Modal operators on equality algebras

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Abstract. The main goal of this paper is to investigate modal operators on equality algebras. To begin with, we introduce the notion of modal operators on equality algebras and investigate some important properties of this operator. As applications, we give a characterization of prelinear equality algebras. In the following, we give the concepts of modal filters and modal congruences of equality algebras and obtain some related results. Moreover, we show that there is one to one correspondence relation between modal filters and modal congruences of a modal equality algebra. Finally, using strong modal filters, we establish the uniform structures on modal equality algebras and we prove that modal equality algebras with uniform topologies are topological modal equality algebras.

Keywords: equality algebra, modal operator, modal filter, modal congruence, uniform structure.

1. Introduction

Fuzzy type theory(FTT) [12, 13, 14] was developed as a fuzzy counterpart of the classical higher-order logic. Since the truth values for algebra is no longer a residuated lattice, a specific algebra which called an EQ-algebra was proposed by Novák and De Baets in [11] which generalizes residuated lattice. In [3], it was mentioned if the product in EQ-algebras is replaced by any other smaller

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binary operation, we still obtains an EQ-algebra. Based on the above reasons, a new algebraic structure was introduced by Jenei in [4], called equality algebra, which consisting of two binary operations meet and equivalence, and constant 1.

In 1981, modal operators on Heyting algebras were introduced and studied as algebraic counter-part of the intuitionistic propositional logic by Macnab [10]. Since then, properties of modal operators were considered on other algebraic structures such as MV-algebra [2], bounded commutative residuated $R\ell - monoids$ (simply called $R\ell - monoids$) [15], commutative reesiduated lattices [9] and so on. The essence of modal operator is closure operator, and closure operator is an important part of the theoretical study of partial order sets.

In this paper, we define modal operators for equality algebras. This paper is structured in five sections. In section 2, we recall the definition of equality algebras and their basic properties that will be used in this paper. In section 3, we introduce the notion of modal operators in equality algebras and investigate some important related properties. And we give some characterizations of prelinear equality algebras. In section 4, we introduce modal filters and modal congruences of modal equality algebras and obtain some important results. And we show that there is a one to one correspondence relation between modal filters and modal congruences of a modal equality algebra. In section 5, we establish uniform structures by the special family of strong modal filters on equality algebras, and then induce uniform topologies. Moreover, we show that modal equality algebras with uniform topologies are topological equality algebras.

2. Preliminaries

In the section, we summarize some definitions and results about equality algebras, which will be used in the following sections of the paper.

Definition 2.1 ([3, 4]). *An algebra structure $(E, \wedge, \sim, 1)$ of the type $(2, 2, 0)$ is called an equality algebra, if it satisfies the following conditions: for all $x, y, z \in E$,*

$(E1)$ *$(E, \wedge, 1)$ is a commutative idempotent integral monoid (i.e. meet semilattice with top element 1).*

$(E2)$ $x \sim y = y \sim x$.

$(E3)$ $x \sim x = 1$.

$(E4)$ $x \sim 1 = x$.

$(E5)$ $x \leq y \leq z$ *implies* $x \sim z \leq y \sim z$ *and* $x \sim z \leq x \sim y$.

$(E6)$ $x \sim y \leq (x \wedge z) \sim (y \wedge z)$.

$(E7)$ $x \sim y \leq (x \sim z) \sim (y \sim z)$.

Where $x \leq y$ if and only if $x \wedge y = x$, for all $x, y \in E$.

The operation $\wedge$ is called meet(infimum) and $\sim$ is an equality operation. Also, other two operations are defined, called implication and equivalence oper-

ation, respectively:

$$x \to y = x \sim (x \wedge y) x \leftrightarrow y = (x \to y) \wedge (y \to x).$$

In what follows, we let $E = (E, \wedge, \sim, 1)$ be an equality algebra.

Proposition 2.2 ([3, 4])**.** *Let E be an equality algebra. Then the following properties hold: for all $x, y, z \in E$:*

(1) $x \sim y \leq x \leftrightarrow y \leq x \to y$,
(2) $x \to y = 1$ *if and only if* $x \leq y$,
(3) $1 \to x = x, x \to 1 = 1, x \to x = 1$,
(4) $x \leq y \to x$,
(5) $x \leq (x \to y) \to y$,
(6) $x \to y \leq (y \to z) \to (x \to z)$,
(7) $x \leq y \to z$ *if and only if* $y \leq x \to z$,
(8) $x \to (y \to z) = y \to (x \to z)$,
(9) $y \leq x$ *implies* $x \leftrightarrow y = x \to y = x \sim y$,
(10) $x \sim y = 1$ *iff* $x = y$.

Proposition 2.3 ([18])**.** *Let E be an equality algebra. Then the following statements hold: for all $x, y, z \in E$,*

(i) $x \leq y$ *implies* $y \to z \leq x \to z, z \to x \leq z \to y$,
(ii) $x \to y = x \to (x \wedge y)$,
(iii) $x \sim y \leq (z \to x) \sim (z \to y)$,
(iv) $x \sim y \leq (z \to x) \to (z \to y)$,
(v) $x \to y \leq (z \to x) \to (z \to y)$,
(vi) $x \to y \leq (x \wedge z) \to (y \wedge z)$,
(vii) $x \to y = ((x \to y) \to y) \to y$.

Definition 2.4 ([18])**.** *Let E be an equality algebra. Then, E is called prelinear, if 1 is the unique upper bound of the set $\{x \to y, y \to x\}$, for all $x, y \in E$.*

Proposition 2.5 ([18])**.** *If E is a prelinear equality algebra, then $(E, \leq)$ is a lattice, where the join operation is given by $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$, for all $x, y \in E$.*

Proposition 2.6 ([19, 20])**.** *An equality algebra E is prelinear iff $(x \to y) \to z \leq ((y \to x) \to z) \to z$, for all $x, y, z \in E$.*

Definition 2.7 ([18])**.** *A lattice equality algebra is an equality algebra which is lattice.*

The following proposition provides some properties of lattice equality algebras.

