Nowhere-zero 3-flows in 4-connected simple graphs with independence number 3

Xiaoxia Zhang*
Department of Mathematics
Xinyang Normal University
Xinyang, 464000
China
xiaoxiazhang_1983@163.com

Yanhua Yang†
Department of Information Engineering
Xinyang Agriculture and Forestry University
Xinyang 464000
China
hl4653312008@126.com

Abstract. It is mainly proved in this paper that every 4-connected simple graph with $\alpha(G) = 3$ admits a nowhere-zero 3-flow, which is a partial result to Tutte's 3-flow conjecture and generalizes the result by Luo *et al.* [Graphs and Combin., (29)(2013)1899–1907] that characterized all graphs with independence number at most 2.
Keywords: nowhere zero 3-flow, independence number.

1. Introduction

Graphs considered in this paper are simple, finite without loops. Terminology and notations not defined here can be seen in the book [1] by Bondy and Murty.

For a graph G, a set S of vertices in G is *independent* if no two vertices of S are adjacent in G. The maximum cardinality of an independent set of vertices of G is called the *independence number* of G and denoted by $\alpha(G)$. If V is a vertex subset of G, we denote by $G - V$ the graph obtained from G by deleting all vertices in V and all incident edges. For two subsets $A, B \subseteq V(G)$, $e(A, B)$ denotes the number of edges with one endpoint in A and the other endpoint in B. For simplicity, if $A = \{v\}$, we write $e(v, B)$ to mean $e(\{v\}, B)$ and if H_1 and H_2 are two subgraphs of G, we write $e(H_1, H_2)$ to mean $e(V(H_1), V(H_2))$. The set of neighbors of x in G is denoted by $N_G(x)$, or simply $N(x)$. In addition, $N[x] = N(x) \cup \{x\}$.

The complete graph on n vertices is denoted by K_n. Let K_n^- denote the graph obtained from K_n by deleting an edge. The *wheel* W_k ($k \geq 2$) is the graph obtained from a k-cycle by adding a new vertex, called the *center* of the

∗. Corresponding author
†. Corresponding author

wheel, which is adjacent to every vertex of the k-cycle. We define W_k to be *odd* (even) if k is odd (or even, respectively).

Let D be an orientation of a graph G, and f a function from $E(G)$ to Z with $-k < f(e) < k$ for every $e \in E(G)$. Then the pair (D, f) is a k-*flow* if it satisfies the Kirchhoff condition $\sum_{e\in E^+(v)} f(e) = \sum_{e\in E^-(v)} f(e)$ for every vertex $v \in V(G)$, where $E^+(v)$ and $E^-(v)$ denote the sets of outgoing and incoming edges of v with respect to D. A k-flow (D, f) is *nowhere-zero* if $f(e) \neq 0$ for every $e \in E(G)$.

Nowhere-zero k-flow was first introduced by Tutte [15]. We shall restrict our attention to the case that $k = 3$ and this paper is mainly motivated by the well-known 3-flow conjecture of Tutte.

Conjecture 1.1 (Tutte [15])**.** *Every* 4*-edge-connected graph admits a nowhere-zero* 3*-flow.*

Let Z_k denote the cyclic group of order k. A graph G is Z_k*-connected* if G has an orientation D such that for any function $b : V(G) \to Z_k$ with $\sum_{v\in V(G)} b(v) = 0$, there is a function $f : E(G) \to Z_k - \{0\}$ such that $b(v) = \sum_{e\in E^+(v)} f(e) - \sum_{e\in E^-(v)} f(e)$ for any $v \in V(G)$, where "$\sum$" refers to the addition in Z_k.

Group connectivity was first introduced by Jaeger *et al.* [9] as a generalization of nowhere-zero flows. Clearly, if G is Z_3-connected, then G admits a nowhere-zero 3-flow.

Many authors are involved in the study of nowhere-zero 3-flows. In particular, Fan and Zhou [6, 7] used degree sum condition and Ore-condition to guarantee the existence of nowhere-zero 3-flows. Recently, Thomassen [14] first made a breakthrough in the 3-flow conjecture. He verified the following theorem.

Theorem 1.2. *Every 8-edge-connected graph admits a nowhere-zero* 3*-flow.*

This result was soon further improved by Lovasz *et al.* [13].

Theorem 1.3. *Every 6-edge-connected graph admits a nowhere-zero* 3*-flow.*

So far, Conjecture 1.1 is still open. Chvátal and Erdös [2] proved a classical result: for a 2-connected simple graph G, if $k(G) \geq \alpha(G)$, then G is hamiltonian. Recently, Han, Lai, Xiong and Yan [8] studied whether a graph with a slightly weaker Chvátal and Erdös condition is supereulerian. It is well-known that every hamiltonian graph is supereulerian which has a nowhere-zero 4-flow. In view of this, we hope to explore that which hamiltonian graphs admit a nowhere-zero 3-flow. Luo *et al.*[10] characterized all the graphs with independence number at most 2 that admit a nowhere-zero 3-flow. However, it is difficult to have a complete characterization for the case of independence number 3. In this paper, we study a slightly stronger Chvátal-Erdös condition (that is $k(G) \geq \alpha(G) + 1$) and nowhere zero 3-flows and show that Conjecture 1.1 holds for a family of graphs, as follows.

Theorem 1.4. *Every 4-connected simple graph with independence number 3 admits a nowhere-zero 3-flow.*

2. Preliminaries

In this section, we establish several lemmas. Some results in [3, 4, 5, 9, 11, 12] on group connectivity are summarized as follows.

Lemma 2.1. *Let A be an abelian group with $|A| \geq 3$. The following results are known.*

(1) K_n *and* K_n^- *are* A*-connected if* $n \geq 5$.

(2) C_n *is* A*-connected if and only if* $|A| \geq n+1$.

(3) W_{2k} *is* Z_3*-connected and* W_{2k+1} *is not* Z_3*-connected, where* k *is a positive integer.*

(4) L_3^+ *in Fig.* 1 *is* Z_3*-connected.*

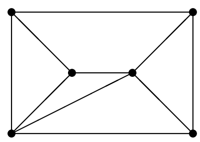

Fig. 1 The graph L_3^+

Let H be a subgraph of G. The *contraction* G/H denotes the graph obtained from G by identifying all vertices in H and then deleting all loops generating in this process.

Lemma 2.2 ([3, 4])**.** *Let* H *be a* Z_3*-connected subgraph of* G*. The following results hold.*

(1) *If* G/H *admits a nowhere-zero 3-flow, then* G *admits a nowhere-zero 3-flow.*

(2) *If* G/H *is* Z_3*-connected, then* G *is* Z_3*-connected.*

Lemma 2.3 ([16])**.** *Let* G *be a* 2*-connected simple graph with* $\delta(G) \geq 4$*. If* $\alpha(G) \leq 2$*, then* G *is* Z_3*-connected.*

Let G be a graph and let $v \in V(G)$ be a vertex of degree $m \geq 4$. Let $N(v) = \{v_1, v_2, \cdots, v_m\}$. The graph $G_{[vv_1,vv_2]}$ is obtained from G by deleting two edges vv_1 and vv_2 and adding a new edge v_1v_2. If $m = 2k$ is even, let

$$M =< \{v_1, v_{k+1}\}, \{v_2, v_{k+2}\}, \cdots, \{v_k, v_{2k}\} >$$

be a way to pair the vertices in $N(v)$. $G_{(v,M)}$ denotes the graph obtained from $G - v$ by adding k new edges v_iv_{k+i}, where $1 \leq i \leq k$.

Lemma 2.4. *Let G be a graph and let $v \in V(G)$ be a vertex of degree $m \geq 4$. Let $N(v) = \{v_1, v_2, \cdots, v_m\}$. Each of the following statements holds.*

(1) *(Lai, [11]) If $G_{[vv_1,vv_2]}$ is Z_3-connected, then G is Z_3-connected.*

(2) *If $m = 2k$ is even and $G_{(v,M)}$ admits a nowhere-zero 3-flow, then G admits a nowhere-zero 3-flow.*

Proof. (2) Let f^* be a nowhere-zero 3-flow of $G_{(v,M)}$. If the newly added edge v_iv_{k+i} is oriented from v_i to v_{k+i}, then the two edges vv_i and vv_{k+i} are oriented from v_i to v and from v to v_{k+i}, respectively.

