilon}, \varphi_{3\varepsilon})$ are their corresponding states solutions which are satisfied the wf (11-13), by setting $(\overrightarrow{\delta\varphi})_\varepsilon = (\delta\varphi_{1\varepsilon}, \delta\varphi_{2\varepsilon}, \delta\varphi_{3\varepsilon}) = \vec{\varphi}_\varepsilon - \vec{\varphi}$, once get

$$\langle \delta\varphi_{1\varepsilon tt}, w_1\rangle + (\nabla\delta\varphi_{1\varepsilon}, \nabla w_1) + (\delta\varphi_{1\varepsilon}, w_1) - (\delta\varphi_{2\varepsilon}, w_1) - (\delta\varphi_{3\varepsilon}, w_1)$$
$$= (\varepsilon\delta\alpha_1, w_1) \tag{47a}$$
$$\delta\varphi_{1\varepsilon}(x,0) = 0 \text{ and } \delta\varphi_{1\varepsilon t}(x,0) = 0 \tag{47b}$$
$$\langle \delta\varphi_{2\varepsilon tt}, w_2\rangle + (\nabla\delta\varphi_{2\varepsilon}, \nabla w_2) + (\delta\varphi_{2\varepsilon}, w_2) + (\delta\varphi_{1\varepsilon}, w_2) + (\delta\varphi_{3\varepsilon}, w_2)$$
$$= (\varepsilon\delta\alpha_2, w_2) \tag{48a}$$
$$\delta\varphi_{2\varepsilon}(x,0) = 0 \text{ and } \delta\varphi_{2\varepsilon t}(x,0) = 0 \tag{48b}$$
$$\langle \delta\varphi_{3\varepsilon tt}, w_3\rangle + \delta\varphi_{3\varepsilon}, \nabla w_3) + (\delta\varphi_{3\varepsilon}, w_3) + (\delta\varphi_{1\varepsilon}, w_3) - (\delta\varphi_{2\varepsilon}, w_3)$$
$$= (\varepsilon\delta\alpha_3, w_2) \tag{49a}$$
$$\delta\varphi_{3\varepsilon}(x,0) = 0 \text{ and } \delta\varphi_{3\varepsilon t}(x,0) = 0. \tag{49b}$$

Substituting $w_i = \delta\varphi_{i\varepsilon t}$, for $i = 1, 2, 3$ in (47a),(48a) & (49a) respectively, adding the three obtained equations, using the same way that it used to get (27), a similar equation will be obtained but with $(\overrightarrow{\delta\varphi})_\varepsilon$ in position of $\vec{\varphi}_n$, and then integration both sides on $[0,t]$, yield to

$$\int_0^t (d/dt)\left[\|(\overrightarrow{\delta\varphi})_{\varepsilon t}(t)\|_Q^2 + \|(\overrightarrow{\delta\varphi})_\varepsilon\|_1^2\right] dt$$
$$\le 2\int_0^t (|\delta\varphi_{2\varepsilon}| + |\delta\varphi_{3\varepsilon}| + \varepsilon|\delta\alpha_1|)|\delta\varphi_1 t| dt + m$$
$$+ 2\int_0^t (|\delta\varphi_1| + |\delta\varphi_2| + \varepsilon|\delta\alpha_2|)\,|\delta\varphi_2 t| dt$$
$$+ 2\int_0^t (|\delta\varphi_{1\varepsilon}| + |\delta\varphi_{3\varepsilon}| + \varepsilon|\delta\alpha_3|)|\delta\varphi_{3\varepsilon t}| dt.$$

Using the definitions of the norms and the relations between them, to get

$$\|\overrightarrow{(\delta\varphi)}_{\varepsilon t}(t)\|_Q^2 + \|\overrightarrow{(\delta\varphi)}_{\varepsilon}(t)\|_1^2 \leq \varepsilon\|\overrightarrow{(\delta\alpha)}(t)\|_Q^2 + 3\int_0^t (\|\overrightarrow{(\delta\varphi)}_{\varepsilon}\|_0^2 + \|\overrightarrow{(\delta\varphi)}_{\varepsilon t}\|_1^2)dt.$$

Applying the B-G inequality, with $a^2 = \varepsilon e^3$, to get $\|\overrightarrow{(\delta\varphi)}_{\varepsilon t}(t)\|_0^2 + \|\overrightarrow{(\delta\varphi)}_{\varepsilon}(t)\|_1^2 \leq a^2\|\overrightarrow{(\delta\alpha)}(t)\|_Q^2, \forall t \in I \Rightarrow \|\overrightarrow{(\delta\varphi)}_{\varepsilon}(t)\|_1^2 \leq a^2\|\overrightarrow{(\delta\alpha)}(t)\|_Q^2, \forall t \in \bar{I}.$ □

From this result and from the definitions of norms the requirement norms are obtained.