Proposition 2.8 ([18])**.** *Let E be a lattice equality algebra. Then the following properties hold, for all $x, y \in E$:*

(i) for all indexed families $\{x_i\}_{i\in I}$ in E, we have $(\vee_{x\in I}) \to y = \wedge_{i\in I}(x_i \to y)$, provided that the infimum and suprimum of $\{x_i\}_{i\in I}$ exist in E;

(ii) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$;

(iii) $x \to y = (x \vee y) \to y$.

Theorem 2.9 ([18])**.** *Any prelinear equality algebra is a distributive lattice.*

The following proposition provides some properties of prelinear equality algebras.

Proposition 2.10 ([18])**.** *Let E be a prelinear equality algebra. Then the following statements holds: for all $x, y, z \in E$,*

(i) $x \leftrightarrow y = x \sim y$;

(ii) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$.

An equality algebra $(E, \wedge, \sim, 1)$ is bounded if there exists an element $0 \in E$ such that $0 \leq x$, for all $x \in E$. In a bounded equality algebra E, we define the negation "$\prime$" on E by, $x' = x \to 0 = x \sim 0$, for all $x \in E$.

Proposition 2.11 ([18])**.** *Let $(E, \wedge, \sim, 0, 1)$ be a bounded lattice equality algebra. Then the following properties hold: for all $x, y \in E$,*

(i) $(x \vee y)' = x' \wedge y'$;

(ii) $x \leq (x')'$;

(iii) $x \to y \leq y' \to x'$, and if it is involutive, then $x \to y = y' \to x'$.

Definition 2.12 ([4])**.** *Let E be an equality algebra, $F \subseteq E$. F is called a filter of E if for all $a, b \in E$,*

(i) $1 \in F$;

(ii) $a \in F, a \leq b \Rightarrow b \in F$;

(iii) $a, a \sim b \in F \Rightarrow b \in F$.

We will denote the set of all filters of E by $F(E)$. Clearly, $\{1\}, E \subseteq F(E)$, and $F(E)$ is closed under arbitrary intersections. And hence we have $(F(E), \subseteq)$ is a complete lattice. A filter F of an equality algebra E is proper if $F \neq E$. A proper filter F is called maximal if $F \subseteq G \subseteq E$ implies $F = G$ for all G proper filter of E. An equality algebra $(E, \wedge, \sim, 1)$ is called simple, if $F(E) = \{\{1\}, E\}$. (See [1, 5])

Definition 2.13 ([4])**.** *A subset θ of $E \times E$ is called congruence of E, if it is an equivalence relation on E and for all $a, a', b, b' \in E$ such that $(a, b), (a', b') \in \theta$ the following hold:*

(i) $(a \wedge a', b \wedge b') \in \theta$;

(ii) $(a \sim a', b \sim b') \in \theta$.

We will denote the set of all congruences of E by $C(E)$.

Let F be a filter of E. Define the congruence relation $\equiv_F$ on E by $x \equiv_F y$ iff $x \sim y \in F$. The set of all congruence class is denote by E/F, i.e. $E/F = \{[x] \mid x \in E\}$, where $[x] = \{x \in E \mid x \equiv_F y\}$. Define $\bullet, \rightharpoonup$ on E/F as follows: $[x] \bullet [y] = [x \wedge y], [x] \rightharpoonup [y] = [x \sim y]$. Therefore, $(E/F, \bullet, \rightharpoonup, [1])$ is an equality algebra which is called a quotient equality algebra of E with respect to F. (See [1])

Proposition 2.14 ([4]). *Let E be an equality algebra. $F \in F(E)$ iff for all $a, b \in E$,*

(i) $1 \in F$;

(ii) $a, a \to b \in F \Rightarrow b \in F$ holds, where $\to$ denotes the implication of E.

The next theorem makes connection between $F(E)$ and $C(E)$.

Theorem 2.15 ([4]). *Let E be an equality algebra, $\theta, \varphi \in C(E), F \in F(E)$. Then:*

(a) $[1]_\theta \in F(E)$, where $[1]_\theta = \{a \mid (a, 1) \in \theta\}$;

(b) $[1]_{\theta_F} = F$;

(c) $\theta_{[1]_\theta} = \theta$;

(d) if $[1]_\theta = [1]_\varphi$, then $\theta = \varphi$.

Theorem 2.16 ([4]). *For $\theta \in C(E)$, we have $(a, b) \in \theta$ iff $(a \sim b, 1) \in \theta$.*

Next, we review some notions about uniformity which will be necessary in the following section.

Let X be a nonempty set and A, B be any subset of $X \times X$. We have the following notation:

(1) $A \circ B = \{(x, y) \in X \times X : (x, z) \in A, (z, y) \in B$, for some $z \in X\}$;

(2) $A^{-1} = \{(x, y) \in X \times X : (y, x) \in A\}$;

(3) $\triangle = \{(x, x) \in X \times X : x \in X\}$.

Definition 2.17 ([16, 7, 8]). *A nonempty collection K of subsets of $X \times X$ is called an uniformity on X, which satisfies the following conditions:*

$(A1)$ $\triangle \subseteq A$ for any $A \in K$;

$(A2)$ if $A \in K$, then $A^{-1} \in K$;

$(A3)$ if $A \in K$, then there exists $B \in K$ such that $B \circ B \subseteq A$;

$(A4)$ if $A, B \in K$, then $A \cap B \in K$;

$(A5)$ if $A \in K$ and $A \subseteq B \subseteq X \times X$, then $B \in K$. Then the pair (X, K) is called an uniform structure (uniform space).

3. Modal operators on equality algebras

In this section, we introduce the notion of modal operators on equality algebras and we investigate some important related properties of modal operators. And in the end, we give some characterizations of prelinear equality algebras.

Definition 3.1. *Let E be an equality algebra. The mapping $f : E \to E$ is called a modal operator if it satisfies the following conditions: for all $x, y \in E$,*

$(f1)$ $x \leq f(x)$;
$(f2)$ $f(f(x)) = f(x)$;
$(f3)$ $f(x \wedge y) = f(x) \wedge f(y)$;
$(f4)$ $f(x \sim y) \leq f(x) \sim f(y)$.

And the pair (E, f) is called an equality algebra with a modal operator (simplify, modal equality algebra).

Let (E, f) be a modal equality algebra. The kernel of f is the set $Ker(f) = \{x \in E \mid f(x) = 1\}$. A modal operator f is said to be faithful if $Ker(f) = \{1\}$.