Define $f : E(G) \to \{1, 2\}$ as

$$f(e) = \begin{cases} f^*(e), & \text{if } e \in E(G) - \{v_1v_{k+1}, v_2v_{k+2}, \cdots, v_kv_{2k}\} \\ f^*(v_iv_{k+i}), & \text{if } e \in \{vv_i, vv_{k+i}\}(1 \leq i \leq k). \end{cases}$$

Then f is a nowhere-zero 3-flow of G. □

Lemma 2.5. *Let G be a simple connected graph with $\delta(G) = 3$ and $\alpha(G) = 2$. If G is not Z_3-connected, then G is one of 19 graphs shown in Fig. 2 or G can be Z_3-contracted to one graph in $\{K_4^{'}, K_4^{''}\}$, where $K_4^{'}$ and $K_4^{''}$ denote the graphs obtained by connecting one edge and two edges between K_1 and K_4, respectively.*

Proof. Let G be a simple graph with $\delta(G) = 3$. Then $\kappa^{'}(G) \leq \delta(G) = 3$. If $\kappa^{'}(G) = 3$, by [Theorem 1.3, 16], $G = H_i$, where $1 \leq i \leq 19$ but $i \notin \{15, 16\}$. If $\kappa^{'}(G) = 1$, let e be a cut edge of G, then $G - e$ has two connected components which are complete graphs since $\alpha(G) = 2$. So $G = H_{15}$ or G can be Z_3-contracted to $K_4^{'}$. Similarly we can prove that if $\kappa^{'}(G) = 2$, then $G = H_{16}$ or G can be Z_3-contracted to $K_4^{''}$. □

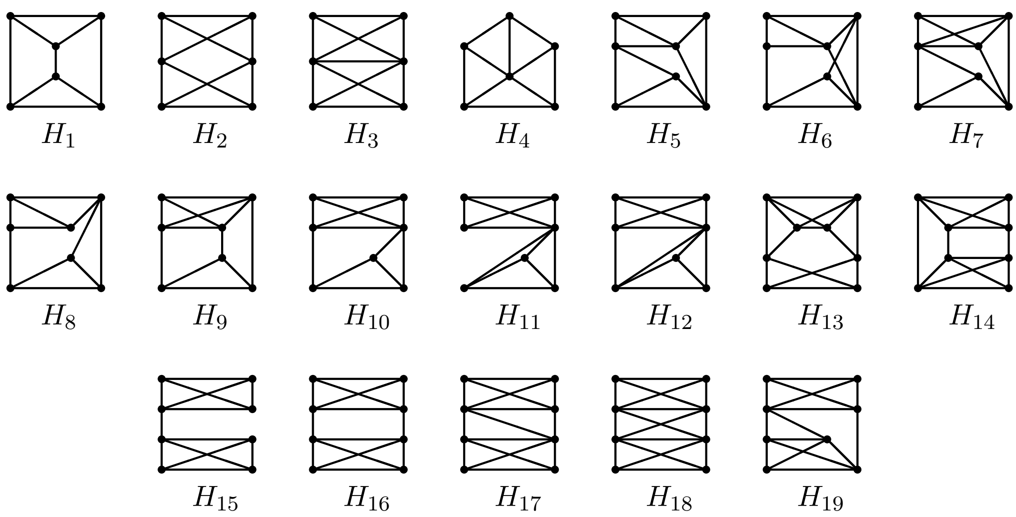

Fig. 2 19 graphs for Lemma 2.5

Lemma 2.6. *Let G be a 4-edge-connected simple graph with $\alpha(G) \le 3$. If G contains a maximal Z_3-connected subgraph H and $\alpha(G-H) = 2$, then G admits a nowhere-zero 3-flow.*

Proof. Let $G^* = G/H$. Then G^* is also a 4-edge-connected simple graph. If G^* is Z_3-connected, then G admits a nowhere-zero 3-flow follows from Lemma 2.2. Next we assume that G^* is not Z_3-connected. By Lemma 2.3, $\alpha(G^*) = 3$. Let v^* be a vertex into which H is contracted. Then $d(v^*) \ge 4$ and $\alpha(G^* - v^*) = \alpha(G-H) = 2$. If $e(v^*, G^* - v^*) = 4$, we consider the graph $G_{(v^*,M)}$, which may contains 2-cycles, where M is any way to pair the vertices in $N(v^*)$. We contract all generating 2-cycles, the resulting graph is denoted by $G^*_{(v^*,M)}$. It is clear that $G^*_{(v^*,M)}$ is a 4-edge connected simple graph and $\alpha(G^*_{(v^*,M)}) \le 2$. By Lemma 2.3, $G^*_{(v^*,M)}$ is Z_3-connected. It follows from Lemmas 2.1 and 2.2 that $G_{(v^*,M)}$ is also Z_3-connected. Hence G admits a nowhere-zero 3-flow follows from Lemma 2.4. Next we suppose that $e(v^*, G^* - v^*) \ge 5$. Note that $\alpha(G^*) = 3$. There must be two vertices in $G^* - v^*$, say v_1 and v_2, such that $\{v^*, v_1, v_2\}$ is an independent set of G^*. This implies that $|V(G^* - v^*)| \ge 7$ and $d(v_i) \ge 4$, where $i \in \{1, 2\}$. If $G^* - v^*$ is not connected, then $G^* - v^*$ has just two connected components which are complete graphs with order more than 4 since $\alpha(G^* - v^*) = 2$. Let G_1 and G_2 be two connected components of $G^* - v^*$. Then $e(v^*, G_1) \ge 2$ or $e(v^*, G_2) \ge 2$. It is clear that $G_1 + v^*$ or $G_2 + v^*$ is Z_3-connected which is contrary to the maximum of H. Hence $G^* - v^*$ is connected and $\delta(G^* - v^*) \ge 3$. It follows from Lemma 2.5 that $G^* - v^* \in \{H_5, H_7, H_{13}, H_{14}, H_{16}, H_{17}, H_{19}\}$ or can be Z_3-contracted to one graph in $\{K_4^{'}, K_4^{''}\}$. It is a routine work to prove that G^* is Z_3-connected, a contradiction. Therefore this lemma holds. □

3. Proof of the main theorem

Proof of Theorem 1.4. Theorem 1.3 tells us that if $\kappa^{'}(G) \ge 6$, then G admits a nowhere-zero 3-flow. So we only need to show that this theorem holds for $4 = \kappa(G) \le \kappa^{'}(G) \le 5$. Let A be a maximal independent set of G and $B = V(G) - A$. Then $|A| = \alpha(G) = 3$ and $|B| \ge 4$ since G is a 4-connected simple graph. It follows that $|V(G)| \ge (3 + 4) = 7$. Next we proceed our proof by induction on $|V(G)|$.

Let $|V(G)| = 7$. Then $|B| = 4$ and $N(x) = B$ for any vertex $x \in A$. Hence G contains $K_{3,4}$ as a spanning subgraph. Furthermore, we get that $G[B]$ is connected. Otherwise A is a cut of G, contrary to that $\kappa(G) = 4$. So there exist two adjacent edges uv and vw in $G[B]$. Let $x, y \in A$. Then $G[\{x, y, u, v, w\}]$ contains W_4 (center at v) as a subgraph. Contracting W_4 and all 2-cycles generating in this process, we can get a K_1. It follows from Lemma 2.1 that G is Z_3-connected, of course, admits a nowhere-zero 3-flow.

Suppose that the theorem is true for any graph $G^{'}$ with $|V(G^{'})| < |V(G)|$. Then we consider the graph G.

Case 1. There exist one vertex $x \in V(G)$ such that $d(x) = 4$.

