Free commutative B-algebras

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Abstract. In this paper, the notion free commutative B-algebras was introduced. In order to characterize free commutative B-algebras and obtain its properties, the concept of external direct product of B-algebras was also introduced.
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1. Introduction

In [8], Neggers, J. and Kim, H. S. introduced the notion of B-algebras. A *B-algebra* is a nonempty set X with a constant 0 and a binary operation " $*$ " satisfying the following axioms for all $x, y, z \in X$: (B1) $x * x = 0$, (B2) $x * 0 = x$, (B3) $(x * y) * z = x * (z * (0 * y))$. B-algebra can be derived from a given group and a group can be derived from a B-algebra. From this relationship, parallel concepts and results were formulated and established. A B-algebra $(X; *, 0)$ is said to be *commutative* if $a * (0 * b) = b * (0 * a)$ for any $a, b \in X$. They also introduced the concepts of subalgebras and normality in [9]. A nonempty subset A of X is said to be a *subalgebra* of X if $a * b \in A$ for all $a, b \in A$. It is said to be *normal* in X if for any $x, y, a, b \in X$ with $x * y, a * b \in A$ implies $(x * a) * (y * b) \in A$.

In [10], Walendziak, A. characterized normality in B-algebras. The concept of B-homomorphism was introduced by Neggers, J., and Kim, H. S. [9].

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A map $\phi : X \to Y$, where $(X; *_X, 0_X)$ and $(Y; *_Y, 0_Y)$ are B-algebras, is called a *B-homomorphism* if $\phi(x *_X y) = \phi(x) *_Y \phi(y)$ for any $x, y \in X$. If ϕ is onto (respectively, one-to-one), then ϕ is called a *B-epimorphism* (respectively, *B-monomorphism*). Moreover, if ϕ is a bijection, then ϕ is called a *B-isomorphism*. The *kernel* of ϕ, denoted by $Ker\ \phi$, is defined to be the set $\{x \in X : \phi(x) = 0\}$. The following properties for a B-algebra are used in this paper : for any $x, y, z \in X$, (P1) $x*y = 0$ implies $x = y$, (P2) $0*(0*x) = x$, (P3) $x*(y*z) = (x*(0*z))*y$, and (P4) $x*y = 0*(y*x)$. If X is commutative, then (P5) $(x*y)*z = (x*z)*y$ and (P6) $x*(x*y) = y$ [8, 11, 6]. In [4], Gonzaga, N. C. and Vilela, J. P. introduced cyclic B-algebras as well as established the laws of exponents for B-algebras. In [1], Endam, J. C. and Teves, R. C. established some properties of cyclic B-algebras. In [7], Lincong, J. A. and Endam, J. C. introduced the concept of direct product of B-algebras and established its structure. From these parallel concepts and results, one may wonder if every concept in groups has its counterpart for B-algebras and if every result is applicable to B-algebras. In this paper, we introduced the concept of free commutative B-algebra. In order to investigate and characterize this type of B-algebra, we also introduced the external direct product of B-algebras. Throughout this paper, X means a B-algebra $(X; *, 0)$.

2. Preliminaries

This section presents some concepts and results needed in this paper. We start with some examples of a B-algebra.

Example 2.1. [[1]] Let $X = \{e, a, b, c\}$ be a set whose binary operation $*$ is shown in the following Cayley Table.

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	a	e	a
c	c	b	a	e

By routine calculations, X is a commutative B-algebra. X is called the *Klein B-algebra* K_4.

Example 2.2. [[7]] Consider the set of integers $\mathbb{Z}$ and the usual subtraction of integers " $-$ ". Then $(\mathbb{Z}; -, 0)$ is a B-algebra. We denote $(\mathbb{Z}; -, 0)$ by $^*\mathbb{Z}$. Since for every $x, y \in \mathbb{Z}$, $x - (0 - y) = x + y = y + x = y - (0 - x)$, $^*\mathbb{Z}$ is commutative.

In [2], the intersection of a family of subalgebras of a B-algebra X is a subalgebra of X.

Definition 2.3 ([4]). *Let $A \subseteq X$ be nonempty and $\{N_\alpha : \alpha \in I\}$ be a collection of subalgebras of X with $A \subseteq N_\alpha$ for all $\alpha \in I$. Then $\bigcap_{\alpha \in I} N_\alpha$ is called the* subalgebra X generated by A*, denoted by $\langle A \rangle_B$. If A is finite, then we say* *that* $\langle A\rangle_B$ *is* finitely generated. *If* $A = \{a\}$*, then* $\langle A\rangle_B = \langle a\rangle_B$ *is called the* cyclic subalgebra of X generated by a. *If* $X = \langle S\rangle_B$*, then* S *is called a* set of generators *for* X*.*

Remark 2.4 ([4])**.** Let S be a subset of X. Then $\langle S\rangle_B$ is the smallest subalgebra of X containing S. If either $S = \emptyset$ or $S = \{0\}$, then $\langle S\rangle_B = \{0\}$. If S is a subalgebra of X, then $\langle S\rangle_B = S$. In particular, $\langle X\rangle_B = X$.

Let $x \in X$. From [8], Neggers, J. and Kim, H. S. defined $x^n = x^{n-1} * (0 * x)$ $(n \geq 1)$ and $x^0 = 0$. Note that $x^1 = x^0 * (0 * x) = 0 * (0 * x) = x$ by (P2). Gonzaga, N. C. and Vilela, J. P. [4] defined $-x = 0 * x$ and $x^{-n} = (-x)^n$ for each $n \geq 1$. In [1], Endam, J. C. and Teves, R. C. defined $x^m = 0 * x^{-m}$ for $m \leq -1$.

By (B3), $(x * y) * z = x * (z * (0 * y))$. For convenience, we directly write $(x * y) * z = x * (z * y^{-1})$ at times. Similarly, by (P3) $x * (y * z) = (x * z^{-1}) * y$.

Definition 2.5 ([3])**.** *The* order *of* $a \in X$*, denoted by* $|a|_B$*, is the smallest positive integer* k *such that* $a^k = 0$*.*

$a \in X$ *is said to be of* finite order *if* $|a|_B < +\infty$*; otherwise, it is of* infinite order*.*

Theorem 2.6 ([3])**.** *Let* $a \in X$*. Then* $|a|_B = |\langle a\rangle_B|$*. In particular, if* $|a|_B = n < +\infty$*, then* $\langle a\rangle_B = \{0, a, a^2, ..., a^{n-1}\}$*.*

Theorem 2.7 ([4])**.** *If* $\emptyset \neq A \subseteq X$*, then* $\langle A\rangle_B$ *consists of all finite products* $(...((a_1^{n_1} * a_2^{n_2}) * a_3^{n_3}) * ... * a_{t-1}^{n_{t-1}}) * a_t^{n_t}$ *where* $a_i \in A$ *and* $n_j \in \mathbb{Z}$ *for all* $i, j = 1, 2, ..., t$*, with* $t < \infty$*.*

Corollary 2.8 ([4])**.** *Let* $x \in X$ *and* $m, n \in \mathbb{Z}$*. Then:*

(*i*) $x^{-n} = (-x)^n = -(x^n) = 0 * x^n$*;*

(*ii*) $x^m * x^n = x^{m-n}$*.*

Theorem 2.9 ([8])**.** *Let* $x \in X$*. Then* $(x^m)^n = x^{mn}$ *for any* $m, n \in \mathbb{Z}$*.*

Corollary 2.10 ([4])**.** *If* $a_i, b_j \in X$ *and* $m_i, n_j \in \mathbb{Z}$ *for* $i = 1, 2, ..., r$ *and* $j = 1, 2, ..., s$*,* $r, s < +\infty$*, then*

$$\begin{aligned}&\left[...((a_1^{m_1} * a_2^{m_2}) * a_3^{m_3}) * ... * a_r^{m_r}\right] * \left[...((b_1^{n_1} * b_2^{n_2}) * b_3^{n_3}) * ... * b_s^{n_s}\right]\\&= (...(((...((a_1^{m_1} * a_2^{m_2}) * a_3^{m_3}) * ... * a_r^{m_r}) * b_s^{-n_s}) * b_{s-1}^{-n_{s-1}}) * ... * b_2^{-n_2}) * b_1^{n_1}.\end{aligned}$$

Definition 2.11 ([2])**.** *Let* H *and* K *be subalgebras of* X*. Define the subset* HK *of* X *to be the set* $HK = \{x \in X : x = h * (0 * k), h \in H, k \in K\}$*.*

Theorem 2.12. *Let* H *and* K *be subalgebras of* X*. Then*

(i) [1] HK *is a subalgebra of* X *if and only if* $HK = \langle H \cup K\rangle_B$*;*

(ii) [2] *HK is a subalgebra of X if and only if $HK = KH$;*

(iii) [2] *$HK = KH$ whenever H and K are normal in X.*

Theorem 2.13 ([10]). *A subalgebra N of X is normal if and only if $x*(x*y) \in N$ for any $x \in X$, $y \in N$.*

Remark 2.14 ([5]). Let N_1 and N_2 be normal subalgebras of X such that $N_1 \cap N_2 = \{0\}$. Then $x * (0 * y) = y * (0 * x)$ for all $x \in N_1$ and $y \in N_2$.

