Some fixed points theorems in intuitionistic Menger spaces

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Abstract. In this paper, we established some common fixed point theorems for two pairs of weakly subsequential continuous (wsc) maps with compatibility of type (E) in an Intuitionistic Menger space (breiefly IM space). We deduce important results in this line by restricting the number of mappings involved. An example is given to support the main result.
Keywords: fixed point, intuitionistic Menger space, compatible maps of type (E), weak subsequential continuous maps (wsc).

1. Introduction

Menger [5] on his famous article titled statistical metric states that large parts of metric geometry can be developed, in particular, a theory of betweenness on the basis of postulates given herein, especially, where the probabilistic situation arises . Thereafter, it became an attractive area of research for the researchers involved in the field of fixed point theory and its applications. Schweizer and Sklar [8, 9] stimulated the study further with their pioneering work on statistical metric spaces. Working on the same line, Sehgal and Bharucha-Reid [10] establised some fixed points theorems involving contraction mappings on probabilistic metric spaces. Stojakovic [12, 13, 14] brought forward the legacy with his work on probabilistic metric spaces and its applications. Singh and Pant [11] gave some fixed point results for commuting maps in probabilistic metric spaces. Steping one milestone ahead, Kutucku *et al.* [6] developed probabilistic metric spaces due to Menger [5] to intutionistic Menger spaces and established common fixed point theorems with the help of continuous t-norm and continuous t-conorm. Rashwan and Heder [7] estabilished some new fixed point results for

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compatible mappings in Menger spaces. Pant *et al.* [3] studied fixed points and its uniqueness for weakly compatible mappings in intuitionistic Menger spaces without any appeal to the continuity of mappings. Leaving aside the condition of continuity, Jain *et al.* [4] came out with some fixed point results for absorbing type of maps in intuitionistic Menger spaces. Leila and Aliouche [18] estabilished a common fixed point theorem for mappings under $\phi-$ contractive conditions on intuitionistic Menger metric spaces.They also studied the existence and uniqueness of the solution to a nonlinear Fredholm integral equation as an application of their result.

Bouhadjera and Thobie [1] proved common fixed point theorems for pairs of subcompatible maps. Singh *et al.* [15] introduced the notion of compatibility of type (E) and proved some common fixed point theorems for it. Recently, Beloul [2] established some fixed point theorems for two pairs of self mappings satisfying contractive conditions by using the weak subsequential mappings with compatibility of type (E). The purpose of this paper is to obtain common fixed point theorems for weak subsequential continuous mappings with compatibility of type (E) in an intuitionistic Menger space.

2. Preliminaries

Definition 2.1 ([3]). A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is said to be t-norm if it satisfies the following conditions:

(i) $*$ is commutative and associative,

(ii) $*$ is continuous

(iii) $a * 1 = a$ for all $a \in [0,1]$ and

(iv) $a * b \leqslant c * d$ whenever $a \leqslant c$ and $b \leqslant d$, for all $a, b, c, d \in [0,1]$.

Definition 2.2 ([3]). A binary operation $\Diamond : [0,1] \times [0,1] \to [0,1]$ is said to be t-conorm if it satisfies the following conditions:

(i) $\Diamond$ is commutative and associative,

(ii) $\Diamond$ is continuous,

(iii) $a \Diamond 0 = a$, for all $a \in [0,1]$ and

(iv) $a \Diamond b \leqslant c \Diamond d$ whenever $a \leqslant c$ and $b \leqslant d$, for all $a, b, c, d \in [0,1]$.

Remark 2.1 ([3]). The concept of triangular norms (t-norms) and triangular conorms (t-conorms) are known as the axiomatic skeltons that we use for characterizing fuzzy intersecion and union respectively. These concepts were originally introduced by Menger [5] in his study of statistical metric spaces.

Definition 2.3 ([3])**.** A distance distribution function is function $F : R \to R^+$ which is left continuous on R, non-decreasing and $\inf_{t\in R} F(t) = 0,\ \sup_{t\in R} F(t) = 1$.