In this case, we concentrate our attention on the graph $G_{(x,M)}$, where M is a way to pair the vertices in $N(x)$. By Lemma 2.4, if $G_{(x,M)}$ admits a nowhere-zero 3-flow, then G admits a nowhere-zero 3-flow. Therefore we suppose that there is no such M that $G_{(x,M)}$ admits a nowhere-zero 3-flow.

Claim 1. $\alpha(G_{(x,M)}) = 3$.

Proof of Claim 1. Note that $\delta(G_{(x,M)}) \geq \delta(G) = 4$ for any way to pair the vertices in $N(x)$. It follows from Lemma 2.3 that if $\alpha(G_{(x,M)}) \leq 2$, then $G_{(x,M)}$ admits a nowhere-zero 3-flow, a contradiction. Hence $3 \leq \alpha(G_{(x,M)}) \leq \alpha(G) = 3$, that is $\alpha(G_{(x,M)}) = 3$.

Note that $4 = \kappa(G) \leq \kappa^{'}(G) \leq \delta(G) = 4$. We have that $\kappa^{'}(G) = 4$. Let S be a minimal vertex cut of G which contains x. Then $|S| \geq 4$. Let G_1 and G_2 be two components(some component maybe not connected) of $G - S$ and $N(x) = \{x_1, x_2, x_3, x_4\}$. Denote by $N_1 = N(x) \cap V(G_1)$, $N_2 = N(x) \cap V(G_2)$ and $N_s = N(x) \cap S$. We assume, without loss of generality, that $|N_1| = 0$. Then $|N_2| \neq 0$, $|N_s| \neq 0$ and $|S| \geq 5$. If $|N_s| \leq 2$, there must be a way M which pairs the vertices of $N(x)$ such that $G_{(x,M)}$ (or $G^*_{(x,M)}$) is simple and $\kappa(G_{(x,M)}) = 4$ (or $\kappa(G^*_{(x,M)}) = 4$), where $G^*_{(x,M)}$ denotes the graph obtained from $G_{(x,M)}$ by contracting some 2-cycles. By Claim 1 and the induction hypothesis, $G_{(x,M)}$ (or $G^*_{(x,M)}$) admits a nowhere-zero 3-flow, a contradiction. Hence $|N_s| = 3$ and $|N_2| = 1$. If $G[N_s] \neq K_3$, as the above, we are done. Next we suppose that $G[N_s] = K_3$. If $e(N_2, N_s) \geq 1$, then $G_{(x,M)}[N(x)]$ is Z_3-connected. Note that $\alpha(G - N[x]) \leq 2$. It follows from Lemma 2.6, $G_{(x,M)}$ admits a nowhere-zero 3-flow, a contradiction. So $e(N_2, N_s) = 0$. Let $N_s = \{x_1, x_2, x_3\}$ and $N_2 = \{x_4\}$. Choose $M = \langle\{x_1, x_2\}, \{x_3, x_4\}\rangle$, then $G_{(x,M)}[N_s]$ is Z_3-connected. Let H be the maximal Z_3-connected subgraph which contains $G_{(x,M)}[N_s]$. Next we shall prove that $\alpha(G_{(x,M)} - H) \leq 2$. Note that $\alpha(G - N[x]) = 2$. Let $v_1, v_2 \in V(G - N[x])$ and $v_1v_2 \notin E(G)$. If $\{v_1, v_2, x_4\}$ is an independent set of three vertices, then $N_s \subseteq N(v_1) \cup N(v_2)$. This implies that $v_1 \in H$ or $v_2 \in H$. Therefore $\alpha(G_{(x,M)} - H) \leq 2$. By Lemma 2.6, $G_{(x,M)}$ admits a nowhere-zero 3-flow, a contradiction. Next we suppose that $|N_1| \geq 1$ and $|N_2| \geq 1$. It follows that $|N_s| \leq 2$. We discuss it in three cases.

Subcase 1.1. $|N_s| = 0$.

If $|N_1| = 1$, $|N_2| = 3$ or $|N_1| = 3$, $|N_2| = 1$, as the above, we are done. So we suppose that $N_1 = \{x_1, x_3\}$, $N_2 = \{x_2, x_4\}$. Choose $M = \langle\{x_1, x_2\}, \{x_3, x_4\}\rangle$, then $G_{(x,M)}$ is simple and $\kappa(G_{(x,M)}) = 4$. By Claim 1 and the induction hypothesis, $G_{(x,M)}$ admits a nowhere-zero 3-flow, a contradiction.

Subcase 1.2. $|N_s| = 1$.

We assume, without loss of generality, that $N_1 = \{x_1, x_3\}$, $N_2 = \{x_2\}$ and $N_s = \{x_4\}$. If $x_1x_4 \notin E(G)$ or $x_3x_4 \notin E(G)$, choose $M = \langle\{x_1, x_4\}, \{x_2, x_3\}\rangle$ or $M = \langle\{x_3, x_4\}, \{x_1, x_2\}\rangle$, then $G_{(x,M)}$ is simple and $\kappa(G_{(x,M)}) = 4$. By

Claim 1 and the induction hypothesis, $G_{(x,M)}$ admits a nowhere-zero 3-flow, a contradiction. Next we suppose that $x_1x_4 \in E(G)$ and $x_3x_4 \in E(G)$.

If $x_1x_3 \in E(G)$, we consider the graph $G_{(x,M)}$, where M is any way to pair the vertices in $N(x)$. It is clear that $G_{(x,M)}[\{x_1, x_3, x_4\}]$ is Z_3-connected. Let H be the maximal Z_3-connected subgraph which contains $G_{(x,M)}[\{x_1, x_3, x_4\}]$. If $\alpha(G_{(x,M)} - H) \leq 2$, by Lemma 2.6, $G_{(x,M)}$ admits a nowhere-zero 3-flow, a contradiction. Hence $\alpha(G_{(x,M)} - H) = 3$. At first, we claim that $x_2x_4 \notin E(G)$. For otherwise, $x_2 \in V(H)$. Thus $\alpha(G_{(x,M)} - H) \leq \alpha(G_{(x,M)} - N(x)) = \alpha(G - N[x]) \leq 2$, a contradiction. If G_1 is a complete graph, then $G_{(x,M)}[V(G_1) \cup \{x_4\}] \subset H$. If $y \in S - V(H)$, then $e(y, G_1) \leq 1$. Note that $\alpha(G_2) \leq 2$. Then $\alpha(G_2 + y) \leq 2$ since $\alpha(G) = 3$. Therefore $\alpha(G_{(x,M)} - H) \leq 2$, a contradiction. Next we assume that G_1 is not a complete graph. Then $\alpha(G_1) = 2$ and $\alpha(G_1 - x_1x_3) = \alpha(G_2) = 1$. Furthermore, there must be $u \in V(G_1 - x_1x_3)$ and $v \in S - \{x, x_4\}$ such that $\{u, v, x_2\}$ is an independent set of three vertices. Note that $e(x_2, \{x_1, x_3, x_4\}) = 0$. Then $\{x_1, x_3, x_4\} \subset N(u) \cup N(v)$. So $u \in H$ or $v \in H$. This yields that $\alpha(G_{(x,M)} - H) \leq 2$, a contradiction.