Example 3.2. Let $(E, \wedge, \sim, 1)$ be an equality algebra. We define mappings $f_1, f_2 : E \to E$ such that $f_1(x) = x$, $f_2(x) = 1, \forall x \in E$. We can see that both f_1 and f_2 are modal operators on E. Also, f_1 is faithful since $Ker(f) = \{1\}$. Moreover, modal operators f_1 and f_2 are called trivial.

Example 3.3. Let $E = \{0, a, b, c, 1\}$ with $0 < c < a, b < 1$. Define the operation $\sim$ on E as follows:

$\sim$	0	c	a	b	1
0	1	0	0	0	0
c	0	1	b	a	c
a	0	b	1	c	a
b	0	a	c	1	b
1	0	c	a	b	1

Then we can see that $(E, \wedge, \sim, 1)$ is an equality algebra. Now, we define a mapping f on E as follows:

$$f(x) = \begin{cases} 0, & x = 0 \\ a, & x = a, c \\ 1, & x = b, 1 \end{cases}.$$

It is easy to check that f is a modal operator on E.

Example 3.4. Let $E = \{0, a, b, 1\}$ with $0 < a < b < 1$. Define the operation $\sim$ on E as follows:

$\sim$	0	a	b	1
0	1	a	0	0
a	a	1	a	a
b	0	a	1	b
1	0	a	b	1

Then we can see that $(E, \wedge, \sim, 1)$ is an equality algebra. Now, we define a mapping f on E as follows:

$$f(x) = \begin{cases} 0, & x = 0 \\ b, & x = a, b \\ 1, & x = 1 \end{cases}.$$

It is also easily to check that f is a modal operator on E.

In what follows, we obtain some important properties of modal operators on equality algebras.

Proposition 3.5. *Let $(E, \wedge, \sim, 0, 1)$ be a bounded equality algebra and $f : E \to E$ be a modal operator. Then for $x, y \in E$ we have:*
(1) $f(1) = 1$;
(2) *If $x \leq y$, then $f(x) \leq f(y)$;*
(3) $f(x \to y) \leq f(x) \to f(y) = f(f(x) \to f(y)) = x \to f(y) = f(x \to f(y))$;
(4) $f(x) \leq (x \to f(0)) \to f(0)$;
(5) $f(x) \wedge f(y) = f(f(x) \wedge f(y))$;
(6) $f(x)' \leq f(x')$;
(7) *If $y \leq x$, then $f(x) \to f(y) = f(x) \sim f(y)$;*
(8) $f(x \sim y) \leq f(x) \to f(y)$.

Proof. (1) By (f1) of Definition 3.1, we have $1 \leq f(1)$. Thus $f(1) = 1$.

(2) For all $x, y \in E$, we have that $x \leq y$ if and only if $x \wedge y = x$. By (f3) of Definition 3.1, if $x \leq y$, then $f(x) = f(x \wedge y) = f(x) \wedge f(y)$. Thus $f(x) \leq f(y)$.

(3) For all $x, y \in E$, by (f4) of Definition 3.1, we have $f(x \to y) = f(x \sim (x \wedge y)) \leq f(x) \sim f(x \wedge y) = f(x) \sim (f(x) \wedge f(y)) = f(x) \to f(y)$. Further by the condition (f1), (f2) of Definition 3.1, we have:
$f(f(x) \to f(y)) \leq f(f(x)) \to f(f(y)) = f(x) \to f(y) \leq x \to f(y) \leq f(x \to f(y)) \leq f(x) \to f(f(y)) = f(x) \to f(y) \leq f(f(x) \to f(y))$. Hence $f(x \to f(y)) = f(f(x) \to f(y)) = f(x) \to f(y) = x \to f(y)$.

(4) By $x \leq (x \to y) \to y$ and (3), we have that $f(x) \leq (f(x) \to f(0)) \to f(0) = (x \to f(0)) \to f(0)$.

(5) $f(x) \wedge f(y) \leq f(f(x) \wedge f(y))$ is valid. Now we show that $f(f(x) \wedge f(y)) \leq f(x) \wedge f(y)$: as $f(x) \wedge f(y) \leq f(x)$ and $f(x) \wedge f(y) \leq f(y)$, by (2), we have $f(f(x) \wedge f(y)) \leq f(f(x))$ and $f(f(x) \wedge f(y)) \leq f(f(y))$, hence $f(f(x) \wedge f(y)) \leq (f(f(x)) \wedge f(f(y)))$. By (f2) of Definition 3.1, we have $f(f(x) \wedge f(y)) \leq f(x) \wedge f(y)$. Therefore $f(x) \wedge f(y) = f(f(x) \wedge f(y))$.

(6) By (f1) and (1) of Proposition 2.3, we have $f(x)' \leq x' \leq f(x')$.

(7) Since $y \leq x$, we have $f(y) \leq f(x)$ and $f(x) \to f(y) = f(x) \sim (f(x) \wedge f(y)) = f(x) \sim f(y)$.

(8) It follows from (1) of Proposition 2.2 and items (2), (3). □

We know that let E be a non-empty set, an isotone mapping $f : E \to E$ is called a closure operator on E if it is such that $f = f^2 \geq id_E$. Then we have,

Remark 3.6. By the definition of a modal operator and Theorem 3.5 (2), every modal operator on an equality algebra E is a closure operator. In general, the converse is not true.

The following example will show that a closure operator on E is not a modal operator.

Example 3.7. Consider the equality algebra E of Example 3.3. Now we define a mapping f on E as follows:

$$f(x) = \begin{cases} c, & x = 0, c \\ 1, & x = a, b, 1 \end{cases}$$

It is easily to check that f is a closure operator on E. However, f is not a modal operator on E since $f(a \wedge b) = f(c) = c$ and $f(a) \wedge f(b) = 1$. Hence $f(a \wedge b) \neq f(a) \wedge f(b)$.

In what follows, we will define residuated equality algebras. And we will give a characterization of modal operator on a residuated equality algebra.