We shall denote by D the family of all distance distribution functions and by H a special distance distribution function in D given by

$$H(t) = \begin{cases} 1, & \text{if } t > 0 \\ 0, & \text{if } t \leqslant 0 \end{cases}.$$

Definition 2.4 ([3])**.** A non-distance distribution function is function $L : R \to R^+$ which is left continuous on R, non-increasing and $\inf_{t\in R} L(t) = 1, \sup_{t\in R} L(t) = 0$.

We shall denote by E the family of all non distance distribution functions and by G a special non distance distribution function in E given by

$$G(t) = \begin{cases} 1, & \text{if } t \leqslant 0 \\ 0, & \text{if } t > 0 \end{cases}.$$

Definition 2.5 ([3])**.** Given an arbitrary set X, a continuous t-norm $*$, a continuous t-conorm $\Diamond$, a probabilistic distance F and a probabilistic non-distance L on X, the 5-tuple $(X, F, L, *, \Diamond)$ is said to be an intuitionistic Menger space if the following conditions are satisfied for all $x, y, z \in X$ and $s, t \geqslant 0$.

(IM1) $F(x, y, t) + L(x, y, t) \leqslant 1$,

(IM2) $F(x, y, 0) = 0$,

(IM3) $F(x, y, t) = H(t)$ iff $x = y$,

(IM4) $F(x, y, t) = F(y, x, t)$,

(IM5) If $F(x, y, t) = 1$ and $F(y, z, s) = 1$, then $F(x, z, t + s) = 1$,

(IM6) $F(x, z, t + s) \geqslant F(x, y, t) * F(y, z, s)$,

(IM7) $L(x, y, 0) = 1$,

(IM8) $L(x, y, t) = G(t)$ iff $x = y$,

(IM9) $L(x, y, t) = L(y, x, t)$,

(IM10) If $L(x, y, t) = 0$ and $L(y, z, s) = 0$, then $L(x, z, t + s) = 0$,

(IM11) $L(x, z, t + s) \leqslant L(x, y, t) \Diamond L(y, z, s)$.

The functions $F(x, y, t)$ and $L(x, y, t)$ denote the degree of nearness and degree of non- nearness between x and y with respect to t, respectively.

Definition 2.6 ([6]). Let $(X, F, L, *, \Diamond)$ be an intuitionistic Menger space with $t * t \geqslant t$ and $(1-t)\Diamond(1-t) \leqslant (1-t)$ for all $0 < t < 1$. A sequence $\{x_n\}$ in X is said to be convergent to $x \in X$ if, for any $\epsilon > 0$ and $k \in (0,1)$, there exists a positive integer N such that $F(x_n, x, \epsilon) > 1 - k$ and $L(x_n, x, \epsilon) < k$ whenever $n \geqslant N$.

Lemma 2.1 ([3]). *Let $(X, F, L, *, \Diamond)$ be an intuitionistic Menger space. If there exists a constant $k \in (0,1)$, and two elements $x, y \in X$ such that for all $t > 0$,*

$$F(x, y, kt) \geqslant F(x, y, t) \text{ and } L(x, y, kt) \leqslant L(x, y, t).$$

Then $x = y$.

Lemma 2.2 ([6]). *Let $(X, F, L, *, \Diamond)$ be an intuitionistic Menger space with $t * t \geqslant t$ and $(1-t)\Diamond(1-t) \leqslant (1-t)$ for all $0 < t < 1$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X converging to x and y respectively. If $t \geqslant 0$ is a point of continuity of $F(x, y, .)$ and $L(x, y, .)$, then $\lim_{n\to\infty} F(x_n, y_n, t) = F(x, y, t)$ and $\lim_{n\to\infty} L(x_n, y_n, t) = L(x, y, t)$.*

Singh *et al.* [15, 16] introduced the notion of compatibility of type (E), A-compatibility of type(E) and S-compatibility of type (E),we redefine all these notions further in the setting of intuitionistic Menger space as follows:

Definition 2.7. Self maps A and S on an intutionistic Menger space $(X, F, L, *, \Diamond)$ are said to be compatible of type (E), if $\lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} SAx_n = Az$ and $\lim_{n\to\infty} A^2x_n = \lim_{n\to\infty} ASx_n = Sz$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