Next we suppose that $x_1x_3 \notin E(G)$. Then G_2 is a complete graph. If $x_2x_4 \in E(G)$, choose $M = \langle\{x_1, x_3\}, \{x_2, x_4\}\rangle$, we consider the graph $G_{(x,M)}$. If there exist one vertex $y' \in S - \{x, x_4\}$ such that $y' \in N(x_2) \cap N(x_4)$ and $y'x_3 \in E(G)$ or $y'x_1 \in E(G)$, then $G[\{x, x_1, x_2, x_4, y'\}] = W_4$ or $G[\{x, x_2, x_3, x_4, y'\}] = W_4$ which is Z_3-connected. So $G[N[x] \cup \{y'\}]$ is Z_3-connected. Note that $\alpha(G - N[x]) \leq 2$. It follows from Lemma 2.6, G is Z_3-connected. Next we assume that $y'x_3 \notin E(G)$ and $y'x_1 \notin E(G)$. Then $G_2 + y$ is also a complete graph since $\alpha(G) = 3$. Choose $M = \langle\{x_1, x_3\}, \{x_2, x_4\}\rangle$, we consider the graph $G_{(x,M)}$. It is clear that $G_{(x,M)}[V(G_2) \cup \{x_4, y'\}]$ is Z_3-connected. Let H' be the maximal Z_3-connected subgraph which contains $G_{(x,M)}[V(G_2) \cup \{x_4, y'\}]$. If $|V(G_2)| \geq 2$, then $\alpha(G_{(x,M)} - H) \leq 2$. By Lemma 2.6, $G_{(x,M)}$ is Z_3-connected, a contradiction. Hence $|V(G_2)| = 1$. Denote by $G^*_{(x,M)} = G_{(x,M)}/H'$. We find that $G^*_{(x,M)}$ is a 4-connected simple graph with $\alpha(G^*_{(x,M)}) \leq 3$. By Lemma 2.3 or the induction hypothesis, $G^*_{(x,M)}$ admits a nowhere-zero 3-flow. It follows from Lemma 2.2 that $G_{(x,M)}$ also admits a nowhere-zero 3-flow, a contradiction. Hence we suppose that $N(x_2) \cap N(x_4) \cap S = \{x\}$. Similarly if $N(x_2) \cap N(x_4) \cap V(G_2) \neq \emptyset$, we can also get that $G_{(x,M)}$ is Z_3-connected, a contradiction. So $N(x_2) \cap N(x_4) \cap V(G_2) = \emptyset$. Note that $\kappa(G) = 4$. Then $|S| \geq 5$. Thus $G_{(x,M)}$ is also a 4-connected simple graph. By induction hypothesis, $G_{(x,M)}$ admits a nowhere-zero 3-flow, a contradiction. Therefore $x_2x_4 \notin E(G)$. If $|S| \geq 5$, as the above, we are done. Hence we assume that $|S| = 4$. If $N(x_1) \cap N(x_4) = \{x\}$ or $N(x_3) \cap N(x_4) = \{x\}$, choose $M = \langle\{x_1, x_4\}, \{x_2, x_3\}\rangle$ or $M = \langle\{x_1, x_2\}, \{x_3, x_4\}\rangle$, then $G_{(x,M)}$ contains only one 2-cycle. Contracting the 2-cycle, the resulting graph is denoted by $G^*_{(x,M)}$. It is easy to prove that $G^*_{(x,M)}$ is also a 4-connected simple with $\alpha(G) = 3$. By induction hypothesis, $G^*_{(x,M)}$ admits a nowhere-zero 3-flow. It follows from Lemma 2.2 that $G_{(x,M)}$, a contradiction. Hence $|N(x_1) \cap N(x_4)| \geq 2$ and $|N(x_3) \cap N(x_4)| \geq 2$. Choose $M = \langle\{x_1, x_4\}, \{x_2, x_3\}\rangle$

or $M = \langle\{x_1, x_2\}, \{x_3, x_4\}\rangle$. It is a routine work to prove that there exist a vertex subset $x_4 \in A \subseteq S$ such that $G_{(x,M)}[V(G_1) \cup A]$ is Z_3-connected and $\alpha(G_{(x,M)} - G_{(x,M)}[V(G_1) \cup A]) \leq 2$. It follows from Lemma 2.6 that $G_{(x,M)}$ is Z_3-connected, a contradiction.

Subcase 1.3. $|N_s| = 2$.

If $|N_s| = 2$, then $|N_1| = 1$ and $|N_2| = 1$. Let $N_1 = \{x_1\}$, $N_2 = \{x_2\}$ and $N_s = \{x_3, x_4\}$.

Claim 2. $x_3x_4 \in E(G)$.

Proof of Claim 2. If $x_3x_4 \notin E(G)$, choose $M = \langle\{x_1, x_2\}, \{x_3, x_4\}\rangle$, then $G_{(x,M)}$ is simple and $\kappa(G_{(x,M)}) = \kappa(G) = 4$. By the induction hypothesis, $G_{(x,M)}$ admits a nowhere-zero 3-flow, a contradiction.

It is clear that $\alpha(G_1) \leq 1$ or $\alpha(G_2) \leq 1$ since $\alpha(G_1) + \alpha(G_2) \leq 3$. We assume, without loss of generality that $G_1 = K_m$.

Claim 3. $m \geq 2$.

Proof of Claim 4. If $m = 1$, then $N(x_1) = S$. Choose $M = \langle\{x_1, x_3\}, \{x_2, x_4\}\rangle$ or $M = \langle\{x_1, x_4\}, \{x_2, x_3\}\rangle$, we consider the graph $G_{(x,M)}$. It is clear that $G_{(x,M)}[\{x_1, x_3, x_4\}]$ is Z_3-connected. Let H be the maximal Z_3-connected subgraph which contains $G_{(x,M)}[\{x_1, x_3, x_4\}]$. If $\alpha(G_{(x,M)} - H) \leq 2$, by Lemma 2.6, $G_{(x,M)}$ is Z_3-connected, a contradiction. So $\alpha(G_{(x,M)} - H) = 3$. If $x_2x_3 \in E(G)$ or $x_2x_4 \in E(G)$, then $x_2 \in H$. Note that $\alpha(G_{(x,M)} - H) \leq \alpha(G - G[N[x]]) = 2$, a contradiction. Hence $x_2x_3 \notin E(G)$ and $x_2x_4 \notin E(G)$. Let $y \in S - H - \{x\}$. Then $yx_3 \notin E(G)$ and $yx_4 \notin E(G)$. If there exist $z \in V(G_2 - x_2)$ such that $\{y, z, x_2\}$ is an independent set of three vertices, then $x_3z \in E(G)$ and $x_4z \in E(G)$. If not, $\{y, z, x_2, x_3\}$ or $\{y, z, x_2, x_4\}$ is an independent set of four vertices, contrary to that $\alpha(G) = 3$. Thus $z \in H$. This implies $\alpha(G_{(x,M)} - H) \leq 2$, a contradiction. Therefore $m \geq 2$.

Claim 4. $|S| = 4$.