Theorem 2.15 ([3]). *Every infinite cyclic B-algebra is isomorphic to the B-algebra $^*\mathbb{Z}$.*

Proposition 2.16 ([9]). *Let $\varphi : X \to Y$ be a B-homomorphism. Then φ is one-to-one if and only if $Ker\ \varphi = \{0_X\}$.*

Definition 2.17 ([5]). *Let $\{X_i : i \in I\}$ be a family of B-algebras. The direct product of this family, denoted by $\prod_{i\in I} X_i$, is the set of all functions $f : I \to \bigcup_{i\in I} X_i$ given by $f(i) \in X_i$ for each $i \in I$ and whose operation is componentwise.*

Theorem 2.18 ([5]). *If $\{X_i : i \in I\}$ is a family of B-algebras, then*

(i) *$\prod_{i\in I} X_i$ is a B-algebra;*

(ii) *X_i is commutative if and only if $\prod_{i\in I} X_i$ is commutative.*

Theorem 2.19 ([5]). *Let $\{f_i : X_i \to Y_i : i \in I\}$ be a family of B-homomorphisms and let $f = \prod f_i$ be the map $\prod_{i\in I} X_i \to \prod_{i\in I} Y_i$, given by $\{a_i\} \mapsto \{f_i(a_i)\}$. Then f is a B-homomorphism such that $f\left(\prod_{i\in I} X_i\right) = \prod_{i\in I} Y_i$ and $Ker\ f = \prod_{i\in I} Ker\ f_i$. Consequently, f is a B-monomorphism [respectively B-epimorphism] if and only if each f_i is.*

Definition 2.20 ([5]). *Let J_1 and J_2 be normal subalgebras of X. Then X is called the* internal product *of J_1 and J_2 if every $a \in A$ can be uniquely expressed as $a = a_1 * (0 * a_2)$, where $a_1 \in J_1$ and $a_2 \in J_2$.*

Theorem 2.21 ([5]). *Let J_1 and J_2 be normal subalgebras of X. Then X is an internal direct product of J_1 and J_2 if and only if $A = J_1J_2$ and $J_1 \cap J_2 = \{0\}$.*

3. Some properties of B-algebras

If N is a subalgebra of a commutative B-algebra X, then by (P6), $x * (x * y) = y \in N$ for any $x \in X$, $y \in N$. By Theorem 2.13, it follows that any subalgebra is normal in a commutative B-algebra. The following Lemma follows from (P4).

Lemma 3.1. *For all $x, y \in X$, $(x * y)^{-1} = y * x$.*

Proposition 3.2. *Let $x, y \in X$. Then for all $n, m \in \mathbb{Z}$, $(x*y^n)*y^m = x*y^{n+m}$.*

Proof. Let $x, y \in X$. Then by (B3) and Corollary 2.8(i) and (ii), $(x * y^n) * y^m = x * (y^m * (0 * y^n)) = x * (y^m * y^{-n}) = x * y^{m-(-n)} = x * y^{m+n}$. □

Proposition 3.3. *Let X be a commutative B-algebra, $x_1, x_2, ..., x_k \in X$ and $n_1, n_2, ..., n_k \in \mathbb{Z}$. Then*

(i)
$$\left[...((x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}) * ... * x_k^{n_k}\right] * x_1^m = (...((x_1^{n_1-m} * x_2^{n_2}) * x_3^{n_3}) * ... * x_{k-1}^{n_{k-1}}) * x_k^{n_k};$$

(ii)
$$\left[...(...((x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}) * ... * x_i^{n_i}) * ... * x_k^{n_k}\right] * x_i^m = (...(...((x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}) * ... * x_i^{n_i+m}) * ... * x_{k-1}^{n_{k-1}}) * x_k^{n_k}.$$

for all $i = 2, ..., k$.

Proof. (i) By (P5) and Corollary 2.8(ii),

$$\begin{aligned}
&\left[...((x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}) * ... * x_k^{n_k}\right] * x_1^m \\
&\quad = \left[...((x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}) * ... * x_1^m\right] * x_k^{n_k} \\
&\quad\ \vdots \\
&\quad = (...((x_1^{n_1} * x_1^m) * x_2^{n_2}) * ... * x_{k-1}^{n_{k-1}}) * x_k^{n_k} \\
&\quad = (...((x_1^{n_1-m} * x_2^{n_2}) * x_3^{n_3}) * ... * x_{k-1}^{n_{k-1}}) * x_k^{n_k}.
\end{aligned}$$

(ii) If $k = 2$ (implying that $i = 2$), then the conclusion follows from Proposition 3.2. We now consider $k > 2$. By Proposition 3.2 and (P5),

$$\begin{aligned}
&\left[...(...((x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}) * ... * x_i^{n_i}) * ... * x_k^{n_k}\right] * x_i^m \\
&\quad = [...(...((x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}) * ... * x_i^{n_i}) * ... * x_i^m\right] * x_k^{n_k} \\
&\quad\ \vdots \\
&\quad = (...((...((x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}) * ... * x_i^{n_i}) * x_i^m) * ... * x_{k-1}^{n_{k-1}}) * x_k^{n_k} \\
&\quad = (...(...((x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}) * ... * x_i^{n_i+m}) * ... * x_{k-1}^{n_{k-1}}) * x_k^{n_k}.
\end{aligned}$$

This completes the proof. □

Proposition 3.4. *Let X be a commutative B-algebra and $x_1, x_2, ..., x_k \in X$. Then for all $n_i, m_i \in \mathbb{Z}$,*

$$\left[...((x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}) * ... * x_k^{n_k}\right] * \left[...((x_1^{m_1} * x_2^{m_2}) * x_3^{m_3}) * ... * x_k^{m_k}\right] = (...((x_1^{n_1-m_1} * x_2^{n_2-m_2}) * x_3^{n_3-m_3}) * ... * x_{k-1}^{m_{k-1}}) * x_k^{n_k-m_k}.$$

Proof. Let X be a commutative B-algebra, $x_1, x_2, ..., x_k \in X$ and $n_i, m_i \in \mathbb{Z}$. By Proposition 3.3 and Corollary 2.10,

$$\begin{aligned}
&\left[...((x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}) * ... * x_k^{n_k}\right] * \left[...((x_1^{m_1} * x_2^{m_2}) * x_3^{m_3}) * ... * x_k^{m_k}\right] \\
&\quad = (...(((...((x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}) * ... * x_k^{n_k}) * x_k^{-m_k}) * x_{k-1}^{-m_{k-1}}) * x_{k-2}^{-m_{k-2}}) * \\
&\qquad ... * x_2^{-m_2}) * x_1^{m_1} \\
&\quad = (...((x_1^{n_1-m_1} * x_2^{n_2-m_2}) * x_3^{n_3-m_3}) * ... * x_{k-1}^{n_{k-1}-m_{k-1}}) * x_k^{n_k-m_k}.
\end{aligned}$$

$\square$

4. Direct product and free commutative B-algebra

Proposition 4.1. *Let $\{X_i : i \in I\}$ be a family of B-algebras and $\{u_i\} \in \prod_{i\in I} X_i$. Then $\{u_i\}^n = \{u_i^n\}$ for all $n \in \mathbb{Z}$.*

Proof. Let $\{X_i : i \in I\}$ be a family of B-algebras. By Theorem 2.18(i), $\left(\prod_{i\in I} X_i; \circledast, \{0\}\right)$ is a B-algebra. Let $\{u_i\} \in \prod_{i\in I} X_i$ and $n \in \mathbb{Z}$.

Case 1. $n = 0$. Then $\{u_i\}^0 = \{0\} = \{0^n\}$.