Lemma 4.2. *With assuming (A), the operator $\vec{\alpha} \mapsto \vec{\varphi}_{\vec{\alpha}}$ is continuous from $L^2(Q)$ in to $L^\infty(I, L^2(\Omega))$ or into*

Proof. Let $\delta\vec{\alpha} = \vec{\bar{\alpha}} - \vec{\alpha}$ and $\delta\vec{\varphi} = \overrightarrow{\bar{\varphi}} - \vec{\varphi}$, where $\overrightarrow{\bar{\varphi}}$ and $\vec{\varphi}$ are the correspond SVS to the CVS $\overrightarrow{\bar{\alpha}}$ and $\vec{\alpha}$ using the first result in Lemma 4.1 , one has $\|\overrightarrow{\bar{\varphi}} - \vec{\varphi}\|_{L^\infty(\tilde{I}, L^2(\Omega))} \leq M\|\vec{\bar{\alpha}} - \vec{\alpha}\|_Q$,

If $\vec{\bar{\alpha}} \xrightarrow[L^2(Q)]{} \vec{\alpha}$ then $\overrightarrow{\bar{\varphi}} \xrightarrow[L^\infty(\tilde{I}, L^2(\Omega))]{} \vec{\varphi}$. Thus the operator $\vec{\alpha} \mapsto \vec{\varphi}_{\vec{\alpha}}$ is Lipschitz continues (LC) from $L^2(\Omega)$ in to $L^\infty(\tilde{I}, L^2(\Omega))$.

Similar way is used to prove this operator is also LC from $L^2(Q)$ into $L^2(Q)$and into $L^2(\tilde{I}, W)$. □

Lemma 4.3 ([10])**.** *The norm $\|\cdot\|_0$ is weakly lower semi continuous (wlsc).*

Lemma 4.4. *The cost function which is given by (10) is wlsc.*

Proof. From Lemma 4.3,$\|\vec{\alpha}\|_{L^2(Q)}$ is wlsc, now when $\vec{\alpha}_k \longrightarrow \vec{\alpha}$ weakly in $L^2(Q)$, then $\vec{\varphi}_k \longrightarrow \vec{\varphi}$ weakly in $L^2(Q)$ (by Lemma 4.1), and then by Lemma 4.3

$$\|\vec{\varphi} - \vec{\varphi}_d\| \leq \lim_{k\longrightarrow\infty} \inf{}_{y_k \in V_k} \|\vec{\varphi}_k - \vec{\varphi}_d\|.$$

Thus $G_0(\vec{\alpha})$ is wlsc. □

Lemma 4.5 ([13])**.** *The norm $\|\cdot\|_0^2$ is strictly convex.*

Theorem 4.1. *Consider the cost function (10), if $G_0(\vec{\alpha})$ is coercive, the set $\vec{A}$is convex. Then there exists a CCOC.*

Proof. Since $G_0(\vec{\alpha}) \geq 0$ and $G_0(\vec{\alpha})$ is coercive, then there exists a minimizing sequence $\{\vec{\alpha}_k\} = (\alpha_{1k}, \alpha_{2k}, \alpha_{3k}) \in \vec{W}_A, \forall k$, s.t. $\lim_{n\longrightarrow\infty} G_0(\vec{\alpha}_k) = inf_{\overrightarrow{\alpha}_k \in \overrightarrow{W}_A} G_0(\overrightarrow{\alpha})$, and $\|\vec{\alpha}_k\| \leq c$, then by ATh there exists a subsequence of $\{\vec{u}_k\}$, for simplicity let be $\{\vec{\alpha}_k\}$, s.t. $\vec{\alpha}_k \longrightarrow \overrightarrow{\alpha}$ weakly in $L^2(Q)$, as $k \longrightarrow \infty$.