Definition 2.8. Self maps A and S on an intuitionistic Menger space $(X, F, L, *, \Diamond)$ are said to be A-compatible of type (E), if $\lim_{n\to\infty} A^2x_n = \lim_{n\to\infty} ASx_n = Sz$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$. Pair A and S are said to be S-compatible of type (E), if $\lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} SAx_n = Az$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

Remark 2.2. It is also interesting to see that if A and S are compatible of type (E), then they are A-Compatible and S-Compatible of type (E), but the converse need not be true (see Example 1 [2]). Bouhadjera and Thobie [1] introduced the concept of subsequential continuity as follows:

Definition 2.9. Self maps A and S of a metric space (X, d) are said to be subsequentially continuous, if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$ and $\lim_{n\to\infty} ASx_n = At$ and $\lim_{n\to\infty} SAx_n = St$.

Motivated by the Definition 2.9 and [2], we define the following in the setting of intutionistic Menger space.

Definition 2.10. Self maps A and S defined on an intuitionistic Menger space $(X, F, L, *, \Diamond)$ are said to be weakly subsequentially continuous (in short wsc), if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$ and $\lim_{n\to\infty} ASx_n = Az$ or $\lim_{n\to\infty} SAx_n = Sz$

Definition 2.11. Self maps A and S defined on an intuitionistic Menger space $(X, F, L, *, \Diamond)$ are said to be S subsequentially continuous, if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$ and $\lim_{n\to\infty} SAx_n = Sz$.

Definition 2.12. Self maps A and S defined on an intuitionistic Menger space $(X, F, L, *, \Diamond)$ are said to be A subsequentially continuous if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$ and $\lim_{n\to\infty} ASx_n = Az$.

Remark 2.3. If the pair of mappings $\{A, S\}$ is A-subsequentially continuous (or S-subsequentially continuous) then it is wsc but never be subsequentially continuous (see Example 3[2]).

Definition 2.13 ([17])**.** Let Ψ be the class of all non decreasing mappings $\psi : [0, 1] \times [0, 1] \to [0, 1]$ and $\Psi : [0, 1] \times [0, 1] \to [0, 1]$ such that

(i) $\lim_{n\to\infty} \psi^n(s) = 1, \forall s \in (0, 1]$,

(ii) $\psi(s) > s, \forall s \in (0, 1)$,

(iii) $\psi(1) = 1$,

(iv) $\lim_{n\to\infty} \eta^n(r) = 0, \forall s \in [0, 1)$,

(v) $\eta(r) < r, \forall s \in (0, 1)$,

(vi) $\eta(0) = 0$. For examples we refer to [17].

3. Main results

Theorem 3.1. *Let A, B, S and T be four self mappings of an intuitionistic Menger space $(X, F, L, *, \Diamond)$ with continuous t-norm $*$ and continuous t-conorm $\Diamond$ satisfying $t * t \geqslant t$ and $(1 - t)\Diamond(1 - t) \leqslant (1 - t)$ for all $0 < t < 1$. Suppose $\psi, \eta \in \Psi$ and there exists a constant $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$ the following conditions are satisfied:*

$$F(Ax, By, kt) \geqslant \psi[\min\{F(Sx, Ty, t), F(Ax, Sx, t), F(By, Ty, t), F(Sx, By, t), F(Ty, Ax, t)\}], \tag{3.1}$$

$$L(Ax, By, kt) \leqslant \eta[\max\{L(Sx, Ty, t), L(Ax, Sx, t), L(By, Ty, t), L(Sx, By, t), L(Ty, Ax, t)\}]. \tag{3.2}$$

If the pairs $\{A, S\}$ and $\{B, T\}$ are weakly subsequential continuous and compatible of type (E), then A, B, S and T have a unique common fixed point in X.