Case 2. $n > 0$. Now,

$$\begin{aligned}
\{u_i\}^n &= \{u_i\}^{n-1} \circledast (\{0\} \circledast \{u_i\}) \\
&= \left[\{u_i\}^{n-2} \circledast (\{0\} \circledast \{u_i\})\right] \circledast \{0 * u_i\} \\
&= \left(\{u_i\}^{n-2} \circledast \{0 * u_i\}\right) \circledast \{0 * u_i\} \\
&\ \ \vdots \\
&= \underbrace{\left[...\left[\left(\{u_i\} \circledast \{0 * u_i\}\right) \circledast \{0 * u_i\}\right] \circledast ... \circledast \{0 * u_i\}\right] \circledast \{0 * u_i\}}_{n \text{ terms}} \\
&= \left[...\left[\left(\{u_i * (0 * u_i)\}\right) \circledast \{0 * u_i\}\right] \circledast ... \circledast \{0 * u_i\}\right] \circledast \{0 * u_i\} \\
&= \left[...\left[\left(\{u_i^2\}\right) \circledast \{0 * u_i\}\right] \circledast ... \circledast \{0 * u_i\}\right] \circledast \{0 * u_i\} \\
&\ \ \vdots \\
&= \{u_i^{n-1}\} \circledast (\{0 * u_i\}) \\
&= \{u_i^{n-1} * (0 * u_i)\} \\
&= \{u_i^n\}.
\end{aligned}$$

Case 3. $n < 0$. Then $-n > 0$. By Case 2 and Corollary 2.8(i), $\{u_i\}^n = \{0\} \circledast \{u_i\}^{-n} = \{0\} \circledast \{u_i^{-n}\} = \{0 * u_i^{-n}\} = \{u_i^n\}$. $\square$

Definition 4.2. *The* external direct product *of a family of B-algebras $\{(X_i; *_i, 0_i) : i \in I\}$, denoted by $\prod_{i\in I}{}^{w} X_i$, is the set of all $f \in \prod_{i\in I} X_i$ such that $f(i) = 0_i$ for all but a finite number of $i \in I$. If all of the B-algebras X_i are commutative, $\prod_{i\in I}{}^{w} X_i$ is usually called the* external direct sum *and is denoted by $\sum_{i\in I} X_i$.*

If I is finite, then the external direct product coincides with direct product. The following remark follows from Definition 4.2 and definition of subalgebra.

Remark 4.3. If $\{X_i : i \in I\}$ is a family of B-algebras, then $\prod_{i\in I}{}^{w} X_i$ is a subalgebra of $\prod_{i\in I} X_i$.

Consider the B-algebra ${}^*\mathbb{Z} = (\mathbb{Z}; -, 0)$. For clarity, we emphasize that ${}^*\mathbb{Z}$ denotes the B-algebra and $\mathbb{Z}$ the set of integers. To avoid confusion, we also emphasize that $\prod^{w} {}^*\mathbb{Z}$ (respectively $\sum {}^*\mathbb{Z}$) would denote the B-algebra and $\prod^{w} \mathbb{Z}$ (respectively $\sum \mathbb{Z}$) the set of elements. The following results are proved similarly as in Theorems 2.18(ii) and 2.19.

Remark 4.4. Let $\{(A_i;*,0) : i \in I\}$ be a family of B-algebras. Then A_i is commutative if and only if $\prod_{i\in I}{}^{w} A_i$ is commutative.

Remark 4.5. Let $\{f_i : X_i \to Y_i : i \in I\}$ be a family of B-homomorphisms and let $f = \prod f_i$ be the map $\prod_{i\in I}{}^{w} X_i \to \prod_{i\in I}{}^{w} Y_i$, given by $\{a_i\} \mapsto \{f_i(a_i)\}$. Then f is a B-homomorphism such that $f\left(\prod_{i\in I}{}^{w} X_i\right) = \prod_{i\in I}{}^{w} Y_i$ and $Ker\ f = \prod_{i\in I}{}^{w} Ker\ f_i$. Consequently, f is a B-monomorphism [respectively B-epimorphism] if and only if each f_i is.

The following remark follows from Theorems 2.12 and 2.21.

Remark 4.6. Let H and K be normal subalgebras of X. Then X is the internal direct product of H and K if and only if $X = \langle H \cup K\rangle_B$ and $H \cap K = \{0\}$.

Theorem 4.7. *Let $\{N_i : i \in I\}$ be a family of normal subalgebras of X such that $X = \left\langle \bigcup_{i\in I} N_i\right\rangle_B$ and $N_k \cap \left\langle \bigcup_{i\neq k} N_i\right\rangle_B = \{0\}$ for each $k \in I$. Then $X \cong \prod_{i\in I}{}^{w} N_i$.*

Proof. Let $x \in N_{i_s}$ and $y \in N_{i_t}$ with $s \neq t$. By Remark 2.14, $x * y^{-1} = x * (0 * y) = y * (0 * x) = y * x^{-1}$. By (B3), Remark 2.14 and (P4) we also have $x^{-1} * y = (0 * x) * y = 0 * (y * (0 * x)) = 0 * (x * (0 * y)) = (0 * y) * x = y^{-1} * x$.

Let $\{u_i\} \in \prod_{i\in I}{}^{w} N_i$. Then there exists $I' = \{i_1, i_2, ..., i_n\} \subseteq I$ such that $u_i = 0$ for all $i \in I \setminus I'$ and $u_{i_1}, u_{i_2}, ..., u_{i_n} \neq 0$. For each $j = 1, 2, ..., n$, there is a normal subalgebra N_{i_j} such that $u_{i_j} \in N_{i_j}$. By hypothesis, it follows that $u_{i_j} \notin N_{i_k}$ for all $j \neq k$. Define $f : \prod_{i\in I}{}^{w} N_i \to X$ by $f(\{u_i\}) = (...((u_{i_1} * u_{i_2}^{-1})u_{i_3}^{-1}) * ... * u_{i_{n-1}}^{-1}) * u_{i_n}^{-1}$. By (B3), (P3), and Corollary 2.8(i), notice that for $a \in N_r$, $b \in N_s$ and $c \in N_t$, where $r, s, t \in I$ are distinct, we have $(a*b^{-1})*c^{-1} = (b*a^{-1})*c^{-1} = b*(c^{-1}*(0*a^{-1})) = b*(c^{-1}*a) = b*(a^{-1}*c) = (b*c^{-1})*a^{-1}$. Similarly, $(a * c^{-1}) * b^{-1} = (c * b^{-1}) * a^{-1} = (c * a^{-1}) * b^{-1}$. This could be extended to any finite number of elements. This implies that the arrangement of the elements does not matter. It follows that f is well-defined.

Let $\{a_i\}, \{b_i\} \in \prod_{i\in I}{}^{w} N_i$. Then there exist $I', I'' \subseteq I$ with $|I'|, |I''| < \infty$ such that $a_i, b_j = 0$ for all $i \in I \setminus I'$ and $j \in I \setminus I''$. Since $|I'|, |I''| < \infty$, $|I' \cup I''| = n < \infty$. Note that the operation $*$ is the same in every N_i. Thus,

$$\begin{aligned} f(\{a_i\} \circledast \{b_i\}) &= f(\{a_i * b_i\}) \\ &= [...[[(a_{i_1} * b_{i_1}) * (a_{i_2} * b_{i_2})^{-1}] * (a_{i_3} * b_{i_3})^{-1}] * \\ &\quad ... * (a_{i_{n-1}} * b_{i_{n-1}})^{-1}] * (a_{i_n} * b_{i_n})^{-1}, \end{aligned}$$

where $i_j \in I' \cup I''$ for all $j = 1, 2, ..., n$. Consider $(a_{i_1} * b_{i_1}) * (a_{i_2} * b_{i_2})^{-1}$. By Lemma 3.1, (B3) and (P3),

$$\begin{aligned} (a_{i_1} * b_{i_1}) * (a_{i_2} * b_{i_2})^{-1} &= (a_{i_1} * b_{i_1}) * (b_{i_2} * a_{i_2}) \\ &= a_{i_1} * ((b_{i_2} * a_{i_2}) * b_{i_1}^{-1}) \end{aligned}$$

$$\begin{aligned} &= a_{i_1} * (b_{i_1} * (b_{i_2} * a_{i_2})^{-1}) \\ &= a_{i_1} * (b_{i_1} * (a_{i_2} * b_{i_2})) \\ &= a_{i_1} * ((b_{i_1} * b_{i_2}^{-1}) * a_{i_2}) \\ &= (a_{i_1} * a_{i_2}^{-1}) * (b_{i_1} * b_{i_2}^{-1}). \end{aligned}$$

Similarly, $[(a_{i_1} * b_{i_1}) * (a_{i_2} * b_{i_2})^{-1}] * (a_{i_3} * b_{i_3})^{-1} = ((a_{i_1} * a_{i_2}^{-1}) * a_{i_3}^{-1}) * ((b_{i_1} * b_{i_2}^{-1}) * b_{i_3}^{-1})$. Continuing in this manner, we obtain $[...[[(a_{i_1} * b_{i_1}) * (a_{i_2} * b_{i_2})^{-1}] * (a_{i_3} * b_{i_3})^{-1}] * ... * (a_{i_{n-1}} * b_{i_{n-1}})] * (a_{i_n} * b_{i_n})^{-1} = [...((a_{i_1} * a_{i_2}^{-1}) * a_{i_3}^{-1}) * ... * a_{i_n}^{-1}] * [...((b_{i_1} * b_{i_2}^{-1}) * b_{i_3}^{-1}) * ... * b_{i_n}^{-1}] = f(\{a_i\}) * f(\{b_i\})$.