Definition 3.8. *Let E be an equality algebra. Then E is called residuated, if* (R) $(x \wedge y) \wedge z = (x \wedge y)$ *iff* $x \wedge (y \sim (y \wedge z)) = x$*, for all* $x, y, z \in E$*.*

Clearly, (R) can be written in a classical way as $x \wedge y \leq z$ iff $x \leq y \to z$.

Example 3.9. Let $E = \{0, a, b, 1\}$ be a chain such that $x \wedge y = min\{x, y\}$, for all $x, y \in E$. Define the operation $\sim$ on E as follows:

$\sim$	0	a	b	1
0	1	0	0	0
a	0	1	a	a
b	0	a	1	b
1	0	a	b	1

Routine calculation shows that $(E, \wedge, \sim, 1)$ is a residuated equality algebra.

Theorem 3.10. *Let E be a residuated equality algebra and $f : E \to E$ be a mapping. Then f is a modal operator on E if and only if, for each $x, y \in E$, it holds:*

(1) $x \to f(y) = f(x) \to f(y)$;
(2) $f(x \sim y) \leq f(x) \sim f(y)$.

Proof. Firstly, assume a mapping f fulfil conditions (1) and (2) of above Theorem, now we will show that f is a modal operator on E, that is, f satisfies the conditions $(f1) - (f4)$ of Definition 3.1: Indeed,

For any $x \in E$, by (1) we have $x \to f(x) = f(x) \to f(x) = 1$. Therefore $x \leq f(x)$. Thus $(f1)$ holds.

For any $x \in E$, it holds $1 = f(x) \to f(x) = f(f(x)) \to f(x)$. This implies $f(f(x)) \leq f(x)$. Then by $(f1)$, we have $f(f(x)) = f(x)$. Thus $(f2)$ holds.

To show that f establish $(f3)$, we first verity the following result:

$$\text{if } x \leq y, \text{ then } f(x) \leq f(y).$$

Indeed, if $x \leq y$, by (1) of Proposition 2.3 and (1) of above Theorem, we have $f(x) \to f(y) = x \to f(y) \geq y \to f(y) = 1$, hence $f(x) \to f(y) = 1$. Therefore $f(x) \leq f(y)$. Now, as $x \wedge y \leq x$ and $x \wedge y \leq y$, applying above result we can obtain that $f(x \wedge y) \leq f(x)$ and $f(x \wedge y) \leq f(y)$, and then $f(x \wedge y) \leq f(x) \wedge f(y)$. Now we show that $f(x) \wedge f(y) \leq f(x \wedge y)$: By $(f1)$, we have $x \wedge y \leq f(x \wedge y)$. As E be a residuated equality algebra, $x \wedge y \leq z$ iff $x \leq y \to z$ holds. Then we get $y \leq x \to f(x \wedge y) = f(x) \to f(x \wedge y)$, so $y \wedge f(x) \leq f(x \wedge y)$. Similarly, we have $f(x) \leq y \to f(x \wedge y) = f(y) \to f(x \wedge y)$, so that $f(x) \wedge f(y) \leq f(x \wedge y)$. Therefore $f(x \wedge y) = f(x) \wedge f(y)$. Thus $(f3)$ holds.

By (2), $(f4)$ is straightforward. Therefore, f is a modal operator on E.

Now if f be a modal operator on E, (1) and (2) is obvious. □

Theorem 3.11. *Let (E, f) be a modal equality algebra. Then the following properties hold:*

(1) *$f(E) = Fix(f)$, where $Fix(f) = \{x \in E \mid x = f(x)\}$.*

(2) *The image $f(E)$ is a subalgebra of E.*

Proof. (1) Assume that $y \in f(E)$, then there exists $x \in E$ such that $f(x) = y$. By $(f2)$ of Definition 3.1, we can obtain that $f(y) = f(f(x)) = f(x) = y$, this is, $y \in Fix(f)$. Conversely, assume that $y \in Fix(f)$. By $Fix(f) = \{x \in E \mid x = f(x)\}$, we have $y \in f(E)$. Therefore $f(E) = Fix(f)$ holds.

(2) For all $x, y \in f(E)$, by $(f3)$ and (1), we have $f(x \wedge y) = f(x) \wedge f(y) = x \wedge y$. So $x \wedge y \in f(E)$. By $(f4)$ and (1), we have $f(x \sim y) \leq f(x) \sim f(y) = x \sim y$. Then by $(f1)$, $x \sim y \leq f(x \sim y)$. Hence, $f(x \sim y) = x \sim y$. Therefore we have $x \sim y \in f(E)$. Hence $f(E)$ is closed under the operation $\wedge$ and $\sim$. And we have $f(1) = 1$, so $1 \in f(E)$. Therefore, $f(E)$ is a subalgebra of E. □

Corollary 3.12. *Let (E, f) be a modal equality algebra. If $f(E) = E$, then f is the identity map on E.*

Proof. If $f(E) = E$, for any $x \in E$, then there exists $x_0 \in E$ such that $f(x_0) = x$. Form $(f2)$, we have $f(x) = f(f(x_0)) = f(x_0) = x$. Therefore, f is the identity map on E. □

Theorem 3.13. *If (E, f) be a modal equality algebra, then $Ker(f) = \{x \in E \mid f(x) = 1\}$ is a filter.*

Proof. By $(f1)$ of Definition 3.1, we have $1 \leq f(1)$. Then $f(1) = 1$, hence $1 \in Ker(f)$.

Let x and $x \to y \in Ker(f)$, then we get $f(x) = 1$ and $f(x \to y) = 1$. By (3) of Proposition 3.5, we can obtain that $1 = f(x \to y) \leq f(x) \to f(y) = 1 \to$

$f(y) = f(y)$, thus $f(y) = 1$. So we get $y \in Ker(f)$. Therefore, by Proposition 2.14, we obtain $Ker(f) = \{x \in E \mid f(x) = 1\}$ is a filter. □

In what follows, the conditions for an equality algebra to be a prelinear equality algebra is given by a modal operator on an equality algebra.