Proof. Since the pair $\{A, S\}$ is weakly subsequential continuous, we can assume that it is A-subsequentially continuous and compatible of type (E). There exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, for some $z \in X$ and $\lim_{n\to\infty} ASx_n = Az$. The compatibility of type (E) implies that $\lim_{n\to\infty} A^2x_n = \lim_{n\to\infty} ASx_n = Sz$ and $\lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} SAx_n = Az$. Therefore $Az = Sz$, whereas in respect of the pair $\{B, T\}$, suppose that it is B-subsequentially continuous.Then, there exists a sequence $\{y_n\}$ in X such that $\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = w$, for some $w \in X$ and $\lim_{n\to\infty} BTy_n = Bw$. The pair $\{B, T\}$ is compatible of type (E), so $\lim_{n\to\infty} B^2y_n = \lim_{n\to\infty} BTy_n = Tw$ and $\lim_{n\to\infty} T^2y_n = \lim_{n\to\infty} TBy_n = Bw$, for some $w \in X$. This gives $Bw = Tw$. Hence z is a coincidence point of the pair $\{A, S\}$ whereas w is a coincidence point of the pair $\{B, T\}$. Now we prove that $z = w$. Choose a $t > 0$ satisfying (3.1) and (3.2). Without loss of generality, we can assume that t and kt are points of continuity of $F(z, w, .)$, $F(z, z, .)$, $F(w, w, .)$, $L(z, w, .)$, $L(z, z, .)$, $L(w, w, .)$, $F(Az, w, .)$, $F(Sz, w, .)$, $F(z, Bz, .)$, $F(z, Tz, .)$, $L(Az, w, .)$, $L(Sz, w, .)$, $L(z, Bz, .)$ and $L(z, Tz, .)$. This is so because these functions are monotonic on $\mathbb{R}$ and hence have at most countable number of discontinuities in $(0, b)$ for any $b > 0$. So we may choose t sufficiently small that $0 < kt < t < b$ and both kt and t are points of continuity of all the functions mentioned above. By putting $x = x_n$ and $y = y_n$ in inequality (3.1), we have

$$F(Ax_n, By_n, kt) \geqslant \psi[\min\{F(Sx_n, Ty_n, t)), F(Ax_n, Sx_n, t)), F(By_n, Ty_n, t), \\ F(Sx_n, By_n, t), F(Ty_n, Ax_n, t)\}].$$

Taking the limit as $n \to \infty$ and using Lemma 2.2, we get

$$F(z, w, kt) \geqslant \psi[\min\{F(z, w, t), F(z, z, t), F(w, w, t), F(z, w, t), F(w, z, t)\}].$$

So, $F(z, w, kt) \geqslant \psi[\ min\{F(z, w, t)), 1, 1, F(z, w, t), F(w, z, t)\}]$. This gives, for all $t > 0$,

$$F(z, w, kt)) \geqslant \psi[F(z, w, t)] \geqslant F(z, w, t). \tag{3.3}$$

Again, by putting $x = x_n$ and $y = y_n$ in inequality (3.2), we have

$$L(Ax_n, By_n, kt) \leqslant \eta[\max\{L(Sx_n, Ty_n, t), L(Ax_n, Sx_n, t), L(By_n, Ty_n, t), \\ L(Sx_n, By_n, t), L(Ty_n, Ax_n, t)\}]$$

Taking the limit as $n \to \infty$, and using Lemma 2.2 we get

$$L(z, w, kt) \leqslant \eta[\max\{L(z, w, t), L(z, z, t), L(w, w, t), L(z, w, t), L(w, z, t)\}].$$

So, we have for all $t > 0$,

$$L(z, w, kt) \leqslant \eta[\max\{L(z, w, t), 0, 0, L(z, w, t), L(w, z, t)\}].$$

$$L(z,w,kt) \leqslant \eta[L(z,w,t)] \leqslant L(z,w,t), \tag{3.4}$$

for all $t > 0$.

By Lemma 2.1, (3.3) and (3.4), we have $z = w$. Now we prove that $Az = z$. By putting $x = z$ and $y = y_n$ in the inequality (3.1), we get $F(Az, By_n, kt) \geqslant \psi[\min\{F(Sz, Ty_n, t), F(Az, Sz, t), F(By_n, Ty_n, t), F(Sz, By_n, t), F(Ty_n, Az, t)\}]$.