Hence, $f(\{a_i\} \circledast \{b_i\}) = f(\{a_i\}) * f(\{b_i\})$. Accordingly, f is a B-homomorphism.

Let $y \in X = \langle \bigcup_{i \in I} N_i \rangle_B$. Then $y = (...((x_1^{m_1} * x_2^{m_2}) * x_3^{m_3}) * ... * x_{n-1}^{m_{n-1}}) * x_n^{m_n}$ for some $x_j \in \bigcup_{i \in I} N_i$, $n_j \in \mathbb{Z}$. For each $j = 1, 2, ..., n$, there exists N_{i_j} such that $x_j \in N_{i_j}$. Since N_{i_j} is a subalgebra, $a_j = x_j^{m_j} \in N_{i_j}$ for each j. Hence, $a_j^{-1} = x_j^{-m_j} \in N_{i_j}$ for each j. By hypothesis, it follows that $a_j \notin N_{i_k}$ for all $j \neq k$. Let $\{u_i\} \in \prod_{i \in I} N_i$ be such that $u_i = 0$ for all $i \neq i_j$ $j = 1, 2, ..., n$, $u_{i_1} = a_1$ and $u_{i_k} = a_k^{-1}$, $k = 2, ..., n$. Since $m < \infty$, $\{u_i\} \in \prod_{i \in I}^{w} N_i$ and

$$\begin{aligned} f(\{u_i\}) &= (...((a_1 * (a_2^{-1})^{-1}) * (a_3^{-1})^{-1}) * ... * (a_{n-1}^{-1})^{-1}) * (a_n^{-1})^{-1} \\ &= (...((a_1 * a_2) * a_3) * ... * a_{n-1}) * a_n \\ &= (...((x_1^{m_1} * x_2^{m_2}) * x_3^{m_3}) * ... * x_{n-1}^{m_{n-1}}) * x_n^{m_n} \\ &= y. \end{aligned}$$

Hence, f is onto.

Let $\{u_i\} \in \prod_{i \in I}^{w} N_i$. Then there exists $I' = \{i_1, i_2, ..., i_n\} \subseteq I$ such that $u_i = 0$ for all $i \in I \setminus I'$ and $u_{i_1}, u_{i_2}, ..., u_{i_n} \neq 0$. Suppose $\{u_i\} \in Ker\ f$. Then $f(\{u_i\}) = 0$. Thus, $(...((u_{i_1} * u_{i_2}^{-1}) u_{i_3}^{-1}) * ... * u_{i_{n-1}}^{-1}) * u_{i_n}^{-1} = 0$. By (P1), it follows that $(...((u_{i_1} * u_{i_2}^{-1}) u_{i_3}^{-1}) * ... * u_{i_{n-2}}^{-1}) * u_{i_{n-1}}^{-1} = u_{i_n}^{-1}$. Since $u_{i_n} \in N_{i_n}$, $u_{i_n}^{-1} = 0 * u_{i_n} \in N_{i_n}$. Hence, $u_{i_n}^{-1} \in N_{i_n} \cap \left\langle \bigcup_{i \neq i_n} N_i \right\rangle_B$. By hypothesis, it follows that $u_{i_n}^{-1} = 0$. Accordingly, $u_{i_n} = (u_{i_n}^{-1})^{-1} = 0^{-1} = 0$. Repeating the process, we have $u_{i_j} = 0$ for all j. Thus, $u_i = 0$ for all $i \in I$. Hence, $\{u_i\} = \{0\}$. By Proposition 2.16, f is one-to-one. Therefore, f is a B-isomorphism. □

In view of Remark 4.6 and Theorem 4.7, we shall extend the notion of internal direct product to an arbitrary family of normal subalgebras.

Definition 4.8. *The B-algebra X satisfying the conditions of Theorem 4.7 is called the* internal direct product *of the family $\{N_i : i \in I\}$ (or the* internal direct sum *if X is commutative).*

Example 4.9. *Consider the Klein B-algebra K_4 described in Example 2.1 and its subalgebras $\langle a \rangle_B = \{e, a\}$ and $\langle b \rangle_B = \{e, b\}$. Since K_4 is commutative, $\langle a \rangle_B$ and $\langle b \rangle_B$ are normal in K_4. Notice that $\langle a \rangle_B \cap \langle b \rangle_B = \{e\}$. Since $e, a, b, c =$*

$a*b \in \langle\langle a\rangle_B \cup \langle b\rangle_B\rangle_B$, $K_4 = \langle\langle a\rangle_B \cup \langle b\rangle_B\rangle_B$. *Therefore,* K_4 *is the internal direct product of* $\langle a\rangle_B$ *and* $\langle b\rangle_B$.

Example 4.10. *Consider the B-algebra* ${}^*\mathbb{Z} = (\mathbb{Z}; -, 0)$ *and* $X = \sum {}^*\mathbb{Z}$ *(external direct sum) where the copies of* ${}^*\mathbb{Z}$ *is indexed by an arbitrary set* I. *For each* $i \in I$, *let* $N_i = \{\{u_j\} \in \sum \mathbb{Z} : u_i \in \mathbb{Z}, u_j = 0 \text{ for all } j \neq i\}$. *Clearly,* N_i *is a subalgebra of* X *for all* $i \in I$. *Since* ${}^*\mathbb{Z}$ *is commutative,* X *is also commutative, by Remark 4.4. Hence,* N_i *is normal in* X *for all* i.

Let $\{u_i\} \in \sum \mathbb{Z}$. *Then* $u_i \in \mathbb{Z}$ *and* $u_i = 0$ *for all but a finite number of* $i \in I$. *Thus, there exists* $t \in \mathbb{Z}^+$ *and* $k_1, k_2, ..., k_t \in I$ *such that* $u_{k_j} \neq 0$ *and* $u_i = 0$ *for all* $i \neq k_j$ *for all* $j = 1, 2, ..., t$.

Consider $k_1 \in I$. *Then for each* $i \in I$, *we can write* $u_i = w'_{1,i} - w_{1,i}$ *where* $w'_{1,i} = u_i$ *and* $w_{1,i} = 0$ *for all* $i \neq k_1$, $w'_{1,k_1} = 0$ *and* $w_{1,k_1} = -u_{y_1}$. *Thus,* $\{u_i\} = \{w'_{1,i} - w_{1,i}\} = \{w'_{1,i}\} * \{w_{1,i}\}$. *Consider* $k_2 \in I$. *Then for each* $i \in I$, *we can write* $w'_{1,i} = w'_{2,i} - w_{2,i}$ *where* $w'_{2,i} = w'_{1,i} = u_i$ *and* $w_{2,i} = 0$ *for all* $i \neq k_2$, $w'_{2,k_2} = 0$ *and* $w_{2,k_2} = -w'_{1,k_2} = -u_{k_2}$. *Thus,*

$$\{u_i\} = \{w'_{1,i}\} * \{w_{1,i}\} = \{w'_{2,i} - w_{2,i}\} * \{w_{1,i}\} = (\{w'_{2,i}\} * \{w_{2,i}\}) * \{w_{1,i}\}.$$

Continuing this process, we have

$$\begin{aligned} \{u_i\} &= (\{w'_{2,i}\} * \{w_{2,i}\}) * \{w_{1,i}\} \\ &= ((\{w'_{3,i}\} * \{w_{3,i}\}) * \{w_{2,i}\}) * \{w_{1,i}\} \\ &\vdots \\ &= \big[...\big((\{w'_{k-1,i}\} * \{w_{t-1,i}\}) * \{w_{t-2,i}\}\big) * ... * \{w_{2,i}\}\big] * \{w_{1,i}\} \end{aligned}$$

where $w_{j,i} = 0$ *for all* $i \neq k_j$ *and* $w_{j,k_j} = -u_{k_j}$ *for all* $j = 1, 2, ..., t-1$. *Since there are* t $k'_j s$, *it follows that* $w'_{t-1,i} = 0$ *for all* $i \neq k_t$ *and* $w'_{t-1,t} = u_{k_t}$. *Let* $w'_{t-1,i} = w_{t,i}$. *Thus,* $\{u_i\} = \big[...\big((\{w_{t,i}\} * \{w_{t-1,i}\}) * \{w_{t-2,i}\}\big) * ... * \{w_{2,i}\}\big] * \{w_{1,i}\}$ *where* $w_{j,i} = 0$ *for all* $i \neq k_j$, $w_{j,k_j} = -u_{k_j}$, $j = 1, 2, ..., t-1$, *and* $w_{t,i} = 0$ *for all* $i \neq k_t$ *and* $w_{t,k_t} = u_{k_t}$. *Hence,* $\{w_{j,i}\} \in N_{j,k_j}$ *for* $j = 1, 2, ..., t$. *Accordingly,* $\{u_i\} \in \langle \bigcup_{i\in I} N_i\rangle_B$. *Thus,* $\sum \mathbb{Z} \subseteq \langle \bigcup_{i\in I} N_i\rangle_B$. *Hence,* $\sum \mathbb{Z} = \langle \bigcup_{i\in I} N_i\rangle_B$.