Theorem 3.14. *Let (E, f) be a modal equality algebra. Then the following conditions are equivalent:*

(1) *E is prelinear;*
(2) *$f(x \vee y) = f(x) \vee f(y)$;*
(3) *$f(x \to y) \vee f(y \to x) = 1$;*
(4) *$f((x \to y) \to z) \leq f((y \to x) \to z) \to z$.*

Proof. (1)⇒(2) Assume that E is a prelinear equality algebra. $f(x) \vee f(y) \leq f(x \vee y)$ can be proved easily by Proposition 2.3 (1). Now we will prove that $f(x \vee y) \leq f(x) \vee f(y)$. Indeed, by Proposition 2.5 and $(f3)$ of Definition 3.1, we have for all $x, y \in E$, $f(x \vee y) = f(((x \to y) \to y) \wedge ((y \to x) \to x)) = f((x \to y) \to y) \wedge f((y \to x) \to x)$. By Proposition 3.5 (3), Proposition 2.3 (1), we have $f(x \vee y) \leq (f(x \to y) \to f(y)) \wedge (f(y \to x) \to f(x)) \leq (f(x \to y) \to (f(x) \vee f(y))) \wedge (f(y \to x) \to (f(x) \vee f(y)))$. Then by Proposition 2.9 and Proposition 2.8 (1), we have $f(x \vee y) \leq (f(x \to y) \vee f(y \to x)) \to (f(x) \vee f(y))$. Now, we will prove that $f(x \to y) \vee f(y \to x) = 1$. Indeed, by (f1) of Definition 3.1 and Proposition 2.5, we have $(x \to y) \vee (y \to x) \leq f(x \to y) \vee f(y \to x)$. Hence by Definition 2.4, we obtain $(x \to y) \vee (y \to x) = 1$, and hence $f(x \to y) \vee f(y \to x) = 1$. Thus, by Proposition 2.2 (3), we have $f(x \vee y) \leq 1 \to (f(x) \vee f(y)) = f(x) \vee f(y)$.

(2)⇒(3) Obviously.

(3)⇒(4) This follows similar to the proof of the equivalence between prelinearity and Proposition 2.6.

(4)⇒(1) Taking modal operator $f = id_E$ and by Proposition 2.6, we can easily prove it. □

4. Modal filters on modal equality algebras

In this section, we will introduce the modal filters and modal congruences of modal equality algebras and obtain some important results. Moreover, we show that there is one to one correspondence relation between modal filters and modal congruences of a modal equality algebra.

Definition 4.1. *Let (E, f) be a modal equality algebra. A non-empty subset $F \subseteq E$ is called a modal filter of (E, f), if F is a filter of E such that $x \in F$ implies $f(x) \in F$, for all $x \in E$.*

We denote the set of all modal filters of (E, f) by $MF(E, f)$.

Example 4.2. Let (E, f) be a modal equality algebra. Then $Ker(f)$ is a modal filter of (E, f).

Example 4.3. Consider the Example 3.3, we can easily check that the modal filter of (E, f) are $\{1\}, \{a, 1\}, \{b, 1\}, \{a, b, c, 1\}$ and E.

Example 4.4. Consider the Example 3.4, we can also easily check that the modal filter of (E, f) are $\{1\}, \{b, 1\}, \{a, b, 1\}$ and E.

Proposition 4.5. *Let (E, f) be a modal equality algebra.*

(1) *If F is a filter of $f(E)$, then $f^{-1}(F)$ is a modal filter of (E, f);*

(2) *If F is a modal filter of (E, f), then $f(F)$ is a filter of $f(E)$.*

Proof. (1) Suppose that F is a filter of $f(E)$. Obviously, $1 \in f^{-1}(F)$. Let $x, y \in E$ such that $x \in f^{-1}(F)$ and $x \leq y$. Then $f(x) \leq f(y)$. Since $f(x) \in F$ and $f(y) \in f(E)$, we can obtain that $f(y) \in F$, that is, $y \in f^{-1}(F)$. If $x, x \sim y \in f^{-1}(F)$. Then $f(x), f(x \sim y) \in F$. Since $f(x \sim y) \leq f(x) \sim f(y)$, then $f(x) \sim f(y) \in F$, thus $f(y) \in F$, that is $y \in f^{-1}(F)$. Therefore, $f^{-1}(F)$ is a filter of E.

If $x \in f^{-1}(F)$, then $f(x) \in F$, so $f(f(x)) = f(x) \in F$, that is, $f(x) \in f^{-1}(F)$. Therefore, $f^{-1}(F)$ is a modal filter of (E, f).

(2) First, we have $f(F) = F \cap f(E)$. Indeed, if $x \in F \cap f(E)$, then we have that $x \in F$ and $x \in f(E)$. Hence $f(x) \in f(F)$ and $f(x) = x$. Thus, we have $x \in f(F)$. It follows that $F \cap f(E) \subseteq f(F)$. Conversely, if $y \in f(F)$, then there exists $x \in F$ such that $y = f(x)$. Since F is a modal filter of (E, f), we have $y = f(x) \in F$. Hence $y \in F \cap f(E)$, $f(F) \subseteq F \cap f(E)$. Therefore, $f(F) = F \cap f(E)$.

Obviously, $1 \in f(F) = F \cap f(E)$. If $x \in f(F) = F \cap f(E), y \in f(E)$ such that $x \leq y$, then $y \in f(F) = F \cap f(E)$. If $x, x \sim y \in f(F) = F \cap f(E)$, then $y \in F$. Since F is a modal filter of (E, f), we have $f(y) \in F \cap f(E)$ and then $y \in f(F) = F \cap f(E)$. Therefore, $f(F)$ is a filter of $f(E)$. □

For any modal filter F of modal equality algebra (E, f). Defined by $f_F : E/F \longrightarrow E/F$ as a mapping $f_F([x]) = [f(x)]$.

Proposition 4.6. *Let (E, f) be a modal equality algebra and F be a modal filter of modal equality algebra (E, f). Then f_F is a modal operator on E/F.*

Proof. First, we will show that f_F is well defined. Indeed, assume that $[x] = [y]$ for $x, y \in E$. Then $(x, y) \in \theta_F$, i.e., $x \sim y \in F$. Since F is a modal filter, then we have $f(x \sim y) \in F$. Now, by $(f4)$ of Definition 3.1, $f(x \sim y) \leq f(x) \sim f(y)$, we have $f(x) \sim f(y) \in F$. Thus, $(f(x), f(y)) \in \theta_F$. And then $[f(x)] = [f(y)]$. Therefore, f_F is well defined. Next, we will prove f_F fulfill Definition 3.1:

$(f1)$ $[x] \leq [f(x)] = f_F([x])$;

$(f2)$ $f_F([x]) = [f(x)] = [f(f(x))] = f_F f_F([x])$;

$(f3)$ $f_F([x] \wedge [y]) = [f(x \wedge y)] = [f(x) \wedge f(y)] = f_F([x]) \wedge f_F([y])$;

$(f4)$ $f_F([x] \sim [y]) = [f(x \sim y)] \leq [f(x) \sim f(y)] = f_F([x]) \sim f_F([y])$.