Letting $n \to \infty$ and using Lemma 2.2, we obtain

$$F(Az, w, kt) \geqslant \psi[\min\{F(Sz, w, t), F(Az, Sz, t), F(w, w, t), F(Sz, w, t), F(w, Az, t)\}].$$

This gives $F(Az, w, kt) \geqslant \psi[min\{F(Sz, w, t), 1, 1, F(Sz, w, t), F(w, Az, t)\}]$. But $Az = Sz$.Thus, for all $t > 0$,

$$F(Az, w, kt) \geqslant \psi[F(Az, w, t)] \geqslant F(Az, w, t). \tag{3.5}$$

Again, by putting $x = z$ and $y = y_n$ in the inequality (3.2), we get $L(Az, By_n, kt) \leqslant \eta[\max\{L(Sz, Ty_n, t), L(Az, Sz, t), L(By_n, Ty_n, t), L(Sz, By_n, t), L(Ty_n, Az, t)\}]$.

Taking the limit as $n \to \infty$ and using Lemma 2.2, we obtain

$$L(Az, w, kt) \leqslant \eta[\max\{L(Sz, w, t), L(Az, Sz, t), L(w, w, t), L(Sz, w, t), L(w, Az, t)\}].$$

This gives $L(Az, w, kt) \leqslant \eta[\max\{L(Sz, w, t), 0, 0, L(Sz, w, t), l(w, Az, t)\}]$ and so for all $t > 0$,

$$L(Az, w, kt) \leqslant \eta[L(Az, w, t)] \leqslant L(Az, w, t) \tag{3.6}$$

By Lemma 2.1, (3.5) and (3.6) we get, $Az = w$. Since $Az = Sz$, we have $Az = Sz = w = z$. Now we prove that $Bz = z$.

By putting $x = x_n$ and $y = z$ in the inequality (3.1), we get $F(Ax_n, Bz, kt) \geqslant \psi[\min\{F(Sx_n, Tz, t), F(Ax_n, Sx_n, t), F(Bz, Tz, t), F(Sx_n, Bz, t), F(Tz, Ax_n, t)\}]$.

Taking the limit as $n \to \infty$ and using Lemma 2.2, we obtain

$$F(z, Bz, kt) \geqslant \psi[\min\{F(z, Tz, t), F(z, z, t), F(Bz, Tz, t), F(z, Bz, t), F(Tz, z, t)\}],$$

$$F(z, Bz, kt)) \geqslant \psi[\min\{F(z, Tz, t), 1, 1, F(z, Bz, t), F(Tz, z, t)\}].$$

Since $z = w$ and $Bw = Tw$, then $Bz = Tz$. Thus, for all $t > 0$,

$$F(z, Bz, kt) \geqslant \psi[F(z, Bz, t)] \geqslant F(z, Bz, t) \tag{3.7}$$

Again, by putting $x = x_n$ and $y = z$ in the inequality (3.2), we get $L(Ax_n, Bz, kt) \leqslant \eta[\max\{L(Sx_n, Tz, t), L(Ax_n, Sx_n, t), L(Bz, Tz, t), L(Sx_n, Bz, t), L(Tz, Ax_n, t)\}]$.

Taking the limit as $n \to \infty$ and using Lemma 2.2, we get

$$L(z, Bz, kt) \leqslant \eta[max\{L(z, Tz, t), L(z, z, t), L(Bz, Tz, t), L(z, Bz, t), L(Tz, z, t)\}],$$

which gives $L(z, Bz, kt) \leqslant \eta[\max\{L(z, Tz, t), 0, 0, L(z, Bz, t), L(Tz, z, t)\}]$ and so for all $t > 0$,

$$L(z, Bz, kt) \leqslant \eta[L(z, Bz, t)] \leqslant L(z, Bz, t) \tag{3.8}$$

By Lemma 2.1, (3.7) and (3.8), we obtain, $Bz = z$. Since $Bz = Tz$, we have $Bz = Tz = z$. So, in all $z = Az = Bz = Sz = Tz$, that is, z is a common fixed point of A, B, S and T.