From Theorem 3.1, corresponding to the sequence of control $\{\vec{\alpha}_k\}$ there exists a sequence of unique solutions $\{\vec{\varphi}_k\} = \vec{\varphi}_{\vec{\alpha}_k}$ such that the norms $\|\vec{\varphi}_k\|_{L^2(\tilde{I},V)}$, $\|\vec{\varphi}_{kt}\|_{L^2(Q)}$ are bounded, then by AT there exist a subsequences of $\{\vec{\varphi}_k\}$, $\{\vec{\varphi}_{kt}\}$ for simplicity let be $\{\vec{\varphi}_k\}$, $\{\vec{\varphi}_{kt}\}$ s.t. $\vec{\varphi}_k \longrightarrow \vec{\varphi}$ weakly in $L^2(\tilde{I}, V), \vec{\varphi}_{kt} \longrightarrow \vec{\varphi}$

weakly in $L^2(Q)$. Now, for each k, the solution $(\varphi_{1k}, \varphi_{2k}, \varphi_{2k})$ satisfies the wf (15-17), multiplying both sides of each equation by $y_i(t), \forall i = 1, 2, 3$ respectively (with $y_i \in C^2[0, T]$, such that $y_i(T) = y_i'(T) = 0, y_i(0) \neq 0, y_i'(0) \neq 0)$. Rewriting the 1^{st} terms in the L.H.S. of each one of them, integrating both sides from 0 to T, finally integrating by parts twice for their 1st terms, one gets the same equations like (30)-(34) with different that each $w_{in} = w_i$ and the term $\int_0^T (p_i + \alpha_i, w_{in}) y_i(t) dt$(forall $i = 1, 2, 3$) in the R.H.S in each one of the obtained equations will be in the form

$$\int_0^T (p_i + \alpha_{ik}, w_i) y_i(t) dt, \forall i = 1, 2, 3, \text{ and } \forall k. \tag{50}$$

Hence, same technique which is used in the proof of Theorem 3.1, will be used to passage the limit (as $k \longrightarrow \infty$), in both sides of the above indicated equations, except the new term (50) which also converge to the following term (since $\alpha_{ik} \longrightarrow \alpha_i$,weakly in $L^2(Q)$).

$$\int_0^T (p_i + \alpha_i, w_i) \varphi_i(t) dt, \forall i = 1, 2, 3, \text{ and } \forall k. \tag{51}$$

From these convergences we get the wf like ((11a),(12a),(13a)).

Also, using the same steps which used in the proof of Theorem (3.1) can be also used here to passage the limits in the initial conditions and to get ((22b),(12b),(13b)), which give us the limit point $(\varphi_1, \varphi_2, \varphi_3)$ is a solution of the SE. At the end since $G_0(\vec{\alpha})$ is W.L.S.C. from Lemma (4.1) and since $\vec{\alpha}_k \longrightarrow \vec{\alpha}$ weakly in $(L^2(\Omega))^3$, then

$$\begin{aligned} G_0(\vec{\alpha}) &\leq \lim_{k \longrightarrow \infty} inf_{\vec{u}_k \in \vec{W}_A} G_0(\vec{\alpha}_k) = \lim_{k \longrightarrow \infty} G_0(\vec{\alpha}_k) = inf G_0(\overrightarrow{\alpha}) \\ &\Longrightarrow G_0(\vec{\alpha}) \leq \inf_{\overrightarrow{\alpha} \in \vec{W}_A} G_0(\overrightarrow{\alpha}) \\ &\Longrightarrow G_0(\vec{\alpha}) = \min_{\overrightarrow{\alpha} \in \vec{W}_A} G_0(\overrightarrow{\alpha}) \end{aligned}$$