Therefore, f_F is a modal operator on E/F. □

Above them, we can obtain that $(E/F, f_F)$ is a modal equality algebra.

Proposition 4.7. *In the modal equality algebra* $(E/Ker(f), f_{Ker(f)})$*, we obtain:*
(1) $[x] \leq [y]$ *iff* $f(x \sim (x \wedge y)) = 1$ *iff* $f(x \to y) = 1$*;*
(2) $[x] = [y]$ *iff* $f(x \sim y) = 1$.

Proof. (1) We have $[x] \leq [y]$ iff $[x] \bullet [y] = [x]$ iff $[x] = [x \wedge y]$ iff $[x] \rightharpoonup [x \wedge y] = [1]$ iff $[x \sim x \wedge y] = [1]$ iff $x \sim x \wedge y \in Ker(f)$ iff $f(x \sim x \wedge y) = 1$ iff $f(x \to y) = 1$.

(2) $[x] = [y]$ iff $[x] \rightharpoonup [y] = [1]$ iff $[x \sim y] = [1]$ iff $x \sim y \in Ker(f)$ iff $f(x \sim y) = 1$. □

Definition 4.8. *Let* (E, f) *be a modal equality algebra and* θ *be a congruence on* E*. Then* θ *is called a modal congruence on* (E, f)*, if* $(x, y) \in \theta$ *implies* $(f(x), f(y)) \in \theta$ *for each* $x, y \in E$.

We denote the set of all modal congruences of (E, f) by $MC(E, f)$.

Next, we will show that there is one to one correspondence relation between modal filters and modal congruences of a modal equality algebra. For obtaining this important result, we need the following proposition.

Proposition 4.9. *Let* (E, f) *be a modal equality algebra and* θ *be a modal congruence on* (E, f)*. Then the following statements hold:*
(1) $[1]_\theta = \{x \in E \mid (x, 1) \in \theta\}$ *is a modal filter of* (E, f)*;*
(2) $(x, y) \in \theta$ *iff* $(x \sim y, 1) \in \theta$.

Proof. (1) Clearly, $[1]_\theta = \{x \in E \mid (x, 1) \in \theta\}$ is a filter of E. Now for each $x \in [1]_\theta$, we have $(x, 1) \in \theta$. Hence $(f(x), f(1)) \in \theta$ and then $(f(x), 1) \in \theta$. Therefore, $f(x) \in [1]_\theta$. This proves that $[1]_\theta$ is a modal filter on (E, f).

(2) If $(x, y) \in \theta$, then $(x \sim x, x \sim y) \in \theta$, that is, $(x \sim y, 1) \in \theta$. Conversely, let $(x \sim y, 1) \in \theta$, then $[x \sim y]_\theta = [1]_\theta$ and then $[x]_\theta \sim [y]_\theta = [1]_\theta$, $[x]_\theta = [y]_\theta$. Hence, $(x, y) \in \theta$. □

Theorem 4.10. *Let* (E, f) *be a modal equality algebra. Then there exists one to one correspondence between* $MF(E, f)$ *and* $MC(E, f)$.

Proof. Define two mappings as follows, $f : MF(E, f) \mapsto MC(E, f)$ by $F \to \theta_F$ and $g : MC(E, f) \mapsto MF(E, f)$ by $\theta \to [1]_\theta$. Let $\theta \in MC(E, f)$. Clearly, $[1]_\theta = \{x \in E \mid (x, 1) \in \theta\}$ is a filter of E. Now for each $x \in [1]_\theta$, we have $(x, 1) \in \theta$. Hence $(f(x), f(1)) \in \theta$ and then $(f(x), 1) \in \theta$. Therefore, $f(x) \in [1]_\theta$. This proves that $[1]_\theta$ is a modal filter on (E, f). Conversely, assume that $F \in MF(E, f)$. Obviously, θ_F is a congruence on E, where $(x, y) \in \theta_F$ iff $x \sim y \in F$. Then for each $(x, y) \in \theta_F$, we have $x \sim y \in F$. Since F is a modal filter, then $f(x \sim y) \in F$. By $(f4)$ of Definition 3.1, we obtain $f(x) \sim f(y) \in F$, thus $(f(x), f(y)) \in \theta_F$. Therefore, θ_F is a modal congruence of (E, f). By the above arguments, we have that f and g are well defined.

For any congruence θ on (E, f), we have $(f \circ g)(\theta) = f([1]_\theta) = \theta_{[1]_\theta}$. By Proposition 4.9, we have $(x, y) \in \theta_{[1]_\theta}$ iff $x \sim y \in [1]_\theta$ iff $(x \sim y, 1) \in \theta$ iff $(x, y) \in \theta$. Hence, $\theta_{[1]_\theta} = \theta$. That is, $f \circ g = id_{MC(E,f)}$. Similarly, we have $g \circ f = id_{MF(E,f)}$. Therefore, there exists one to one correspondence between $MF(E, f)$ and $MC(E, f)$. □

5. Uniformity structures on modal equality algebras

In this section, we will consider the uniformity structures on the modal equality algebra as an application of the strong modal filters of modal equality algebras. For this, we will introduce the concept of strong modal filters of modal equality algebras firstly.

Definition 5.1. *Let (E, f) be a modal equality algebra. A non-empty subset $F \subseteq E$ is called a strong modal filter of (E, f), if F is a filter of E such that $x \in F$ iff $f(x) \in F$, for all $x \in E$.*

We denote the set of all strong modal filters of (E, f) by $SMF(E, f)$.

Note that for each modal equality algebra (E, f), we can easily check that each strong modal filter of (E, f) is a modal filter, not vice versa. Next, we will give an example to proof this result.