Suppose there exists $\{u_i\} \in N_k \cap \left\langle \bigcup_{q\neq k} N_q \right\rangle_B$ *such that* $\{u_i\} \neq \{0\}$. *Then* $\{u_i\} \in N_k$, $\{u_i\} \in \left\langle \bigcup_{q\neq k} N_q \right\rangle_B$ *and there exists* $r \in I$ *such that* $u_r \neq 0$. *Thus,* $u_i = 0$ *for all* $i \neq k$ *and* $u_k \in \mathbb{Z}$, $u_k \neq 0$. *Also,*

$$\{u_i\} = (...((\{w_{1,i}\}^{n_1} \circledast \{w_{2,i}\}^{n_2}) \circledast \{w_{3,i}\}^{n_3}) \circledast ... \circledast \{w_{t-1,i}\}^{n_{t-1}}) \circledast \{w_{t,i}\}^{n_t}$$

for some $\{w_{s,i}\} \in \bigcup_{q\neq k} N_q$, $s = 1, 2, ..., t$. *Hence, for each* s, *there exists* N_{q_s} *such that* $\{w_{s,i}\} \in N_{q_s}$, $q_s \neq k$. *It follows that* $w_{s,q_k} = 0$. *By Proposition 4.1,* $\{w_{s,i}\}^{n_s} = \{w_{s,i}{}^{n_s}\}$. *Thus,* $w_{s,q_k}{}^{n_s} = 0^{n_s} = 0$ *for all* $s = 1, 2, ..., t$.

Consider $\{w_{1,i}\}^{n_1} \circledast \{w_{2,i}\}^{n_2}$. *We have* $w_{1,q_k}{}^{n_1} = 0$ *and* $w_{2,q_k}{}^{n_2} = 0$. *It follows that by Proposition 4.1 and definition of* $\circledast$, $\{w_{1,i}\}^{n_1} \circledast \{w_{2,i}\}^{n_2} = \{w_{1,i}{}^{n_1}\} \circledast$

$\{w_{2,i}{}^{n_2}\}$ *where* $w_{1,q_k}{}^{n_1} * w_{2,q_k}{}^{n_2} = 0*0 = 0$, *by (B1). Continuing in this manner, it follows that*

$$(...((w_{1,q_k}{}^{n_1} * w_{2,q_k}{}^{n_2}) * w_{3,q_k}{}^{n_3}) * ... * w_{t-1,q_k}{}^{n_{t-1}}) * w_{t,q_k}{}^{n_t} \quad = \quad 0$$

But $u_k \neq 0$ *and* $u_k = (...((w_{1,q_k}{}^{n_1} * w_{2,q_k}{}^{n_2}) * w_{3,q_k}{}^{n_3}) * ... * w_{t-1,q_k}{}^{n_{t-1}}) * w_{t,q_k}{}^{n_t}$, *a contradiction. Hence,* $\{u_i\} = \{0\}$. *Therefore, X is the internal direct direct sum of the family* $N_i = \{\{u_j\} \in \sum \mathbb{Z} : u_i \in \mathbb{Z}, u_j = 0 \text{ for all } j \neq i\}$.

Definition 4.11. *Let X be a B-algebra and $S \subseteq X$. A combination of elements of S is a finite product* $[...[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * ... * x_{k-1}^{n_{k-1}}] * x_k^{n_k}$ *where $x_i \in S$ and $n_i \in \mathbb{Z}$ for all $i = 1, 2, ..., k$, with $k < \infty$.*

Definition 4.12. *Let $(X; *, 0)$ be a B-algebra. A subset $S \subseteq X$ is said to be a basis for X if $X = \langle S \rangle_B$ and for distinct $x_1, x_2, ..., x_k \in S$ and $n_i \in \mathbb{Z}$,* $[...[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * ... * x_{k-1}^{n_{k-1}}] * x_k^{n_k} = 0$ *implies $n_i = 0$ for all $i = 1, 2, ..., k$.*

Definition 4.13. *Let X be a commutative B-algebra. X is said to be a free commutative B-algebra if it has a nonempty basis.*

Example 4.14. Consider the B-algebra $^*\mathbb{Z} = (\mathbb{Z}, -, 0)$ and $\{1\} \subseteq \mathbb{Z}$. Note that $^*\mathbb{Z} = \langle 1 \rangle_B$. Also, for all $n \in \mathbb{Z}$, $1^n = 0$ if and only if $n = 0$. Thus, $\{1\}$ is a basis for $\mathbb{Z}$. Therefore, $^*\mathbb{Z} = (\mathbb{Z}; -, 0)$ is a free commutative B-algebra.

Example 4.15. By Remark 2.4, $\langle \emptyset \rangle_B = \{0\}$. Thus, $\{0\}$ is not a free commutative B-algebra.

Lemma 4.16. *Let $(F; *, 0)$ be a free commutative B-algebra with basis S. Then $0 \notin S$.*

Proof. Suppose $0 \in S$. Note that $0^1 = 0$. Since $1 \neq 0$, we get a contradiction to the definition of a basis. Hence, $0 \notin S$. □

Theorem 4.17. *Let $(F; *, 0)$ be a free commutative B-algebra with basis S. Then the representation of every element of F as a combination of elements of S is unique.*

Proof. Let $(F; *, 0)$ be a free commutative B-algebra with basis S and $u \in F$, $u \neq 0$. Then $u = [...[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * ... * x_{k-1}^{n_{k-1}}] * x_k^{n_k}$ for some $x_i \in S$, $n_i \in \mathbb{Z}$ and $x_i \neq 0$ for all i, by Lemma 4.16. Supppose $x_i = x_j$ for some $i \neq j$. Without loss of generality, we assume that $i < j$. By Proposition 3.3, we may assume that $x_i \neq x_j$ for all $i \neq j$. Suppose

$$\begin{aligned} u &= [...[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * ... * x_{k-1}^{n_{k-1}}] * x_k^{n_k} \quad \text{and} \\ u &= [...[(y_1^{m_1} * y_2^{m_2}) * y_3^{m_3}] * ... * y_{t-1}^{m_{t-1}}] * y_t^{m_t} \end{aligned}$$

for some $x_i, y_j \in S$, $n_i, m_j \in \mathbb{Z}$, $i = 1, 2, ..., k$, $j = 1, 2, ..., t$. Again, we may assume that $x_p \neq x_q$ and $y_p \neq y_q$ for all $p \neq q$. If $k > t$, then we may write

$$\begin{aligned} u &= [...[(y_1^{m_1} * y_2^{m_2}) * y_3^{m_3}] * ... * y_{t-1}^{m_{t-1}}] * y_t^{m_t} \\ &= \underbrace{[...[[[...[(y_1^{m_1} * y_2^{m_2}) * y_3^{m_3}] * ... * y_t^{m_t}] * 0] * 0] * ... * 0] * 0}_{k \text{ terms}} \\ &= [...[[...[(y_1^{m_1} * y_2^{m_2}) * y_3^{m_3}] * ... * y_t^{m_t}] * y_{t+1}^0] * ... * y_{k-1}^0] * y_k^0 \end{aligned}$$

where $y_i \in S$ for each $i = t+1, t+2, ..., k$. Thus, we may assume that $k = t$.

Case 1: $x_i = y_j$ for all $i, j = 1, 2, ..., k$. By Proposition 3.4,

$$\begin{aligned} 0 &= u * u \\ &= \left[...[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * ... * x_k^{n_k}\right] * \left[...[(y_1^{m_1} * y_2^{m_2}) * y_3^{m_3}] * ... * y_k^{m_k}\right] \\ &= \left[...[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * ... * x_k^{n_k}\right] * \left[...[(x_1^{m_1} * x_2^{m_2}) * x_3^{m_3}] * ... * x_k^{m_k}\right] \\ &= [...[(x_1^{n_1-m_1} * x_2^{n_2-m_2}) * x_3^{n_3-m_3}] * ... * x_{k-1}^{n_{k-1}-m_{k-1}}] * x_k^{n_k-m_k}. \end{aligned}$$

Since S is a basis, it follows that $n_i - m_i = 0$, that is, $n_i = m_i$ for all i.