Thus $\vec{\alpha}$ is a CCOC. □

5. The NTHO

Theorem 5.1. *Consider $G_0(\vec{\alpha})$ which is given by (10) and the TAEs of the STE (1-9) are given by*

$$\eta_{1tt} - \nabla \eta_1 + \eta_1 + \eta_2 + \eta_3 = (\varphi_1 - \varphi_{1d}), \text{ on } Q, \tag{52a}$$

$$\eta_1 = 0 on \sigma, \eta_1(x, T) = \eta_{1t}(x, T) = 0, \text{ on } \Omega \tag{52b}$$

$$\eta_{2tt} - \nabla \eta_2 + \eta_2 - \eta_1 - \eta_3 = (\varphi_2 - \varphi_{2d}), \text{ on } Q, \tag{53a}$$

$$\eta_2 = 0 on \sigma, \eta_2(x, T) = \eta_2 t(x, T) = 0 \text{ on } \Omega \tag{53b}$$

$$\eta_{3tt} - \nabla \eta_3 + \eta_3 - \eta_1 + \eta_2 = (\varphi_3 - \varphi_{3d}), \text{ on } Q, \tag{54a}$$

$$\eta_3 = 0 on \sigma, \eta_3(x, T) = \eta_3 t(x, T) = 0 \text{ on } \Omega. \tag{54b}$$

And the Hamiltonian is defined:

$$H(x,t,\vec{\varphi},\vec{\alpha},\vec{\eta}) = \sum_{i=1}^{3}(\eta_i(p_i(x,t)+\alpha_i) + (1/2)\sum_{i=1}^{3}(\|\varphi_i-\varphi_{id}\|_Q^2 + \beta/2)\|\alpha_i\|_Q^2).$$

Then for $\vec{\alpha'} \in \vec{W}$, the directional derivative of G is given by

$$DG(\vec{\alpha},\overrightarrow{\overrightarrow{\alpha}}-\vec{\alpha}) = \lim_{\epsilon\longrightarrow 0}\frac{G(\vec{\alpha}+\epsilon(\delta\alpha)) - G(\vec{\alpha})}{\epsilon} = \int_Q H_{\vec{\alpha}}(x,t,\vec{\varphi},\vec{\alpha},\vec{\eta}).(\overrightarrow{\overrightarrow{\alpha}}-\vec{\alpha})dxdt.$$

Proof. At first let, the wf of the adjoint equations is given, $\forall w_1, w_2, w_3 \in W$,(a.e. on I) by

(55a) $\langle\eta_{1tt},w_1\rangle+(\nabla\eta_1,\nabla w_1)+(\eta_1,w_1)+(\eta_2,w_1)+(\eta_3,w_1)=(\varphi_1-\varphi_{1d},w_1)$,

(55b) $(\eta_1(T),w_1) = (\eta_{1t}(T),w_1) = 0$,

(56a) $\langle\eta_{2tt},w_2\rangle+(\nabla\eta_2,\nabla w_2)+(\eta_2,w_2)-(\eta_1,w_2)-(\eta_3,w_1)=(\varphi_2-\varphi_{2d},w_2)$,

(56b) $(\eta_2(T),w_2) = (\eta_{2t}(T),w_2) = 0$,

(57a) $\langle\eta_{3tt},w_3\rangle+(\nabla\eta_3,\nabla w_3)+(\eta_3,w_3)-(\eta_1,w_3)+(\eta_2,w_3)=(\varphi_3-\varphi_{3d},w_3)$,

(57b) $(\eta_3(T),w_3) = (\eta_3 t(T),w_3) = 0$.

The proof of existence of a unique solution $\vec{\eta} = (\eta_1,\eta_2,\eta_3) \in L^2(Q)$ for the wf (55-57) can be done by using the same manner which is used in the proof of Theorem 3.1,

Substituting $w_i = \delta\varphi_{i\epsilon}$ for $i = 1,2,3$ in (55a) ,(56a),(57a) respectively, integrating both sides for $0 \le t \le T$, to get

$$\int_0^T \langle\delta\varphi_{1\epsilon},\eta_{1tt}\rangle dt + \int_0^T [(\nabla\nabla\eta_1,\nabla\delta\varphi_{1\epsilon}) + (\eta_1,\delta\varphi_{1\epsilon}) + (\eta_2,\delta\varphi_{1\epsilon})$$
$$+ (\eta_3,\delta\varphi_{1\epsilon})]dt = \int_0^T (\varphi_1-\varphi_{1d},\delta\varphi_{1\epsilon})dt, \tag{58}$$