To prove uniqueness, let $z^* \in X$ be such that $z^* = Az^* = Bz^* = Sz^* = Tz^*$. By putting $x = z$ and $y = z^*$ in the inequality (3.1), we get $F(Az, Bz^*, kt) \geqslant \psi[\min\{F(Sz, Tz^*, t), F(Az, Sz, t), F(Bz^*, Tz^*, t), F(Sz, Bz^*, t), F(Tz^*, Az, t)\}]$.

That is, $F(z, z^*, kt) \geqslant \psi[\min\{F(z, z^*, t), F(z, z, t), F(z^*, z^*, t), F(z, z^*, t), F(z^*, z, t)\}]$. So, $F(z, z^*, kt) \geqslant \psi[\min\{F(z, z^*, t), 1, 1, F(z, z^*, t), F(z^*, z, t)\}]$. Thus, for all $t > 0$,

$$F(z, z^*, kt) \geqslant \psi[F(z, z^*, t)] \geqslant F(z, z^*, t). \tag{3.9}$$

By putting $x = z$ and $y = z^*$ in the inequality (3.2), we get $L(Az, Bz^*, kt) \leqslant \eta[\max\{L(Sz, Tz^*, t), L(Az, Sz, t), L(Bz^*, Tz^*, t), L(Sz, Bz^*, t), L(Tz^*, Az, t)\}]$.

That is, $L(z, z^*, kt) \leqslant \eta[\max\{L(z, z^*, t), L(z, z, t), L(z^*, z^*, t), L(z, z^*, t), L(z^*, z, t)\}]$. So, $L(z, z^*, kt) \leqslant \eta[\max\{L(z, z^*, t), 0, 0, L(z, z^*, t), L(z^*, z, t)\}]$. Thus, for all $t > 0$,

$$L(z, z^*, kt) \leqslant \eta[L(z, z^*, t)] \leqslant L(z, z^*, t). \tag{3.10}$$

By Lemma 2.1, (3.9) and (3.10) we get, $z = z^*$. □

If we put $A = B$ in Theorem 3.1 we have the following corollary for three mappings:

Corollary 3.1. *Let A, S and T be three self mappings of an intuitionistic Menger space $(X, F, L, *, \Diamond)$ with continuous t-norm $*$ and continuous t-conorm $\Diamond$ satisfying $t * t \geqslant t$ and $(1-t)\Diamond(1-t) \leqslant (1-t)$ for all $0 < t < 1$. Suppose $\psi, \eta \in \Psi$ and there exists a constant $k \in (0,1)$ such that for all $x, y \in X$ and $t > 0$ the following conditions are satisfied:*

$$\begin{aligned} F(Ax, Ay, kt) \geqslant \psi[\min\{&F(Sx, Ty, t), F(Ax, Sx, t),\\ &F(Ay, Ty, t), F(Sx, Ay, t), F(Ty, Ax, t)\}],\\ L(Ax, Ay, kt) \leqslant \eta[\max\{&L(Sx, Ty, t), L(Ax, Sx, t),\\ &L(Ay, Ty, t), L(Sx, Ay, t), L(Ty, Ax, t)\}]. \end{aligned}$$

If the pairs $\{A, S\}$ and $\{A, T\}$ are weakly subsequential continuous and compatible of type (E), then A, S and T have a unique common fixed point in X.

Alternatively, if we set $S = T$ in Theorem 3.1, we'll have the following corollary for three self mappings:

Corollary 3.2. *Let A, B and S be four self mappings of an intuitionistic Menger space $(X, F, L, *, \lozenge)$ with continuous t-norm $*$ and continuous t-conorm $\lozenge$ satisfying $t * t \geqslant t$ and $(1-t)\lozenge(1-t) \leqslant (1-t)$ for all $0 < t < 1$. Suppose $\psi, \eta \in \Psi$ and there exists a constant $k \in (0,1)$ such that for all $x, y \in X$ and $t > 0$ the following conditions are satisfied:*

$$F(Ax, By, kt) \geqslant \psi[\min\{F(Sx, Sy, t), F(Ax, Sx, t), F(By, Sy, t), F(Sx, By, t), F(Sy, Ax, t)\}],$$
$$L(Ax, By, kt) \leqslant \eta[\max\{L(Sx, Sy, t), L(Ax, Sx, t), L(By, Sy, t), L(Sx, By, t), L(Sy, Ax, t)\}].$$

If the pairs $\{A, S\}$ and $\{B, S\}$ are weakly subsequential continuous and compatible of type (E), then A, B and S have a unique common fixed point in X.