Case 2. $x_i \neq y_i$ for some $i = 1, 2, ..., k$. By Corollary 2.10, Proposition 3.3 and (P5) for $i = 1$,

$$\begin{aligned} 0 &= \left[...[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * ... * x_k^{n_k}\right] * \left[...[(y_1^{m_1} * x_2^{m_2}) * x_3^{m_3}] * ... * x_k^{m_k}\right] \\ &= [...[[[...[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * ... * x_k^{n_k}] * x_k^{-m_k}] * x_{k-1}^{-m_{k-1}}] * ... * x_2^{-m_2}] * y_1^{m_1} \\ &= [...[(x_1^{n_1} * x_2^{n_2-m_2}) * x_3^{n_3-m_3}] * ... * x_k^{n_k-m_k}] * y_1^{m_1} \\ &= [...[[(x_1^{n_1} * y_1^{m_1}) * x_2^{n_2-m_2}] * x_3^{n_3-m_3}] * ... * x_{k-1}^{n_{k-1}-m_{k-1}}] * x_k^{n_k-m_k}. \end{aligned}$$

Thus, $n_j - m_j = 0$, that is, $n_j = m_j$ for all $j = 2, 3, ..., k$ and $n_1, m_1 = 0$. By Corollary 2.10, Proposition 3.3 and (P5) for $i = 2, 3, ..., k$,

$$\begin{aligned} 0 &= u * u \\ &= \Big[...[...[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * ... * x_i^{n_i}] * ... * x_k^{n_k}\Big] * \\ &\quad \Big[...[...[(x_1^{m_1} * x_2^{m_2}) * x_3^{m_3}] * ... * y_i^{m_i}] * ... * x_k^{m_k}\Big] \\ &= [...[[[[...[[[[...[...[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * ... * x_i^{n_i}] * ... * x_{k-1}^{n_{k-1}}] * x_k^{n_k}] * \\ &\quad x_k^{-m_k}] * x_{k-1}^{-m_{k-1}}] * ...] * x_{i+1}^{-m_{i+1}}] * y_i^{-m_i}] * x_{i-1}^{-m_{i-1}}] * ... * x_2^{-m_2}] * x_1^{m_1} \\ &= [...[[[[...[(x_1^{n_1-m_1} * x_2^{n_2-m_2}) * x_3^{n_3-m_3}] * ... * x_{i-1}^{n_{i-1}-m_{i-1}}] * x_i^{n_i}] * y_i^{-m_i}] * \\ &\quad x_{i+1}^{n_{i+1}-m_{i+1}}] * ... * x_{k-1}^{n_{k-1}-m_{k-1}}] * x_k^{n_k-m_k}. \end{aligned}$$

Thus, $n_j - m_j = 0$, that is, $n_j = m_j$ for all $j \neq i$ and $n_i, -m_i = 0$, that is, $n_i, m_i = 0$. Hence, $n_i, m_i = 0$, for all $i = 1, 2, ..., k$. Consequently, $x_i^{n_i} = x_i^0 = 0 = y_i^0 = x_i^{n_i}$. Hence,

$$u = [...[[...[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * ... * 0] * x_{i+1}^{n_{i+1}}] * ... * x_{k-1}^{n_{k-1}}] * x_k^{n_k} \quad \text{and}$$
$$u = [...[[...[(y_1^{m_1} * y_2^{m_2}) * y_3^{m_3}] * ... * 0] * y_{i+1}^{m_{i+1}}] * ... * y_{k-1}^{m_{k-1}}] * y_k^{m_k}$$

with $x_j = y_j$ and $n_j = m_j$ for all $j \neq i$. By (B2), 0 plays no role in each equality. Accordingly, we may disregard each i^{th} term. If follows that if

$$\begin{array}{rcl} u & = & [...[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * ... * x_{k-1}^{n_{k-1}}] * x_k^{n_k} \quad \text{and} \\ u & = & [...[(y_1^{m_1} * y_2^{m_2}) * y_3^{m_3}] * ... * y_{k-1}^{m_{k-1}}] * y_k^{m_k}, \end{array}$$

then $x_j = y_j$ and $n_j = m_j$ for all j. This proves the uniqueness of the representation. □

The following theorem is a characterization of a free commutative B-algebra.

Theorem 4.18. *Let $(F; *, 0)$ be a commutative B-algebra. Then the following are equivalent:*

(i) *F has a nonempty basis;*

(ii) *F is the internal direct sum of a family of infinite cyclic subalgebras;*

(iii) *F is (isomorphic to) a direct sum of copies of the B-algebra $^*\mathbb{Z}$.*

Proof. *(i)* $\Rightarrow$ *(ii)* Suppose F has a nonempty basis S. Let $x \in S$ and $n \in \mathbb{Z}$. By the definition of a basis, it follows that if $x^n = 0$, then $n = 0$. Thus, $|x|_B = +\infty$. Hence, by Theorem 2.6, it follows that $\langle x \rangle_B$ is an infinite cyclic subalgebra of F. Since F is commutative, $\langle x \rangle_B$ is normal in F. Since for each $x \in S$, $\langle x \rangle_B \subseteq \langle S \rangle_B = F$, it is clear that $F = \left\langle \bigcup_{x \in S} \langle x \rangle_B \right\rangle_B$. Suppose there exists $z \in S$ such that $\langle z \rangle_B \cap \left\langle \bigcup_{\substack{x \in S \\ x \neq z}} \langle x \rangle_B \right\rangle_B \neq \langle 0 \rangle_B = \{0\}$. Then there exists $w \in \langle z \rangle_B \cap \left\langle \bigcup_{\substack{x \in S \\ x \neq z}} \langle x \rangle_B \right\rangle_B$ such that $w \neq 0$. Thus, $w \in \langle z \rangle_B$ and $w \in \left\langle \bigcup_{\substack{x \in S \\ x \neq z}} \langle x \rangle_B \right\rangle_B$, $w \neq 0$. Hence, $w = z^n$ for some $n \in \mathbb{Z}$ and $w = [...[(y_1^{n_1} * y_2^{n_2}) * y_3^{n_3}] * ... * y_{k-1}^{n_{k-1}}] * y_k^{n_k}$ for some distinct $y_1, y_2, ..., y_k \in \bigcup_{\substack{x \in S \\ x \neq z}} \langle x \rangle_B$, $n_i \in \mathbb{Z}$. Note that if $y_i = 0$ for some $i = 2, 3, ..., k$, then y_i plays no role in the combination. If $y_1 = 0$, then $y_1^{n_1} * y_2^{n_2} = 0 * y_2^{n_2} = y_2^{-n_2}$. In these cases, we just change the variables as well as the indexing. Hence, we may assume that $y_i \neq 0$ for all i. Now, for each i, there exists $x_i \in S \backslash \{z\}$ such that $y_i \in \langle x_i \rangle$. Hence, $y_i = x_i^{m_i}$ for some $m_i \in \mathbb{Z}$. By Theorem 2.9,

$$\begin{array}{rcl} z^n & = & w \\ & = & [...[(y_1^{n_1} * y_2^{n_2}) * y_3^{n_3}] * ... * y_{k-1}^{n_{k-1}}] * y_k^{n_k} \\ & = & [...[((x_1^{m_1})^{n_1} * (x_2^{m_2})^{n_2}) * (x_3^{m_3})^{n_3}] * ... * (x_{k-1}^{m_{k-1}})^{n_{k-1}}] * (x_k^{m_k})^{n_k} \\ & = & [...[(x_1^{m_1 n_1} * x_2^{m_2 n_2}) * x_3^{m_3 n_3}] * ... * x_{k-1}^{m_{k-1} n_{k-1}}] * x_k^{m_k n_k}, \end{array}$$

$x_i \in S \backslash \{z\}$, $m_i n_i \in \mathbb{Z}$. Thus,

$$0 \quad = \quad z^n * z^n = [...[(x_1^{m_1 n_1} * x_2^{m_2 n_2}) * x_3^{m_3 n_3}] * ... * x_k^{m_k n_k}] * z^n$$

with $z \neq x_1, x_2, ..., x_k$. If the $x_i's$ are distinct, it follows that $m_i n_i, n = 0$ for all i since S is a basis. If $x_p = x_t$ for some $p \neq t$, then by Proposition 3.3, we have

$$\begin{aligned} 0 &= [...[...[...[(x_1^{m_1n_1} * x_2^{m_2n_2}) * x_3^{m_3n_3}] * ... * x_p^{m_pn_p}] * ... * x_t^{m_tn_t}] * \\ &\quad ... * x_k^{m_kn_k}] * z^n \\ &= [...[[...[...[(x_1^{m_1n_1} * x_2^{m_2n_2}) * x_3^{m_3n_3}] * ... * x_i^{m_pn_p \pm m_tn_t}] * ... * x_{t-1}^{m_{t-1}n_{t-1}}] * \\ &\quad x_{t+1}^{m_{t+1}n_{t+1}}] * ... * x_k^{m_kn_k}] * z^n. \end{aligned}$$

Thus, $m_i n_i = 0$ for all $i \neq p, t$ and $n, m_p n_p \pm m_t n_t = 0$. In both cases, $n = 0$. Thus, $w = z^n = z^0 = 0$, a contradiction. Hence, $\langle z \rangle_B \cap \left\langle \bigcup_{\substack{x \in S \\ x \neq z}} \langle x \rangle_B \right\rangle_B = \{0\}$. It follows that $F = \sum_{x \in S} \langle x \rangle_B$.