Example 5.2. Let E be an equality algebra such that $|E| > 1$ and a mapping f be defined by $f(x) = 1, \forall x \in E$. Then we can check that (E, f) is a modal equality algebra. Now, we can see that $F = \{1\}$ is a modal filter but is not a strong modal filter. Since for $x \neq 1, x \in E$, we have $f(x) = 1 \in F$, but $x \notin F$.

In the next, we will consider the uniformity structure on the modal equality algebra as an application of the strong modal filters of modal equality algebras.

Theorem 5.3. *Let (E, f) be a modal equality algebra and F be a strong modal filter of modal equality algebra (E, f). Define $U_F = \{(x, y) \in E \times E \mid f(x \sim y) \in F\}$ and $K^\star = \{U_F \mid F \in SMF(E, f)\}$. Then $K^\star$ satisfies the conditions $(A1) - (A4)$ of Definition 2.17.*

Proof. Now, we will prove that $K^\star$ satisfies the conditions $(A1) - (A4)$:

(A1) Let $U_F \in K^\star, (x, x) \in \triangle$. As $x \sim x = 1$, then $f(x \sim x) = f(1) = 1 \in F$. Thus we have $(x, x) \in U_F$. Thus, $\triangle \subseteq U_F$. Therefore, $(A1)$ holds.

(A2) If $U_F \in K^\star$, We know that $(x, y) \in U_F$ iff $f(x \sim y) \in F$ iff $f(y \sim x) \in F$ iff $(y, x) \in U_F$ iff $(x, y) \in U_F^{-1}$. Thus, $U_F^{-1} = U_F$. Thus, $U_F^{-1} \in K^\star$. Therefore, $(A2)$ holds.

(A3) Let $\Lambda(F) = \{F_a \mid F_a \subseteq F\}$ be the collection of strong modal filters contained in F. Clearly, $\Lambda(F)$ is not empty. Let G be strong modal filter generated by $\cup F_a$. Then $U_G \in K^\star$. Next, we will show that $U_G \circ U_G \subseteq U_F$:

If $(x, y) \in U_G \circ U_G$, then there exists $z \in E$ such that $(x, z) \in U_G$ and $(z, y) \in U_G$. And we know $f(x \sim z) \in G$ and $f(z \sim y) \in G$. Then by G is a strong modal filter, we have $x \sim z \in G$ and $z \sim y \in G$. By $(E7)$ of Definitin 2.1, we have $(x, y) \in G$. That is, $f(x \sim y) \in G$. Since G be strong modal filter generated by $\cup F_a$ and $\cup F_a \subseteq F$, then $G \subseteq F$. Thus $f(x \sim y) \in F$ and then $(x, y) \in U_F$. Therefore, $U_G \circ U_G \subseteq U_F$. Therefore, $(A3)$ holds.

(A4) We should proof that for all $U_F, U_G \in K^\star$, $U_F \cap U_G = U_{F \cap G}$. Indeed, if $(x, y) \in U_F \cap U_G$, then we have that $(x, y) \in U_F$ and $(x, y) \in U_G$. Hence $f(x \sim y) \in F$ and $f(x \sim y) \in G$. Thus, $f(x \sim y) \in F \cap G$ and then $(x, y) \in U_{F \cap G}$.

Therefore, $U_F \cap U_G \subseteq U_{F\cap G}$. Similarly, we obtain $U_F \cap U_G \supseteq U_{F\cap G}$. That is, $U_F \cap U_G = U_{F\cap G}$ holds. Therefore, $(A4)$ holds. $\Box$

Theorem 5.4. *Let (E, f) be a modal equality algebra. Define $\mathcal{K} = \{U \in E\times E \mid U_F \subseteq U,\ for\ some\ U_F \in K^\star\}$. Then $((E, f), \mathcal{K})$ is an uniform structure.*

Proof. By above Theorem 5.3, we can know that $\mathcal{K}$ fulfill the conditions $(A1)-(A4)$ of Definition 2.17. Now, we prove that $\mathcal{K}$ fulfill $(A5)$. If $U \in \mathcal{K}, U \subseteq V \subseteq E \times E$. Then there exists $U_F \in K^\star$ such that $U_F \subseteq U \subseteq V$. Thus, $V \in \mathcal{K}$. Therefore, $\mathcal{K}$ is an uniformity on (E, f) and hence $((E, f), \mathcal{K})$ is an uniform structure. $\Box$

Now, we define $U[x] = \{y \in E \mid (x, y) \in U\}$, for all $x \in E, U \in \mathcal{K}$. Clearly, if $V \subseteq U$, then $V[x] \subseteq U[x]$. And then the next theorem shows that we can obtain a topology on the modal equality algebra (E, f).

Theorem 5.5. *Let (E, f) be a modal equality algebra. Define $T = \{Y \subseteq E \mid \forall\ x \in Y, \exists\ U \in \mathcal{K}, U[x] \subseteq Y\}$. Then T is a topology on the modal equality algebra (E, f).*

Proof. Obviously, $\emptyset, E \in T$. And it clear that T is closed under arbitrary union. Next, we will show that T is closed under finite intersection. Let $Y, W \in T$ such that $x \in Y \cap W$. Then there exist $U, V \in \mathcal{K}$ such that $U[x] \subseteq Y$ and $V[x] \subseteq W$. Now, let $N = U\cap V$, then $N \in \mathcal{K}$. And then $N[x] \subseteq U[x]\cap V[x] \subseteq Y\cap W$. Thus $N[x] \subseteq Y \cap W$, and then $Y \cap W \in T$. Therefore, T is a topology on (E, f). $\Box$

Note that for any $x \in E$, $U[x]$ is a neighborhood of x.

Let Σ be an arbitrary family of strong modal filters of an equality algebra E which is closed under intersection. Then the topology T_Σ obtained from Theorem 5.5 is called an uniform topology on (E, f) induced by Σ. And if $\Sigma = \{F\}$, we denote it by T_F.

In what following, we show that modal equality algebras with uniform topologies are topological modal equality algebras. Firstly, we give the definition as follows.