$$\int_0^T \langle\delta\varphi_{2\epsilon},\eta_{2tt}\rangle dt + \int_0^T [(\nabla\eta_2,\nabla\delta\varphi_{2\epsilon}) + (\eta_2,\delta\varphi_{2\epsilon}) - (\eta_1,\delta\varphi_{2\epsilon})$$
$$- (\eta_3,\delta\varphi_{2\epsilon})]dt = \int_0^T (\varphi_2-\varphi_{2d},\delta\varphi_{2\epsilon})dt, \tag{59}$$

$$\int_0^T \langle\delta\varphi_{3\epsilon},{}_{3tt}\rangle dt + \int_0^T [(\nabla\eta_3,\nabla\delta\varphi_{2\epsilon}) + (\eta_3,\delta\varphi_{3\epsilon}) - (\eta_1,\delta\varphi_{3\epsilon})$$
$$+ (\eta_2,\delta\varphi_{3\epsilon})]dt = \int_0^T (\varphi_3-\varphi_{3d},\delta\varphi_{3\epsilon})dt. \tag{60}$$

Now, let $\vec{\alpha}, \overrightarrow{\overrightarrow{\alpha}} \in L^2(Q)$, $\overrightarrow{\delta\alpha} = \overrightarrow{\overrightarrow{\alpha}} - \vec{\alpha}$, for $\epsilon > 0, \vec{\alpha}_\epsilon = \vec{\alpha} + \epsilon\overrightarrow{\delta\alpha} \in L^2(Q)$, then by Theorem 3.1, $\vec{\varphi} = \vec{\varphi}_{\vec{\alpha}} \& \vec{\varphi}_\epsilon = \vec{\varphi}_{\vec{\alpha}_\epsilon}$ are their corresponding solutions. Setting $\overrightarrow{\delta\varphi}_\epsilon = (\delta\varphi_{1\epsilon},\delta\varphi_{2\epsilon},\delta\varphi_{3\epsilon}) = \vec{\varphi}_\epsilon - \vec{\varphi}$, substituting $w_i = \eta_i$, for $i = 1,2,3$

in (47a),(48a),(49a) respectively, integrating both sides for $0 \leq t \leq T$, then integrating by parts twice the 1^{st} term in the L.H.S. of each one of the obtained equation, once get

$$\begin{aligned}&\int_0^T \langle \delta\varphi_1\epsilon, \eta_{1tt}\rangle dt + \int_0^T [(\nabla\delta\varphi_1\epsilon, \nabla\eta_1) + (\delta\varphi_1\epsilon, \eta_1) - (\delta\varphi_2\epsilon, \eta_1)\\ &- (\delta\varphi_3\epsilon, \eta_1)]dt = \int_0^T (\epsilon\delta\delta_1, \eta_1)dt,\end{aligned} \tag{61}$$

$$\begin{aligned}&\int_0^T \langle \delta\varphi_2\epsilon, \eta_{2tt}\rangle dt + \int_0^T [(\nabla\delta\varphi_2\epsilon, \nabla\eta_2) + (\delta\varphi_2\epsilon, \eta_2) + (\delta\varphi_1\epsilon, \eta_2)\\ &+ (\delta\varphi_3\epsilon, \eta_2)]dt = \int_0^T (\epsilon\delta\alpha_2, \eta_2)dt,\end{aligned} \tag{62}$$

$$\begin{aligned}&\int_0^T \langle \delta\varphi_3\epsilon, \eta_{3tt} dt + \int_0^T [(\nabla\delta\varphi_3\epsilon, \nabla\eta_3) + (\delta\varphi_3\epsilon, \eta_3) + (\delta\varphi_1\epsilon, \eta_3)\\ &- (\delta\varphi_2\epsilon, \eta_3)]dt = \int_0^T (\epsilon\delta\alpha_3, \eta_3)dt,\end{aligned} \tag{63}$$

$$\begin{aligned}&\epsilon\int_0^T [(\delta\alpha_1, \eta_1) + (\delta\alpha_2, \eta_2) + (\delta\alpha_3, \eta_3)]dt = \int_0^T [(\varphi_1 - \varphi_{1d}, \delta\varphi_1\epsilon)\\ &+ (\varphi_2 - \varphi_{2d}, \delta\varphi_2\epsilon) + (\varphi_3 - \varphi_{3d}, \delta\varphi_2\epsilon)]dt,\end{aligned} \tag{64}$$

where $O_1(\epsilon) \longrightarrow 0$, as $\epsilon \longrightarrow 0$, with $O_1(\epsilon) = \|(\delta\varphi)_\epsilon\|_Q + \epsilon\|(\delta\alpha)\|_Q$