$(ii) \Rightarrow (iii)$ Let $S \subseteq F$ such that $F = \sum_{x \in S} \langle x \rangle_B$, where $\langle x \rangle_B$ is infinite for each $x \in S$. By Theorem 2.15, it follows that $\langle x \rangle_B \cong {}^*\mathbb{Z}$. Thus, for each $x \in S$, there exists a B-isomorphism $f_x : \langle x \rangle_B \to {}^*\mathbb{Z}$. Let $f = \prod_{x \in S} f_x$ be the map $\sum_{x \in S} \langle x \rangle_B \to \sum {}^*\mathbb{Z}$ given by $f(\{u_x\}) = \{f_x(u_x)\}$ and where the copies of ${}^*\mathbb{Z}$ are indexed by S. By Remark 4.5, f is a one-to-one and onto B-homomorphism, since each f_x is. Hence, $F = \sum_{x \in S} \langle x \rangle_B \cong \sum {}^*\mathbb{Z}$.

$(iii) \Rightarrow (i)$ Suppose $F \cong \sum {}^*\mathbb{Z}$ where the copies of ${}^*\mathbb{Z}$ are indexed by a nonempty set S. Then for all $\{u_x\} \in \sum \mathbb{Z}$, $x \in S$, $u_x = 0$ for all but a finite number of $x \in S$. Note that $\sum {}^*\mathbb{Z} = (\sum \mathbb{Z}; *, \{0\})$ is a B-algebra where $\{u_x\} * \{v_x\} = \{u_x - v_x\}$ for all $\{u_x\}, \{v_x\} \in \sum \mathbb{Z}$.

For each $y \in S$, let $\alpha_y = \{u_x\} \in \sum \mathbb{Z}$ such that $u_x = 0$ for all $x \neq y$ and $u_y = 1$. Let $A = \{\alpha_y : y \in S\}$. Since $S \neq \emptyset$, it follows that $\emptyset \neq A \subseteq \sum \mathbb{Z}$. We show that A is a basis for $\sum {}^*\mathbb{Z}$.

Let $a \in \sum \mathbb{Z}$. Then $a = \{u_x\}$ where $x \in S$, $u_x \in \mathbb{Z}$ and $u_x = 0$ for all but a finite $x \in S$. Thus, there exists $k \in \mathbb{Z}^+$ and $y_1, y_2, ..., y_k \in S$ such that $u_{y_i} \neq 0$ and $u_x = 0$ for all $x \neq y_i$ for all $i = 1, 2, ..., k$.

Consider $y_1 \in S$. Then for each $x \in S$, we can write $u_x = w'_{1,x} - w_{1,x}$ where $w'_{1,x} = u_x$ and $w_{1,x} = 0$ for all $x \neq y_1$, $w'_{1,y_1} = 0$ and $w_{1,y_1} = -u_{y_1}$. Thus, $a = \{u_x\} = \{w'_{1,x} - w_{1,x}\} = \{w'_{1,x}\} * \{w_{1,x}\}$. Consider $y_2 \in S$. Then for each $x \in S$, we can write $w'_{1,x} = w'_{2,x} - w_{2,x}$ where $w'_{2,x} = w'_{1,x} = u_x$ and $w_{2,x} = 0$ for all $x \neq y_2$, $w'_{2,y_2} = 0$ and $w_{2,y_2} = -w'_{1,y_2} = -u_{y_2}$. Thus,

$$a = \{w'_{1,x}\} * \{w_{1,x}\} = \{w'_{2,x} - w_{2,x}\} * \{w_{1,x}\} = (\{w'_{2,x}\} * \{w_{2,x}\}) * \{w_{1,x}\}.$$

Continuing this process, we have

$$\begin{aligned} a &= (\{w'_{2,x}\} * \{w_{2,x}\}) * \{w_{1,x}\} \\ &= ((\{w'_{3,x}\} * \{w_{3,x}\}) * \{w_{2,x}\}) * \{w_{1,x}\} \\ &\vdots \\ &= \big[...\big((\{w'_{k-1,x}\} * \{w_{k-1,x}\}) * \{w_{k-2,x}\}\big) * ... * \{w_{2,x}\}\big] * \{w_{1,x}\} \end{aligned}$$

where $w_{i,x} = 0$ for all $x \neq y_i$ and $w_{i,y_i} = -u_{y_i}$ for all $i = 1, 2, ..., k-1$. Since there are k $y_i's$, it follows that $w'_{k-1,x} = 0$ for all $x \neq y_k$ and $w'_{k-1,k} = u_{y_k}$. Let

$w'_{k-1,x} = w_{k,x}$. Thus, $a = \big[...\big((\{w_{k,x}\} * \{w_{k-1,x}\}) * \{w_{k-2,x}\}\big) * ... * \{w_{2,x}\}\big] * \{w_{1,x}\}$ where $w_{i,x} = 0$ for all $x \neq y_i$, $w_{i,y_i} = -u_{y_i}$, $i = 1, 2, ..., k-1$, and $w_{k,x} = 0$ for all $x \neq y_k$ and $w_{k,y_k} = u_{y_k}$.

Since $1^n = n$ and $0^n = 0$ for all $n \in \mathbb{Z}$, it follows that $w_{i,x} = 0 = 0^{u_{y_i}}$ for all $x \neq y_i$, $w_{i,y_i} = -u_{y_i} = 1^{-u_{y_i}}$, $i = 1, 2, ..., k-1$ and $w_{k,x} = 0 = 0^{u_{y_k}}$ for all $x \neq y_k$ and $w_{k,y_k} = u_{y_k} = 1^{u_{y_k}}$.

For each i, let $\alpha_{y_i} = \{v_{i,x}\}$. Then $v_{i,x} = 0$ for all $x \neq y_i$ and $v_{i,y_i} = 1$, $i = 1, 2, ..., k$. Thus, $(v_{i,x})^{-u_{y_i}} = 0^{-u_{y_i}} = 0$ for all $x \neq y_i$ and $(v_{i,y_i})^{-u_{y_i}} = 1^{-u_{y_i}} = -u_{y_i}$, $i = 1, 2, ..., k-1$ and $(v_{k,x})^{u_{y_k}} = 0^{u_{y_k}} = 0$ for all $x \neq y_k$ and $(v_{k,y_k})^{u_{y_k}} = 1^{u_{y_k}} = u_{y_k}$, Hence, $w_{i,x} = (v_{i,x})^{-u_{y_i}}$ for all $i = 1, 2, ..., k-1$ and $w_{k,x} = (v_{k,x})^{u_{y_k}}$. Thus, by Proposition 4.1, $\{w_{i,x}\} = \{(v_{i,x})^{-u_{y_i}}\} = \{v_{i,x}\}^{-u_{y_i}} = (\alpha_{y_i})^{-u_{y_i}}$ for all $i = 1, 2, ..., k-1$ and similarly, $\{w_{k,x}\} = (\alpha_{y_k})^{u_{y_k}}$. Thus,

$$\begin{aligned} a &= \big[...\big((\{w_{k,x}\} * \{w_{k-1,x}\}) * \{w_{k-2,x}\}\big) * ... * \{w_{2,x}\}\big] * \{w_{1,x}\} \\ &= \big[...\big((\alpha_{y_k})^{u_{y_k}} * (\alpha_{y_{k-1}})^{-u_{y_{k-1}}}\big) * ... * (\alpha_{y_2})^{-u_{y_2}}\big] * (\alpha_{y_1})^{-u_{y_1}} \end{aligned}$$

where $u_{y_i} \in \mathbb{Z}$. By Theorem 2.7, it follows that $a \in \langle A \rangle_B$. Accordingly, $\sum \mathbb{Z} \subseteq \langle A \rangle_B$. Since $A \subseteq \sum \mathbb{Z}$, $\langle A \rangle_B \subseteq \sum \mathbb{Z}$. Thus, $\sum \mathbb{Z} = \langle A \rangle_B$.