Definition 5.6. *Let (E, f) be a modal equality algebra and T be a topology on (E, f). Then $((E, f), T)$ is called a topological modal equality algebra if the operators $\wedge, \sim$ are continuous.*

Note that the operator $\diamond \in \{\wedge, \sim\}$ is continuous iff for any $x, y \in E$ and any neighborhood C of $x \diamond y$ there exist neighborhoods A and B of x and y, respectively, such that $A \diamond B \subseteq C$.

Theorem 5.7. *Let (E, f) be a modal equality algebra and Σ be an arbitrary family of strong modal filters of E which is closed under intersection. Then the structure $((E, f), T_\Sigma)$ is a topological modal equality algebra.*

Proof. By the definition of a topological modal equality algebra, we will show that $\diamond$ is continuous, where $\diamond = \{\wedge, \sim\}$. Assume $x \diamond y \in Y$, where $x, y \in E$ and Y is an open subset of E. Then there exists $U \in \mathcal{K}, U[x \diamond y] \subseteq Y$ and there exists a strong modal filter F such that $U_F \in K^\star$, $U_F \subseteq U$. Next, we will prove that the following relation holds:

$$U_F[x] \diamond U_F[y] \subseteq U_F[x \diamond y] \subseteq U[x \diamond y].$$

Let $h \diamond k \in U_F[x] \diamond U_F[y]$ such that $h \in U_F[x]$ and $k \in U_F[y]$. Hence, we have $(x, h) \in U_F$ and $(y, k) \in U_F$, and then $f(x \sim h) \in F, f(y \sim k) \in F$. Since F is a strong modal filter, we can obtain that $x \sim h \in F$ and $y \sim k \in F$. Therefore, $(x \diamond y) \sim (h \diamond k) \in F$. And then $f((x \diamond y) \sim (h \diamond k)) \in F$. Therefore $(x \diamond y, h \diamond k) \in U_F$ and then $(x \diamond y, h \diamond k) \in U$. Hence, $h \diamond k \in U_F[x \diamond y] \subseteq U[x \diamond y]$. That is, the above relation holds. Clearly, $U_F[x]$ and $U_F[y]$ are neighborhoods of x and y, respectively. Therefore, the operator $\diamond$ is continuous. Thus the structure $((E, f), T_\Sigma)$ is a topological modal equality algebra. $\square$

6. Conclusion

The study of equality algebras is motivated by the goal to develop appropriate algebraic semantics for fuzzy type theory as we mentioned in the introduction, so a concept of fuzzy type theory should be introduced. In this paper, we applied these ideas to propose the notion of modal operators on equality algebras. Also, we obtained some properties of modal equality algebras. As applications, we give a characterization of prelinear equality algebras. And we give the concepts of modal filters and modal congruences of equality algebras and obtain some related results. Moreover, we show that there is one to one correspondence relation between modal filters and modal congruences of a modal equality algebra. Finally, using strong modal filters, we establish the uniform structures on modal equality algebras. In particular, we prove that modal equality algebras with uniform topologies are topological modal equality algebras.

Future research will focus on characterizing modal filter generated by a subset of a modal equality algebra in terms of fuzzy equality operation and researching their related properties.

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References

[1] L.C. Ciungu, *Internal states on equality algebras*, Soft Computing, 19 (2015), 939-953.

[2] M. Harlenderová, J. Rachůnek, *Modal operators on MV-algebras*, Mathematica Bohem, 131 (2006), 39-48.

[3] S. Jenei, *Equality algebras*, Studia Logica, 100 (2012), 1201-1209.

[4] S. Jenei, *Equality algebras*, Proceedings of the CINTI2010 conference (11th IEEE International Symposium on Computational Intelligence and Informatics), November 18-20, 2010, Budapest, 2010.

[5] S. Jenei, L. Korodi, *Pseudo equality algebras*, Archive for Mathematical Logic, 52 (2013), 469-481.

[6] K.D. Joshi, *Introduction to general topology*, New Age International Publisher, India, 1983.

[7] Y.B. Jun, E.H. Ron, *On uniformities of BCK-algebras*, Communications of the Korean Mathematical Society, 10 (1995), 11-14.

[8] Y.B. Jun, H.S. Kim, *Uniformity structures in positive implicative algebras*, International Mathematical Journal, 2 (2002), 215-218.

[9] M. Kondo, *Modal operators on commutative residuated lattices*, Mathematica Slovaca, 61 (2011), 1-14.

[10] D.S. Macnab, *Modal operators on Heyting algebras*, Algebra Universalis, 12 (1981), 5-29.

[11] V. Novák, B. De Baets, *EQ-algebras*, Fuzzy Sets Systems, 160 (2009), 2956-2978.

[12] V. Novák, *On fuzzy type theory*, Fuzzy Sets Systems, 149 (2005), 235-273.

[13] V. Novák, *EQ-algebra based fuzzy type theory and its exetensions*, Logic Journal of the IGPL, 19 (2011), 512-542.

[14] V. Novák, *Fuzzy type theory as higher-order fuzzy logic*, Proceedings of the 6th International Conference on Intelligent Technologies, 2005.

[15] J. Rachůnek, D. Šalounová, *Modal operators on bounded commutative residuated* $R\ell-monoids$, Mathematica Slovaca, 57 (2007), 321-332.

[16] A.B. Saeid, H. Babaei, M. Haveshki, *Uniform Topology on Hilbert algebras*, Kyungpook Mathematical Journal, 45 (2005), 405-411.

[17] J.T. Wang, X.L. Xin, Y.B. Jun, *Very true operations on equality algebras*, Journal of Computational Analysis and Applications, 24 (2018), 507-521.

[18] F. Zebardast, R.A. Borzooei, M. Aaly Kologani, *Results on equality algebras*, Information Science, 381 (2017), 270-282.

[19] M. El-Zekey, V. Novák, R. Mesiar, *On good EQ-algebras*, Fuzzy Sets and Systems, 178 (2011), 1-23.

[20] M. El-Zekey, *Representable good EQ-algebras*, Soft Computing, 14 (2010), 1011-1023.

[21] X. Zhu, J. Yang, A.B. Saeid, *Extreme fuzzy filters and its application on BL-algebras*, Journal of Intelligent and Fuzzy Systems, 33 (2017), 3733-3741

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