Proof of Claim 3. If $|S| \geq 5$, by the minimum property of S, then $d(x_i) \geq 5$, where $i \in \{1, 2, 3, 4\}$. Firstly, we assume that $x_1x_3 \in E(G)$ and $x_1x_4 \in E(G)$. If $x_2x_3 \in E(G)$ or $x_2x_4 \in E(G)$, choose $M = \langle\{x_1, x_3\}, \{x_2, x_4\}\rangle$, then $G_{(x,M)}[N(x)]$ is Z_3-connected. Note that $\alpha(G - N[x]) \leq 2$. By Lemma 2.6, $G_{(x,M)}$ is Z_3-connected, a contradiction. So $x_2x_3 \notin E(G)$ and $x_2x_4 \notin E(G)$. Let $V_1 = \{v | v \in S - \{x, x_3, x_4\}$ and $G[V(G_1) \cup \{v\}]$ is a complete graph $\}$. If there exist one vertex $u \in V(G_1 - x_1) \cup V_1$ such that $ux_3 \in E(G)$ or $ux_4 \in E(G)$, choose $M = \langle\{x_1, x_3\}, \{x_2, x_4\}\rangle$, then $G_{(x,M)}[V(G_1) \cup V_1 \cup \{x_3, x_4\}]$ is Z_3-connected and $\alpha(G_{(x,M)} - G_{(x,M)}[V(G_1) \cup V_1 \cup \{x_3, x_4\}]) \leq 2$. By Lemma 2.6, $G_{(x,M)}$ is Z_3-connected, a contradiction. Hence $vx_3 \notin E(G)$ and $vx_4 \notin E(G)$ for each $v \in V(G_1 - x_1) \cup V_1$. Let $V_1' = G_2 + S - V_1 - \{x, x_2, x_3, x_4\}$. If there exist one vertex $w \in V_1'$ such that $\{v, w, x_2\}$ is an independent set, then $wx_3 \in E(G)$ and $wx_4 \in E(G)$. For otherwise, $\{v, x_3, x_2, w\}$ or $\{v, x_4, x_2, w\}$ is an independent set of four vertices, contrary to that $\alpha(G)=3$. Let $V_2=\{w|w\in V_1'$ and $\{v,w,x_2\}$ is an independent set$\}$. Then $G_{(x,M)}[V_2\cup\{x_1,x_3,x_4\}]$ is Z_3-connected. Note that $\alpha(G_{(x,M)}-G_{(x,M)}[V_2\cup\{x_1,x_3,x_4\}])\le 2$. By Lemma 2.6, $G_{(x,M)}$ is Z_3-connected, a contradiction. Secondly, we assume, without loss of generality, that $x_1x_3\in E(G)$, but $x_1x_4\notin E(G)$. If $x_2x_3\notin E(G)$, choose $M=\langle\{x_1,x_4\},\{x_2,x_3\}\rangle$, then $\kappa(G_{(x,M)})=\kappa(G)=4$. By the induction hypothesis, $G_{(x,M)}$ admits a nowhere-zero 3-flow, a contradiction. Hence $x_2x_3\in E(G)$. If $N(x_1)\cap N(x_3)\cap S=\{x\}$, choose $M=\langle\{x_1,x_3\},\{x_2,x_4\}\rangle$, we can prove that $G_{(x,M)}[V(G_1)\cup\{x_3,x_4\}]$ is Z_3-connected or $G_{(x,M)}$ is a 4-connected simple graph with $\alpha(G_{(x,M)})=3$ since $|S|\ge 5$. By Lemma 2.6 or the induction hypothesis, $G_{(x,M)}$ admits a nowhere-zero 3-flow, a contradiction. So $|N(x_1)\cap N(x_3)\cap S|\ge 2$. Similarly $|N(x_3)\cap N(x_4)\cap S|\ge 2$ and $|N(x_2)\cap N(x_3)\cap S|\ge 2$. Let $y_1\in N(x_1)\cap N(x_3)$, $y_2\in N(x_3)\cap N(x_4)$ and $y_3\in N(x_2)\cap N(x_3)$. If $y_1=y_2$, then $G[\{x,x_1,x_3,x_4,y_1\}]=K_4$ (center at x_4) which is Z_3-connected. It follows that $G[N[x]\cup\{y_1\}]$ is Z_3-connected. Denote by $G^*=G/G[N[x]\cup\{y_1\}]$. Then $\alpha(G^*)\le 2$ and $\delta(G^*)\ge 4$. By Lemmas 2.3 and 2.2, G is Z_3-connected. Similarly if $y_1=y_3$ or $y_2=y_3$, G is also Z_3-connected. So we suppose that $y_1\ne y_2\ne y_3$. Note that $\alpha(G)=3$. Then $G[\{y_1,y_2,y_3\}]$ contains at least one edge. We assume, without loss of generality, that $y_1y_2\in E(G)$. Then $G[N[x]\cup\{y_1,y_2,y_3\}]$ is a triangularly connected graph. It is easy to prove that $G_{(x,M)}[N[x]\cup\{y_1,y_2,y_3\}]$ is Z_3-connected, where M is any way to pair the vertices of $N(x)$. As the above, we can prove that $G_{(x,M)}$ is Z_3-connected, a contradiction. Thirdly, we assume that $x_1x_3\notin E(G)$ and $x_1x_4\notin E(G)$. If $x_2x_3\notin E(G)$ or $x_2x_4\notin E(G)$, choose $M=\langle\{x_1,x_4\},\{x_2,x_3\}\rangle$ or $M=\langle\{x_1,x_3\},\{x_2,x_4\}\rangle$, then $G_{(x,M)}$ is a 4-connected simple graph with $\alpha(G_{(x,M)})=3$ since $|S|\ge 5$. As the above, we are done. Next we suppose that $x_2x_3\in E(G)$ and $x_2x_4\in E(G)$. Choose $M=\langle\{x_1,x_3\},\{x_2,x_4\}\rangle$. It is clear that $G_{(x,M)}[\{x_2,x_3,x_4\}]$ is Z_3-connected. Let H be the maximal Z_3-connected subgraph which contains $G_{(x,M)}[\{x_2,x_3,x_4\}]$. Next we shall prove that $\alpha(G_{(x,M)}-H)\le 2$. On the one hand, $\alpha(G-N[x])=2$. On the other hand, if there exist two vertices $u_1,u_2\in G_{(x,M)}-H$ such that $\{x_1,u_1,u_2\}$ is an independent set, then there must be one vertex in $\{x_2,x_3,x_4\}$, say x_3 such that $\{x_1,x_3,u_1,u_2\}$ is an independent set of four vertices, contrary to that $\alpha(G)=3$. It follows from Lemma 2.6 that $G_{(x,M)}$ is Z_3-connected, a contradiction. Therefore $|S|=4$.

Note that $\kappa(G)=4$. Then $e(x_i,G_1-x_1)\ge 1$, where $i\in\{3,4\}$. If $x_1x_3\in E(G)$ or $x_1x_4\in E(G)$, choose $M=\langle\{x_1,x_3\},\{x_2,x_4\}\rangle$ or $M=\langle\{x_1,x_4\},\{x_2,x_3\}\rangle$, respectively. It is easy to prove that $G_{(x,M)}[V(G_1)\cup\{x_1,x_3,x_4\}]$ is Z_3-connected. If there exist one vertex, say $v\in V(G_1)$ such that $vy\notin E(G)$, then $\alpha(G_2+y)\le 2$. If $vy\in E(G)$ for each $v\in V(G_1)$, then $G_{(x,M)}[V(G_1)\cup\{x_1,x_3,x_4,y\}]$ is Z_3-connected. By Lemma 2.6, in either case, $G_{(x,M)}$ is Z_3-connected, a contradiction. So $x_1x_3\notin E(G)$ and $x_1x_4\notin E(G)$. Thus $m\ge 3$. If $e(\{x_3,x_4\},G_1)\ge 3$, choose $M=\langle\{x_1,x_2\},\{x_3,x_4\}\rangle$. It is clear that $G[V(G_1)\cup\{x_3,x_4\}]$ is Z_3-connected. By Lemma 2.6, $G_{(x,M)}$ is Z_3-connected, a contradiction. So $e(\{x_3,x_4\},G_1)=2$. Thus $\alpha(G[V(G_2)\cup\{x,x_3,x_4\}])\le 2$ and $\alpha(G[V(G_2)\cup$