Let $\alpha_{y_1}, \alpha_{y_2}, ..., \alpha_{y_k}$, be distinct elements of A with

$$\big[...\big[((\alpha_{y_1})^{n_1} * (\alpha_{y_2})^{n_2}) * (\alpha_{y_3})^{n_3}\big] * ... * (\alpha_{y_{k-1}})^{n_{k-1}}\big] * (\alpha_{y_k})^{n_k} = \{0\},$$

$y_i \in S$, $n_i \in \mathbb{Z}$, $i = 1, 2, ..., k$. By definition of elements of A, it follows that $y_1, y_2, ..., y_k$ are also distinct elements of A.

For each i, $\alpha_{y_i} = \{w_{i,x}\}$ where $w_{i,x} = 0$ for all $x \neq y_i$ and $w_{i,y_i} = 1$. Now, by Proposition 4.1, $(\alpha_{y_1})^{n_1} * (\alpha_{y_2})^{n_2} = \{w_{1,x}\}^{n_1} * \{w_{2,x}\}^{n_2} = \{(w_{1,x})^{n_1}\} * \{(w_{2,x})^{n_2}\} = \{(w_{1,x})^{n_1} - (w_{2,x})^{n_2}\}$. For all $x \neq y_1, y_2$, $(w_{1,x})^{n_1} - (w_{2,x})^{n_2} = 0^{n_1} - 0^{n_2} = 0$. For $x = y_1$, $(w_{1,y_1})^{n_1} - (w_{2,y_1})^{n_2} = 1^{n_1} - 0^{n_2} = n_1 - 0 = n_1$. For $x = y_2$, $(w_{1,y_2})^{n_1} - (w_{2,y_2})^{n_2} = 0^{n_1} - 1^{n_2} = 0 - n_2 = -n_2$.

Let $(\alpha_{y_1})^{n_1} * (\alpha_{y_2})^{n_2} = \{v_x\}$. Then for all $x \neq y_1, y_2$, $v_x = 0$, $v_{y_1} = n_1$ and $v_{y_2} = -n_2$. By Proposition 4.1, $((\alpha_{y_1})^{n_1} * (\alpha_{y_2})^{n_2}) * (\alpha_{y_3})^{n_3} = \{v_x\} * \{w_{3,x}\}^{n_3} = \{v_x\} * \{(w_{3,x})^{n_3}\} = \{v_x - (w_{3,x})^{n_3}\}$. For all $x \neq y_1, y_2, y_3$, $v_x - (w_{3,x})^{n_3} = 0 - 0^{n_3} = 0$. For $x = y_1$, $v_{y_1} - (w_{3,y_1})^{n_3} = n_1 - 0^{n_3} = n_1 - 0 = n_1$. For $x = y_2$, $v_{y_2} - (w_{3,y_2})^{n_3} = -n_2 - 0^{n_3} = -n_2 - 0 = -n_2$. For $x = y_3$, $v_{y_3} - (w_{3,y_3})^{n_3} = 0 - 1^{n_3} = 0 - n_3 = -n_3$.

Continuing this process, we obtain

$$\big[...\big[((\alpha_{y_1})^{n_1} * (\alpha_{y_2})^{n_2}) * (\alpha_{y_3})^{n_3}\big] * ... * (\alpha_{y_{k-1}})^{n_{k-1}}\big] * (\alpha_{y_k})^{n_k} = \{z_x\},$$

where $z_x = 0$ for all $x \neq y_1, y_2, ..., y_k$, $z_{y_1} = n_1$ and $z_{y_i} = -n_i$ for all $i = 2, 3, ..., k$. By hypothesis, $\{z_x\} = \{0\}$. Hence, $0 = z_{y_1} = n_1$, $0 = z_{y_1} = -n_i$ for all $i = 2, 3, ..., k$. Accordingly, $n_i = 0$ for all $i = 1, 2, ..., k$.

Therefore, A is a basis for $\sum {}^*\mathbb{Z}$. Since $F \cong \sum {}^*\mathbb{Z}$, it follows that F also has a nonempty basis. □

Theorem 4.19. *Let $(F;*,0)$ be a free commutative B-algebra. Then there exists a nonempty set S and a function $\iota : S \to F$ with the following property: given a commutative B-algebra Y and a function $f : S \to Y$, there exists a unique B-homomorphism $g : F \to Y$ such that $g\iota = f$.*

Proof. Let $S \neq \emptyset$ be basis for F and consider the inclusion map $\iota : S \to F$ given by $\iota(x) = x$ for all $x \in S$. Let $(Y;*',0')$ be a commutative B-algebra and $f : S \to Y$. Let $u \in F$. Then $u = [...[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * ...] * x_k^{n_k}$ for some $x_i \in S$, $n_i \in \mathbb{Z}$. Define $g : F \to Y$ by

$$\begin{aligned} g(u) &= g\big(...[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * ... * x_k^{n_k}\big) \\ &= [...[(f(x_1)^{n_1} *' f(x_2)^{n_2}) *' f(x_3)^{n_3}] *' ... *' f(x_{k-1})^{n_{k-1}}] *' f(x_k)^{n_k} \end{aligned}$$

for all $u = [...[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * ... * x_{k-1}^{n_{k-1}}] * x_k^{n_k} \in F$.

Let $u, v \in F$. Then

$$\begin{aligned} u &= [...[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * ... * x_{k-1}^{n_{k-1}}] * x_k^{n_k} \quad \text{and} \\ v &= [...[(y_1^{m_1} * y_2^{m_2}) * y_3^{m_3}] * ... * y_{t-1}^{m_{t-1}}] * y_t^{m_t} \end{aligned}$$

for some $x_i, y_j \in S$, $n_i, m_j \in \mathbb{Z}$. Suppose $u = v$. By Theorem 4.17, $k = t$, $x_i = y_i$, and $n_i = m_i$ for all i. By the well-definedness of f, $f(x_i) = f(y_i)$ for all i. Accordingly,

$$\begin{aligned} g(u) &= [...[(f(x_1)^{n_1} *' f(x_2)^{n_2}) *' f(x_3)^{n_3}] *' ... *' f(x_{k-1})^{n_{k-1}}] *' f(x_k)^{n_k} \\ &= [...[(f(y_1)^{m_1} *' f(y_2)^{m_2}) *' f(y_3)^{m_3}] *' ... *' f(y_{k-1})^{n_{k-1}}] *' f(y_k)^{m_k} \\ &= g(v). \end{aligned}$$

Thus, g is well-defined and for all $x \in S$, $g\iota(x) = g(\iota(x)) = g(x) = f(x)$. By Corollary 2.10,

$$\begin{aligned} g(u * v) &= g\Big(\big(...[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * ... * x_k^{n_k}\big) * \\ &\quad \big(...[(y_1^{m_1} * y_2^{m_2}) * y_3^{m_3}] * ... * y_t^{m_t}\big)\Big) \\ &= g\Big([...[[[...[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * \\ &\quad ... * x_k^{n_k}] * y_s^{-m_t}] * y_{t-1}^{-m_{t-1}}] * ... * y_2^{-m_2}] * y_1^{m_1}\Big) \\ &= [...[[[...[(f(x_1)^{n_1} *' f(x_2)^{n_2}) *' f(x_3)^{n_3}] *' ... *' f(x_k)^{n_k}] *' \\ &\quad f(y_t)^{-m_t}] *' f(y_{t-1})^{-m_{t-1}}] *' ... *' f(y_2)^{-m_2}] *' f(y_1)^{m_1} \\ &= \big(...[(f(x_1)^{n_1} *' f(x_2)^{n_2}) *' f(x_3)^{n_3}] *' ... *' f(x_k)^{n_k}\big) *' \\ &\quad \big(...[(f(y_1)^{m_1} *' f(y_2)^{m_2}) *' f(y_3)^{m_3}] *' ... *' f(y_t)^{m_t}\big) \\ &= g(u) *' g(v). \end{aligned}$$

Hence, g is a B-homomorphism.

Suppose there exists $g' : F \to Y$ such that $g'\iota = f$. Since S generates F, g' is completely determined by its action on S. Let $x \in S$. Then

$$g'(x) = g'(\iota(x)) = g'\iota(x) = f(x) = g\iota(x) = g(\iota(x)) = g(x).$$

Hence, $g = g'$. Accordingly, g is unique. □

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