If we put $S = T$ in Corollary 3.1 , we have the following result for two self mappings:

Corollary 3.3. *Let A, S be two self mappings of an intuitionistic Menger space $(X, F, L, *, \lozenge)$ with continuous t-norm $*$ and continuous t-conorm $\lozenge$ satisfying $t * t \geqslant t$ and $(1-t)\lozenge(1-t) \leqslant (1-t)$ for all $0 < t < 1$. Suppose $\psi, \eta \in \Psi$ and there exists a constant $k \in (0,1)$ such that for all $x, y \in X$ and $t > 0$ the following conditions are satisfied:*

$$F(Ax, Ay, kt) \geqslant \psi[\min\{F(Sx, Sy, t), F(Ax, Sx, t), F(Ay, Sy, t), F(Sx, Ay, t), F(Sy, Ax, t)\}],$$
$$L(Ax, Ay, kt) \leqslant \eta[\max\{L(Sx, Sy, t), L(Ax, Sx, t), L(Ay, Sy, t), L(Sx, Ay, t), L(Sy, Ax, t)\}].$$

If the pair $\{A, S\}$ is weakly subsequential continuous and compatible of type (E), then A and S have a unique common fixed point in X.

Example 3.1. Let $X = [0, 2]$ with metric $d(x, y) = |x - y|$ and for each $t \in (0, 1)$, define

$$F(x, y, t) = \begin{cases} \frac{t}{t + |x-y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0, \end{cases}.$$

$$L(x, y, t) = \begin{cases} \frac{|x-y|}{t + |x-y|}, & \text{if } t > 0; \\ 1, & \text{if } t = 0, \end{cases}$$

for all $x, y \in X$. Clearly $(X, F, L, *, \lozenge)$ is an intuitionistic Menger space where $*$ is defined by $t * t \geqslant t$ and $\lozenge$ defined by $(1-t)\lozenge(1-t) \leqslant (1-t)$. Let us define mappings A, B, S and T as follows:

$$A(x) = B(x) = \begin{cases} 1, & 0 \leqslant x \leqslant 1 \\ \frac{3}{4}, & 1 < x \leqslant 2 \end{cases}.$$

$$S(x) = T(x) = \begin{cases} \frac{x+1}{2}, & 0 \leqslant x \leqslant 1 \\ 2, & 1 < x \leqslant 2 \end{cases}.$$

Let us consider a sequence $\{x_n\}$ in X defined by $x_n = 1 - \frac{1}{n}$ for $n \in N$ such that $\lim_{n\to\infty} Ax_n = 1 = \lim_{n\to\infty} Sx_n$ and $\lim_{n\to\infty} ASx_n = 1 = A(1)$; $\lim_{n\to\infty} A^2x_n = 1 = S(1)$, $\lim_{n\to\infty} S^2x_n = 1 = A(1)$. Hence $\{A, S\}$ is weakly subsequentially continuous and compatible of type (E). Proceeding in the same way, we can easily show that$\{B, T\}$ is weakly subsequentially continuous and compatible of type (E). For each $r, s \in [0, 1]$ and $\psi, \eta \in \Psi$, define the maps $\psi, \eta : [0, 1] \to [0, 1]$ as $\psi(s) = \frac{2s}{s+1}$ and $\eta(r) = \frac{r}{2r-1}$.