$\{x, y, x_3, x_4\}]) \leq 2$ if $e(y, G_1) < |V(G_1)|$. When $m \geq 5$, G_1 is Z_3-connected follows from Lemma 2.1. Choose $M = \langle\{x_1, x_2\}, \{x_3, x_4\}\rangle$, then $G[V(G_1) \cup \{x_3, x_4\}]$ is Z_3-connected, we are done. Therefore we suppose that $3 \leq m \leq 4$. If $G_2 = K_n$, by the symmetry, $3 \leq n \leq 4$, $x_2x_i \notin E(G)$ and $e(\{x_3, x_4\}, G_2) = 2$, were $i \in \{3, 4\}$. If $x_3y \in E(G)$ or $x_4y \in E(G)$, choose $M = \langle\{x_1, x_2\}, \{x_3, x_4\}\rangle$, then $G[V(G_1) \cup \{y, x_3, x_4\}]$ is Z_3-connected. It follows from Lemmas 2.6 that $G_{(x,M)}$ is Z_3-connected, a contradiction. Hence we assume that $x_3y \notin E(G)$ and $x_4y \notin E(G)$. Let $G_1 = G_2 = K_3$. Then $G[V(G_1 \cup \{y\})] = K_4$ and $G[V(G_2 \cup \{y\})] = K_4$. Thus G is the graph (a) showed in Figure 3 which is even. So G admits a nowhere-zero 3-flow. Let $G_1 = G_2 = K_4$. If there exist $w_1 \in (V(G_1) - x_1)$ and $w_2 \in (V(G_2) - x_2)$ such that $w_1y \notin E(G)$ and $w_2y \notin E(G)$, then $\{x, y, w_1, w_2\}$ is an independent set of four vertices, contrary to that $\alpha(G) = 3$. Hence, $e(y, G_1) = 4$ or $e(y, G_2) = 4$. Thus, G contains Z_3-connected subgraph W_4. By Lemma 2.6, G admits a nowhere-zero 3-flow. At last, we consider that $G_1 = K_3$ and $G_2 = K_4$. Then G is even or the graph (b) showed in Figure 3, which has a $\{C_3, C_4, C_3 \oplus C_4\}$-composition. So G also admits a nowhere-zero 3-flow. Next we suppose that $G_2 \neq K_n$. It is easy to check that $G_2 - x_2 = K_l$ since $\alpha(G) = 3$, where $l \geq 2$. Note that $\kappa(G) = 4$ and $\alpha(G_2) = 2$. We have that $1 \leq e(x_2, K_l) \leq l - 1$. Note that $\alpha(G_1 + y) \leq 2$. If there exist some way M which pairs the vertices of $N(x)$ such that $G_{(x,M)}[V(G_2) \cup \{x_3, x_4\}]$ is Z_3-connected, then $G_{(x,M)}$ is Z_3-connected follows from Lemma 2.6, a contradiction. Next we suppose that there is no such M that $G_{(x,M)}[V(G_2) \cup \{x_3, x_4\}]$ is Z_3-connected. If $x_2x_3 \in E(G)$ and $x_2x_4 \in E(G)$, choose $M = \langle\{x_1, x_2\}, \{x_3, x_4\}\rangle$, then $G_{(x,M)}[\{x_2, x_3, x_4\}]$ is Z_3-connected. Note that $\kappa(G) = 4$. Then $e(\{x_2, x_3, x_4\}, K_l) \geq 3$. It is clear that if $l \geq 4$ or $l = 2$, then $G_{(x,M)}[V(G_2) \cup \{x_3, x_4\}]$ is Z_3-connected, a contradiction. Hence $l = 3$. If $e(\{x_2, x_3, x_4\}, K_l) \geq 4$, as the above, we are done. Next we assume that $e(\{x_2, x_3, x_4\}, K_l) = 3$. In this case, $G[V(K_l) \cup \{y\}] = K_4$. Thus $G_{(x,M)}[V(G_2) \cup \{x_3, x_4, y\}]$ is Z_3-connected, a contradiction. Therefore $x_2x_3 \notin E(G)$ or $x_2x_4 \notin E(G)$. As the above, if $e(\{x_3, x_4\}, K_l) \geq 3$, we need only consider the case of $l = 3$ and $e(\{x_3, x_4\}, K_l) = 3$. Choose $M = \langle\{x_1, x_2\}, \{x_3, x_4\}\rangle$, it is easy to check that $G_{(x,M)}[V(K_l) \cup \{x_3, x_4\}]$ is Z_3-connected or can be contracted to K_4. For the former, we are done; for the letter, $e(x_2, V(K_l) \cup \{x_3, x_4) \geq 3$ if $x_2y \notin E(G)$ and $e(x_2y, V(K_l) \cup \{x_3, x_4) \geq 4$ if $x_2y \in E(G)$. We get that $G_{(x,M)}[V(G_2) \cup \{x_3, x_4, y\}]$ is Z_3-connected follows from Lemmas 2.1 and 2.2, a contradiction. Hence $e(\{x_3, x_4\}, K_l) = 2$. If $m = 2$, then $G_2 = K_3$, a contradiction. If $m \geq 5$, it is clear that $G_{(x,M)}[V(G_2) \cup \{x_3, x_4\}]$ is Z_3-connected, a contradiction. Therefore $3 \leq m \leq 4$. Note that $\kappa(G) = 4$. Then x_3 and x_4 have no common neighbor vertex in K_l. For $l = 3$, let $G_2 - x_2 = G[\{u_1, u_2, u_3\}]$, then $1 \leq e(x_2, \{u_1, u_2, u_3\}) \leq 2$. We assume, without loss of generality, that $x_3u_2 \in E(G)$, $x_4u_3 \in E(G)$ and $x_2u_1 \in E(G)$. If $x_2x_3 \notin E(G)$, then $x_2u_3 \in E(G)$. If not, $\{x_1, x_2, x_3, u_3\}$ constructs an independent set of four vertices, a contradiction. Similarly, if $x_2x_4 \notin E(G)$, then $x_2u_2 \in E(G)$. Note that $\alpha(G_2) = 2$. Then $x_2x_3 \in E(G)$ or $x_2x_4 \in E(G)$. Choose $M = \langle\{x_1, x_3\}, \{x_2, x_4\}\rangle$ or $M =$

$\langle\{x_1, x_4\}, \{x_2, x_3\}\rangle$, we also get that $G_{(x,M)}$ is Z_3-connected, a contradiction. For $l = 4$, let $G_2 - x_2 = G[\{u_1, u_2, u_3, u_4\}]$, then $1 \leq e(x_2, \{u_1, u_2, u_3, u_4\}) \leq 3$. If $e(x_2, \{u_1, u_2, u_3, u_4\}) = 3$, then $G[\{x_2, u_1, u_2, u_3, u_4\}]$ contains W_4 as a spanning subgraph. By Lemma 2.1, $G[\{x_2, u_1, u_2, u_3, u_4\}]$ is Z_3-connected. Choose $M = \langle\{x_1, x_2\}, \{x_3, x_4\}\rangle$, it is clear that $G_{(x,M)}[V(G_2) \cup \{x_3, x_4\}]$ is Z_3-connected, a contradiction. Hence $1 \leq e(x_2, \{u_1, u_2, u_3, u_4\}) \leq 2$. Thus there must be $x_2x_3 \in E(G)$ and $x_2x_4 \in E(G)$ since $\alpha(G) = 3$, a contradiction.

(a) (b)

Fig. 3 Two graphs which admits a nowhere-zero 3-flow.

Case 2. $d(v) \geq 5$ for any vertex $v \in V(G)$.

We first consider that $\kappa'(G) = 4$. There must be a nontrivial minimal edge cut E_c such that $|E_c| = 4$. Let G_1 and G_2 be two components of $G - E_c$.

Subcase 2.1. Either G_1 or G_2 is Z_3-connected.

We assume, without loss of generality, that G_1 is Z_3-connected. Denote by $G_1^\star = G/G_1$. By Lemma 2.2, it is sufficient to show that $G_1^\star$ admits a nowhere-zero 3-flow. Clearly $\alpha(G_1^\star) \leq \alpha(G)$. Note that $\delta(G_1^\star) \geq \kappa'(G_1^\star) \geq 4$. If $\alpha(G_1^\star) \leq 2$, by Lemma 2.3, $G_1^\star$ admits a nowhere-zero 3-flow. Next we assume that $\alpha(G_1^\star) = 3$. If $\kappa(G_1^\star) \geq \kappa(G) = 4$, by the induction hypothesis, $G_1^\star$ also admits a nowhere-zero 3-flow. Therefore it is crucial to consider that $\kappa(G_1^\star) < \kappa(G)$.