On the other hand, we have

$$\begin{aligned}G_0(\vec{\alpha}_\eta) - G_0(\vec{\alpha}) &= \int_Q ((\varphi_1 - \varphi_{1d})\delta\varphi_{1\epsilon} + \epsilon\beta\alpha_1\beta\alpha_1)dxdt\\ &+ \int_Q ((\varphi_2 - \varphi_{2d})\delta\varphi_2\epsilon + \epsilon\beta\alpha_2\delta\alpha_2)dxdt\\ &+ \int_Q ((\varphi_3 - \varphi_{3d})\delta\varphi_{3\epsilon} + \epsilon\beta\alpha_3\delta\alpha_3)dxdt + O_1(\epsilon),\end{aligned} \tag{65}$$

where $O_1(\epsilon) = \|\overrightarrow{\delta\varphi}_\epsilon\|_Q + \epsilon\|\overrightarrow{\delta\alpha}\|_Q$, with $O_1(\epsilon) \longrightarrow 0$, as $\epsilon \longrightarrow 0$ Now, setting (64) in (65), one have that

$$G_0(\vec{\alpha}_\epsilon) - G_0(\vec{\alpha}) = \epsilon\int_Q [(\eta_1 + \beta\alpha_1)\delta\alpha_1 + (\eta_2 + \beta\alpha_2)\delta\alpha_2 + (\eta_3 + \beta\alpha_3)\delta\alpha_3]dxdt + O_1(\epsilon).$$

Finally, dividing both sides of the above equality by ϵ, then taking the limit $\epsilon \longrightarrow 0$, once get

$$DG(\vec{\alpha}, \vec{\bar{\alpha}} - \vec{\alpha}) = \int_Q H_{\vec{u}}(x, t, \vec{\varphi}, \vec{\alpha}, \vec{\eta}).(\vec{\bar{\alpha}} - \vec{\alpha})dxdt = \int_Q (\vec{\eta} + \beta\vec{\alpha}) \cdot \overrightarrow{\delta\alpha} dxdt,$$

where $H_{\vec{\alpha}}(x, t, \vec{\varphi}, \vec{\alpha}, \vec{\eta}) = (\eta_1 + \beta\alpha_1\eta_2 + \beta\alpha_2\eta_3 + \beta\alpha_3)^T$.

□

Theorem 5.2. *The CCOC of the above problem is* $DG(\vec{\alpha}, \vec{\bar{\alpha}} - \vec{\alpha}) = \vec{\eta} + \beta\vec{\alpha} = 0$

Proof. If $\vec{\alpha}$ is an CCOC of the problem, then $G_0(\vec{\bar{\alpha}}) = min_{\vec{\alpha} \in \vec{W}_A} G_0(\vec{\alpha}), \forall \vec{\alpha} \in L^2(Q)$, i.e. $DG(\vec{\alpha}, \vec{\bar{\alpha}} - \vec{\alpha}) = 0 \Longrightarrow \vec{\eta} + \beta\vec{\bar{\alpha}} = 0, \delta\vec{\alpha} = \vec{w} - \vec{\bar{\alpha}} \Longrightarrow$ The NCO is $(\vec{\eta} + \beta\vec{\bar{\alpha}}, \delta\vec{\alpha}) \geq 0 \Longrightarrow (\vec{\eta} + \vec{\bar{\alpha}}, \vec{w}) \geq (\vec{\eta} + \beta\vec{\bar{\alpha}}, \vec{\bar{\alpha}}), \forall \vec{w} \in L^2(Q)$. □

6. Conclusions

The GM is employed to prove the existence and unique theorem for a SVS of the TSPDEs of parabolic type for fixed CCCV as well as The existence and uniqueness solution of the TAEs associated with the TSPDEs. The existence of a CCOCV associated with the considered TLPDEs of parabolic type is proved. The derivation of the Directional derivative (DDV) for the cost functional is obtained; finally, the necessity theorem (necessity conditions) for optimality (NTHO) of this OCP is sated and proved.