Next, for $k = \frac{3}{4}$, we'll consider the different cases for which the following inequalities holds:

$$F(Ax, By, kt) \geqslant \psi[\min\{F(Sx, Ty, t), F(Ax, Sx, t), F(By, Ty, t), F(Sx, By, t), F(Ty, Ax, t)\}]$$

and

$$L(Ax, By, kt) \leqslant \eta[\max\{L(Sx, Ty, t), L(Ax, Sx, t), L(By, Ty, t), L(Sx, By, t), L(Ty, Ax, t)\}].$$

1. If $x, y \in [0, 1]$, we have $F(Ax, By, kt) = 1 \geqslant \psi(\frac{t}{t+\frac{|x-y|}{2}}) \geqslant \frac{t}{t+\frac{|x-y|}{2}} = F(Sx, Ty, t)$ and $L(Ax, By, kt) = 0 \leqslant \eta(\frac{\frac{|x-y|}{2}}{t+\frac{|x-y|}{2}}) \leqslant \frac{\frac{|x-y|}{2}}{t+\frac{|x-y|}{2}} = L(Sx, Ty, t)$.

Where $\psi(\frac{t}{t+\frac{|x-y|}{2}}) = \frac{2t}{2t+\frac{|x-y|}{2}}$ and $\eta(\frac{\frac{|x-y|}{2}}{t+\frac{|x-y|}{2}}) = \frac{\frac{|x-y|}{2}}{\frac{|x-y|}{2}-t}$.

2. If $x \in [0, 1]$ and $y \in (1, 2]$ we have $F(Ax, By, kt) = \frac{kt}{kt+0.25} \geqslant \psi(\frac{t}{t+1.25}) \geqslant \frac{t}{t+1.25} = F(By, Ty, t)$ and $L(Ax, By, kt) = \frac{1}{kt+0.25} \leqslant \eta(\frac{1.25}{t+1.25}) \leqslant \frac{1.25}{t+1.25} = L(By, Ty, t)$.

Where $\psi(\frac{t}{t+1.25}) = \frac{2t}{2t+1.25}$ and $\eta(\frac{1.25}{t+1.25}) = \frac{1.25}{t-1.25}$.

3. If $x \in (1, 2]$ and $y \in [0, 1]$ we have $F(Ax, By, kt) = \frac{kt}{kt+0.25} \geqslant \psi(\frac{t}{t+1.25}) \geqslant \frac{t}{t+1.25} = F(Ax, Sx, t)$ and $L(Ax, By, kt) = \frac{0.25}{kt+0.25} \leqslant \eta(\frac{1.25}{t+1.25}) \leqslant \frac{1.25}{t+1.25} = L(Ax, Sx, t)$.

Where $\psi(\frac{t}{t+1.25}) = \frac{2t}{2t+1.25}$ and $\eta(\frac{1.25}{t+1.25}) = \frac{1.25}{t-1.25}$.

4. If $x, y \in (1, 2]$, we have $F(Ax, By, kt) = 1 \geqslant \psi(\frac{t}{t+1.25}) \geqslant \frac{t}{t+1.25} = F(Sx, By, t)$ and $L(Ax, By, kt) = 0 \leqslant \eta(\frac{1.25}{t+1.25}) \leqslant \frac{1.25}{t+1.25} = L(Sx, By, t)$.

Where $\psi(\frac{t}{t+1.25}) = \frac{2t}{2t+1.25}$ and $\eta(\frac{1.25}{t+1.25}) = \frac{1.25}{t-1.25}$.

Thus, all the conditions of Theorem (3.1) are satisfied and A,B,S,T have a unique common fixed point $x = 1$.

Theorem 3.2. *Let A, B, S and T be four self mappings of an intuitionistic Menger space* $(X, F, L, *, \Diamond)$ *with continuous t-norm* $*$ *and continuous t-conorm* $\Diamond$ *satisfying* $t * t \geqslant t$ *and* $(1-t)\Diamond(1-t) \leqslant (1-t)$ *for all* $0 < t < 1$. *Suppose* $\psi, \eta \in \Psi$ *and there exists a constant* $k \in (0, 1)$ *such that for all* $x, y \in X$ *and* $t > 0$ *satisfying* (3.1) *and* (3.2). *Assume that* (*i*) *the pair* $\{A, S\}$ *is* A*-compatible of type (E) and* A*-subsequentially continuous.*

(*ii*) *the pair* $\{B, T\}$ *is* B*-compatible of type (E) and* B*-subsequentially continuous.*

Then A*,* B*,* S *and* T *have a unique common fixed point in* X*.*

Proof. The proof is obvious as on the lines of Theorem 3.1. □

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