Let v^* be the new vertex into which G_1 is contracted and let V^* be a nontrivial minimal vertex cut of $G_1^\star$. Then $v^* \in V^*$. For otherwise, V^* is also a nontrivial minimum vertex cut of G, contrary to that $\kappa(G_1^\star) < \kappa(G)$. Let $V_1^* = V^* - v^*$. Then V_1^* is a nontrivial minimal vertex cut of G_2. If $|V_1^*| = 1$, we can get $\kappa(G) \leq 3$, a contradiction. If $|V_1^*| \geq 3$, then $|V^*| \geq 4$. It follows that $\kappa(G_1^\star) \geq \kappa(G) = 4$, contrary to that $\kappa(G_1^\star) < \kappa(G)$. Hence $|V_1^*| = 2$. Let G_{21} and G_{22} be two connected components of $G_2 - V_1^*$. If $\alpha(G_{21}) = 2$ or $\alpha(G_{22}) = 2$, we can get that $\alpha(G) = 4$, contrary to that $\alpha(G) = 3$. Therefore both G_{21}

and G_{22} are complete graphs. We assume, without loss of generality, that $E_c = \{x_1y_1, x_2y_2, x_3y_3, x_4y_4\}$. Then $x_i \neq x_j$ and $y_i \neq y_j$, where $1 \leq i < j \leq 4$. Let $X = \{x_1, x_2, x_3, x_4\} \subset V(G_1)$ and $Y = \{y_1, y_2, y_3, y_4\} \subset V(G_2)$. Then $e(X, G_{21}) = 2$ and $e(X, G_{22}) = 2$ since $\kappa(G) = 4$. Let $G_{21} = K_m$ and $G_{22} = K_n$. Then $m \geq 4$ and $n \geq 4$ since $d(v) \geq 5$ for any vertex $v \in V(G)$. If $m \geq 5$, by Lemma 2.1(1), G_{21} is Z_3-connected. If $m = 4$, then $e(V_1^*, G_{21}) \geq 6$. So there must be one vertex, say $u^* \in V_1^*$ such that $e(u^*, G_{21}) \geq 3$. It is clear that $G_{21} + u^*$ contains W_4 as a spanning subgraph. It follows from Lemma 2.1 that $G_{21} + u^*$ is Z_3-connected. Therefore it is easy to prove that $G_1^\star$ admits a nowhere-zero 3-flow.

Subcase 2.2. Neither G_1 nor G_2 are Z_3-connected.

If neither G_1 nor G_2 are Z_3-connected, we can immediately get the following claim.

Claim 5. G_i contains no Z_3-connected subgraph H_i with $|V(G_i - H_i)| \leq 3$, where $i \in \{1, 2\}$.

Proof of Claim 5. We prove by contradiction and suppose that G_i contains Z_3-connected subgraph H_i with $|V(G_i - H_i)| \leq 3$. Let $G_i^* = G_i/H_i$ and w_i^* be the new vertex into which H_i is contracted. Then $d(w_i^*) \geq 4$ since $\kappa'(G) \geq 4$. Note that $|V(G_i - H_i)| \leq 3$. So we have that $|V(G_i^*)| \leq 4$ and G_i^* contains 2-cycles. It is easy to prove that G_i^* is Z_3-connected. By Lemma 2.2, G_i is Z_3-connected, a contradiction. Hence, this claim holds.

We have that $|V(G_1)| \geq 6$ and $|V(G_2)| \geq 6$ since $\delta(G) \geq 5$. It follows that $\alpha(G_1) = \alpha(G_2) = 2$. Let A be a maximal independent set of G_1 and $B = \{y | yx \notin E(G), y \in V(G_2), x \in A\}$. Then there are at most two vertices in B whose neighbors in A. Hence $|B| \geq (6-2) = 4$ and $G[B]$ is a complete graph since $\alpha(G) = 3$. If $|B| \geq 5$, by Lemma 2.1, $G[B]$ is Z_3-connected, contrary to Claim 5. So $|B| = 4$. It is just that $|A \cap \{x_1, x_2, x_3, x_4\}| = 2$ and $|V(G_2)| = 6$. We assume, without loss of generality, that $A = \{x_1, x_2\}$. Then for $1 \leq i \leq 2$, $e(y_i, B) \geq 2$ since $\delta(G) \geq 4$. Note that $G[B] = K_4$. If there exist some y_i such that $e(y_i, B) \geq 3$, then $G[B \cup \{y_i\}]$ contains Z_3-connected spanning subgraph W_4, contrary to Claim 5. Hence $e(y_1, B) = 2$ and $e(y_2, B) = 2$ while $y_1y_2 \in E(G)$. Let $B = \{y_3, y_4, y_5, y_6\}$. If $|N(y_1) \cap N(y_2)| \leq 1$, then G_2 contains L_3^+ or W_4 as a subgraph which is Z_3-connected follows from Lemma 2.1(3)(4), contrary to Claim 5. Therefore we assume that $|N(y_1) \cap N(y_2)| = 2$. Then $N(y_1) \cap N(y_2) = \{y_5, y_6\}$. If $|V(G_1)| \geq 7$, then G_1 contains K_m as a subgraph, contrary to Claim 5, where $m \geq |V(G_1)| - 2 \geq 5$. So $|V(G_1)| = 6$. Let $V(G_1) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. Then $G[\{x_2, x_3, x_5, x_6\}] = K_4$ since $y_1y_4 \notin E(G)$ and $G[\{x_2, x_4, x_5, x_6\}] = K_4$ since $y_1y_3 \notin E(G)$. It is clear that G_1 contains K_5^- as a subgraph, contrary to Claim 5.

Next we consider that $\kappa'(G) = 5$. Let $S = \{v_1, v_2, v_3, v_4\}$ be a minimal cut of G. Then $G[S]$ contains at least one edge, say v_1v_2. Let G_1 and G_2 be two components of $G - S$. Note that $\alpha(G) = 3$. Then one of G_1 and G_2 must be a complete graph. We assume, without loss of generality, that $G_1 = K_m$. Thus $\alpha(G_2) \leq 2$. Let $A \subseteq S$ and $A \neq \emptyset$. We have the following claim.

Claim 6. If $G[V(G_1) \cup A]$ is Z_3-connected, then G is also Z_3-connected.

Proof of Claim 6. Let $G^* = G/G[V(G_1) \cup A]$. Note that $\alpha(G_2) \leq 2$. If $A = S$, by Lemma 2.6, G^* is Z_3-connected. If $A \subset S$, then $e(v, G_1) = 1$, for any vertex $v \in S - A$. This yields that $\alpha(G_2 + v) \leq 2$ since $\alpha(G) = 3$. Hence $\alpha(G_2 + S - A) \leq 2$. By Lemma 2.6, G^* is Z_3-connected. Therefore if $G[V(G_1) \cup A]$ is Z_3-connected, G is also Z_3-connected follows from Lemma 2.2.