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References

[1] G. Akudibillah, A. Pandey, J. Medlock, *Optimal control for HIV treatment*, Mathematical Biosciences and Engineering, 16 (2019), 373-96.

[2] P. Kormushev, B. Ugurlu, D.G. Caldwell, N.G. Tsagarakis, *Learning to exploit passive compliance for energy-efficient gait generation on a compliant humanoid,* Autonomous Robots, 43 (2019), 7995.

[3] L. Kahina, P. Spiteri, F. Demim, A. Mohamed, A. Nemra, F. Messine, *Application optimal control for a problem aircraft flight*, Journal OF Engineering Science and Technology Review, 11 (2018), 156164.

[4] V. Oliinyk, O. Kozmenko, I. Wiebe, S. Kozmenko, *Optimal control over the process of innovative product diffusion: the case of sony corporation,* Economics and Sociology, 11 (2018), 265-285.

[5] Xu Bin, Xu Chen, Xiuhui Huang, Lili Tao, *A multistrategy-based multi-objective differential evolution for optimal control in chemical processes,* Hindawi: Complexity, Volume 2018, Article ID 2317860, 22 pages.

[6] Donghai Hu, Yanzhi Yan, and Xiaoming Xu, *Energy saving optimal design and control of electromagnetic brake on passenger car*, Mech. Sci., 10 (2019), 5770, 2019.

[7] S. Hamid, H.B. Sahib, H.A. Fawzi, A.Y. Nori, *Medication adherence and glycemic control in newly diagnosed diabetic patients,* International Journal of Research in Pharmaceutical Sciences. Int. J. Res. Pharm. Sci., 9 (2018), 816-820.

[8] C. Brett, A. Dedner, C. Elliott, *Optimal control of elliptic PDEs at points,* IMA Journal of Numerical Analysis, 36 (2015), 1-34.

[9] I. Chryssoverghi, J. Al-Hawasy, *The continuous classical optimal control problem of semi linear parabolic equations (CCOCP)*, Journal of Kerbala University, 2010, 8, 3.

[10] J. Al-Hawasy, *The continuous classical optimal control problem of a non-linear hyperbolic partial differential equations (CCOCP),* Al-Mustansiriyah Journal of Science, 19 (2008), 96-110.

[11] E.H. Al-Rawdhanee, *The continuous classical optimal control of a couple non-linear elliptic partial differential equations,* M.Sc. thesis, Al-Mustansiriyah University, Baghdad, Iraq, 2015.

[12] J. Al-Hawasy, G.M. Kadhem, *The continuous classical optimal control for a coupled nonlinear parabolic partial differential equations with equality and inequality constraints,* Journal of Al-Nahrain University, 19 (2016), 173-186.

[13] J. Al-Hawasy, *The continuous classical optimal control of a couple non-linear hyperbolic partial differential equations with equality and inequality constraints,* Iraqi Journal of Science, 2016, 57, 2C, 1528-1538.

[14] J. Al-Hawasy, S.J. Al-Qaisi, *The continuous classical optimal boundary control of a couple linear elliptic partial differential equations,* Special Issue, 1st Scientific International Conference, College of Science, Al-Nahrain University, 21-22/11/2017, Part I, 137-142.

[15] J. Al-Hawasy, A.A. Naeif, *The continuous classical boundary optimal control of a couple linear of parabolic partial differential equations,* Al-Mustansiriyah Journal of Science, 29 (2018), 118-126.

[16] J. Al-Hawasy, *The continuous classical boundary optimal control of couple nonlinear hyperbolic boundary value problem with equality and inequality constraints,* Baghdad Science Journal, 2019 (accepted).

[17] J. Al-Hawasy, D.A. Jasim, *The continuous classical optimal control problems for triple elliptic partial differential equations,* accepted for publishing in IHJPAS.

[18] J. Al-Hawasy, M.A.K. Jaber, *The continuous classical optimal control problems for triple parabolic partial differential equations,* submitted for publishing in IHJPAS.

[19] F. Albiac, N.J. Kalton, *Topics in Banach space theory,* 2nd ed., France, 2015.

[20] R. Temam, *Navier-Stokes equations*, North-Holand Publishing Company, 1977.

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