Note that $\delta(G) \geq \kappa'(G) = 5$. So $m \geq 2$. If $m \geq 5$, by Lemma 2.1 G_1 is Z_3-connected. Note that $e(V(G_1), S) \geq 5$. Then there must be one vertex $v \in S$ such that $e(v, V(G_1)) \geq 2$. Thus $G[V(G_1) \cup \{v\}]$ is Z_3-connected. By Claim 6, G is Z_3-connected. If $m = 4$, then $e(V(G_1), S) \geq 8$. If there exist one vertex $v \in S$ such that $e(v, V(G_1)) \geq 3$, then $G[V(G_1) \cup \{v\}] = K_5$ or K_5^- which is Z_3-connected follows from Lemma 2.1. By Claim 6, G is Z_3-connected. Next we suppose that $e(v, V(G_1)) = 2$ for each vertex $v \in S$. It follows that $e(\{v_1, v_2\}, V(G_1)) = 4$. If $|N_{G_1}(v_1) \cap N_{G_1}(v_2)| \leq 1$, then $G[V(G_1) \cup \{v_1, v_2\}]$ contains L_3^+ or W_4 as a subgraph. It follows from Lemma 2.1 that $G[V(G_1) \cup \{v_1, v_2\}]$ is Z_3-connected. By Claim 6, G is Z_3-connected. Next we assume that $|N_{G_1}(v_1) \cap N_{G_1}(v_2)| = 2$. Then $|N_{G_1}(v_3) \cap N_{G_1}(v_4)| = 2$. If $v_iv_j \in E(G)$, then $G[V(G_1) \cup S]$ is Z_3-connected, where $i \in \{1, 2\}, j \in \{3, 4\}$. By Lemma 2.6, G is Z_3-connected. Next we assume that $v_iv_j \notin E(G)$, where $i \in \{1, 2\}, j \in \{3, 4\}$. Let $N_{G_1}(v_3) = \{x, y\}$. We consider the graph $G_{[v_3x, v_3y]}$ which contains a 2-cycle xyx. Denote by $H_1 = G_1 + S - v_3 + xy$. It is clear that H_1 is a Z_3-connected subgraph of $G_{[v_3x, v_3y]}$. Note that $e(S - v_3, G_2) \geq 7$ and $\kappa'(G) = 5$. Then $G_{[v_3x, v_3y]} - v_3$ is a 4-edge-connected graph. By Lemma 2.6, $G_{[v_3x, v_3y]} - v_3$ is Z_3-connected. So is $G_{[v_3x, v_3y]}$ since $e(v_3, G_2) \geq 3$. It follows from Lemma 2.4 that G is Z_3-connected. If $m = 3$, then $e(V(G_1), S) \geq 9$. Let $D_3 = \{v | v \in S$ and $e(v, G_1) = 3\}$. Then $|D_3| \geq 1$. If $|D_3| \geq 2$, then G contains Z_3-connected subgraph K_5^-. By Lemma 2.6, G is Z_3-connected. Hence we suppose that $|D_3| = 1$. Then $e(v, G_1) \geq 2$ for each vertex $v \in S$. So $4 \leq e(\{v_1, v_2\}, V(G_1)) \leq 5$. Let $A = \{v_1, v_2\}$. We shall show that $G[V(G_1) \cup A]$ is Z_3-connected. If $e(\{v_1, v_2\}, V(G_1)) = 5$, then $G[V(G_1) \cup A]$ contains K_5^- as a spanning subgraph. If $e(\{v_1, v_2\}, V(G_1)) = 4$, then $e(v_i, G_1) = 2$, where $i \in \{1, 2\}$. Note that $e(u, S) \geq 3$ for each vertex $u \in V(G_1)$. Then $|N_{G_1}(v_1) \cap N_{G_1}(v_2)| \leq 1$. Thus $G[V(G_1) \cup A]$ contains W_4 as a spanning subgraph. Therefore $G[V(G_1) \cup A]$ is Z_3-connected follows from Lemma 2.1. By Claim 6, G is Z_3-connected. At last, we assume that $m = 2$. Let $G_1 = u_1u_2$. Then $N(u_1) = \{u_2, v_1, v_2, v_3, v_4\}$ and $N(u_2) = \{u_1, v_1, v_2, v_3, v_4\}$. We consider the graph $G_{[u_1v_1, u_1v_2]}$ which contains a 2-cycle $v_1v_2v_1$. Let H_2 be the maximal Z_3-connected subgraph of $G_{[u_1v_1, u_1v_2]}[V(G_1) \cup S]$. If $v_iv_j \in E(G)$, then $H_2 = G_{[u_1v_1, u_1v_2]}[V(G_1) \cup S]$, where $i \in \{1, 2\}, j \in \{3, 4\}$. By Lemma 2.6, G^* is Z_3-connected. It follows from Lemmas 2.1 and 2.4 that G is Z_3-connected. Next we assume that $v_iv_j \notin E(G)$, where $i \in \{1, 2\}, j \in \{3, 4\}$. Thus $H_2 = v_1u_2v_2 + v_1v_2$ and $v^*v_j \in E(G^*)$, where

$j \in \{3, 4\}$. Note that $d_{G^*}(v_3) \geq 5$. Next we consider the graph $G^*_{[v_3u_1, v_3v^*]}$ which contains a 2-cycle u_1v^*. Let H^* be the maximal Z_3-connected subgraph which contains a 2-cycle u_1v^* of $G^*_{[v_3u_1, v_3v^*]}$ and v^{**} be the vertex into which H^* is contracted. Then $\alpha(G_2 + v^{**}) \leq 2$. For otherwise, there are two vertices in G_2, say x, y such that $\{v^{**}, x, y\}$ is an independent set of three vertices. Note that $v_1v_4 \notin E(G)$. Thus $\{v_1, v_4, x, y\}$ is an independent set of four vertices, contrary to that $\alpha(G) = 3$. Hence $\alpha(G_2 + v^{**} - v_3) \leq 2$. Denote by $G^{**} = G^*_{[v_3u_1, v_3v^*]}/H^*$. Then $G^{**} = G_2 + v^{**}$. Note that $\delta(G^{**} - v_3) \geq 4$. By Lemma 2.3, $G^{**} - v_3$ is Z_3-connected. So is G^{**} since $d_{G^{**}}(v_3) \geq 3$. It follows from Lemmas 2.1 and 2.4 that both G^* and G are Z_3-connected, a contradiction.

Therefore we conclude that if $\alpha(G) = 3$, then G admits a nowhere-zero 3-flow, this completes the proof.

Acknowledgements

Supported by the Natural Science Foundation of China (11701496,11526176), Nanhu Scholars Program for Young Scholars of XYNU, Doctoral Scientific Research Startup Fund of Xinyang Normal University (0201431) and Youth Sustentation Fund of Xinyang Normal University (2014-QN-047).

Supported by Youth Sustentation Fund of Xinyang Agriculture and Forestry University (201401009).

References

[1] J. A. Bondy and U. S. R. Murty, *Graph theory with application,* North-Holland, New York, 1976.

[2] V. Chvátal, P. Erdös, *A note on Hamiltonian circuits,* Discrete Math., 2 (1972), 111-113.

[3] J. J. Chen, E. Eschen and H. J. Lai, *Group connectivity of certain graphs,* Ars Combin., 89 (2008), 141-158.

[4] M. DeVos, R. Xu and G. Yu, *Nowhere-zero Z_3-flows through Z_3-connectivity,* Discrete Math., 306 (2006), 26-30.

[5] G. Fan, H. J. Lai, R. Xu, C. Q. Zhang and C. Zhou, *Nowhere-zero 3-flows in triangularly connected graphs*, Journal of Combin. Theory Ser. B, 98 (2008), 1325-1336.

[6] G. Fan and C. Zhou, *Degree sum and Nowhere-zero 3-flows,* Discrete Math., 308 (2008), 6233-6240.

[7] G. Fan and C. Zhou, *Ore condition and Nowhere-zero 3-flows*, SIAM J. Discrete Math., 22 (2008), 288-294.

[8] L. Han, H. -J. Lai, L. Xiong and H. Yan, *The Chvátal-Erdös condition for supereulerian graphs and the Hamiltonian index*, Discrete Math., 310 (2010), 2082-2090.

[9] F. Jaeger, N. Linial, C. Payan and M. Tarsi, *Group connectivity of graphs-a nonhomogeneous analogue of Nowhere-zero flow properties,* J. Combin. Theory, Ser. B, 56 (1992), 165-182.

[10] R. Luo, Z. Miao and R. Xu, *Nowhere-zero 3-flows of graphs with independence number two,* Graphs and Combin., 29 (2013), 1899-1907.

[11] H. J. Lai, *Group connectivity of 3-edge-connected chordal graphs,* Graphs and Combin., 16 (2000), 165-176.

[12] R. Luo, R. Xu, J. Yin and G. Yu, *Ore-condition and* Z_3*-connectivity,* European Journal of Combin., 29 (2008), 1587-1595.

[13] L. M. Lovasz, C. Thomassen, Y. Wu and C. Q. Zhang, *Nowhere-zero 3-flows and modulo k-orientations*, J. Combin. Theory Ser. B, 103 (2013), 587-598.

[14] C. Thomassen, *The weak 3-flow conjecture and the weak circular flow conjecture,* J. Combin. Theory, Ser. B, 102 (2012), 521-529.

[15] W. T. Tutte, *A contribution on the theory of chromatic polynomials,* Canad. J. Math., 6 (1954), 80-91.

[16] F. Yang, X. Li and L. Li, Z_3*-connectivity with independent number 2*, Graphs and Combin., 32 (2016), 419-429.

Accepted: 20.03.2018