On exact sequences of the rigid fibrations

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Abstract. In 2002, Biss investigated on a kind of fibration which is called rigid covering fibration (we rename it by rigid fibration) with properties similar to covering spaces. In this paper, we obtain a relation between arbitrary topological spaces and its rigid fibrations. Using this relation we obtain a commutative diagram of homotopy groups and quasi-topological homotopy groups and deduce some results in this field.

Keywords: fibration, rigid covering fibration, topological homotopy group, exact sequence.

1. Introduction and motivation

The quasi-topological nth homotopy group of the pointed space (X, x), denoted by $\pi_n^{qtop}(X, x)$, is a quasi-topological group of the familiar homotopy group $\pi_n(X, x)$ which is endowed with the quotient topology induced by the natural surjective map $q : \Omega^n(X, x) \to \pi_n(X, x)$, where $\Omega^n(X, x)$ is the nth loop space of (X, x) with the compact-open topology (see [1, 2, 4]).

In 2002, Biss [1] investigated on a kind of fibration which is called rigid covering fibration (we rename it by rigid fibration) with properties similar to

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covering spaces. He proved that there is a universal rigid fibration for a space if and only if its topological fundamental group is totally path disconnected [1, Theorem 4.3]. The rigid fibrations generalize classical covering spaces, the hypothesis being that the topological fundamental group is totally disconnected rather than discrete as in the classical theory. Since rigid fibrations have unique path lifting, one can apply standard arguments to extend a number of classical theorems about covering spaces to the rigid fibration setting. For instance, every morphism in the category of rigid fibrations over X is itself a rigid fibration [1, Lemma 4.5]. Also, Biss proved that the coset space of any subgroup $\pi \leq \pi_1^{qtop}(HE)$ has no nonconstant paths, where HE is the Hawaiian earring space. Therefore, for any subgroup π, there is a rigid fibration $p : E \longrightarrow HE$ with $p_*\pi_1(E) = \pi$. On the other hand Brazas [3] introduced the $\mathcal{C}$-covering map for a space X which has a unique lifting property with respect to maps on the objects of $\mathcal{C}$, where $\mathcal{C}$ is the category of path-connected spaces having the unit disk as an object, and then he defined the category of $\mathcal{C}$-covering maps for a space X denoted by $Cov_{\mathcal{C}}(X)$. The $\mathcal{C}$-covering maps generalize classical covering maps. In this paper, we consider the category of rigid fibration maps of a space X, $RF(X)$, and show that it is a subcategory of $wCov_{\mathcal{C}}(X)$. Then we obtain some results in rigid fibrations. Also, we consider rigid fibrations of two topological spaces X and Y and obtain a relation between these spaces and its rigid fibrations. More precisely, if $p : E_H \longrightarrow X$ and $q : E_G \longrightarrow Y$ are two rigid fibrations of X and Y, respectively, such that $p_*\pi_1(E_H) = H$ and $q_*\pi_1(E_G) = G$ and $g : X \longrightarrow Y$ is a continuous map such that $g_*(H) \subseteq G$, then there is a map $\tilde{f} : E_H \longrightarrow E_G$ such that $q \circ f = g \circ p$ and vise versa. Then with the above conditions, we obtain the following commutative diagram in Set_* with exact rows:

$$
\begin{array}{ccccccccccc}
\cdots & \longrightarrow & \pi_n(F_1) & \xrightarrow{i_*} & \pi_n(E_H) & \xrightarrow{p_*} & \pi_n(X) & \xrightarrow{d} & \pi_{n-1}(F_1) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle (f|_{F_1})_*} & & {\scriptstyle f_*}\downarrow & & {\scriptstyle g_*}\downarrow & & {\scriptstyle (f|_{F_1})_*}\downarrow & & \\
\cdots & \longrightarrow & \pi_n(F_2) & \xrightarrow{j_*} & \pi_n(E_G) & \xrightarrow{q_*} & \pi_n(Y) & \xrightarrow{d'} & \pi_{n-1}(F_2) & \longrightarrow & \cdots .
\end{array}
$$

In follow, by using this diagram, we deduce some results about homotopy groups and quasi-topological homotopy groups.

2. Preliminaries

In this section, we recall some of the main notion and results of rigid fibrations.

Definition 2.1 ([1, Definition 2.1]). *Let X be a topological space. A fibration $p : E \longrightarrow X$ is said to be a covering fibration if $p_* : \pi_i(E) \longrightarrow \pi_i(X)$ is an isomorphism, for $i \geq 2$, and an injection for $i = 1$.*

We recall that a fibration $p : E \longrightarrow X$ has unique path lifting property if for any path $\gamma : I \longrightarrow X$ in X and $e \in E$ with $p(e) = \gamma(0)$, there is a unique lift $\tilde{\gamma} : I \longrightarrow E$ with $\gamma = p \circ \tilde{\gamma}$ and $\tilde{\gamma}(0) = e$. One can show that a fibration has unique path lifting property if and only if all of the fibers have no nonconstant paths [6, Theorem 2.2.5].

Definition 2.2 ([1, Definition 4.1])**.** *Let X be a topological space. A fibration $p : E \longrightarrow X$ is called a rigid covering fibration if it is a covering fibration and if, in addition, each fiber has no nonconstant paths.*

If we use the homotopy sequence of fibrations for rigid covering fibrations, we see that the condition "covering fibration" in Definition 2.2 can be replaced with "fibration". Therefore we can say that A fibration $p : E \longrightarrow X$ is called a rigid covering fibration (rename it to rigid fibration) if each fiber has no nonconstant paths.

Theorem 2.3 ([1, Theorem 4.3])**.** *Let X be a space, and let $\pi < \pi_1(X)$ be a subgroup of the fundamental group of X. If the left coset space $\pi_1^{qtop}(X)/\pi$ has no nonconstant paths, then there is a rigid covering fibration $p : E \longrightarrow X$ with $p_*\pi_1(E) = \pi$.*

For a pointed map $f : (X, x_0) \longrightarrow (Y, y_0)$, the mapping fiber is the pointed space $Mf = \{(x, \omega) \in X \times Y^I : \omega(0) = y_0 \text{ and } \omega(1) = f(x)\}$, the base point of this space is (x_0, ω_0), where ω_0 is the constant path at y_0. Also, there are an injection $k : \Omega(Y, y_0) \longrightarrow Mf$ given by $k(\omega) = (x_0, \omega)$ and an obvious map $\lambda : X \longrightarrow Mf$ by $\lambda(x) = (x, \omega_0)$ (see [5]).

3. Main results

In this section, we obtain some results in rigid fibrations and intend then to obtain a relation between arbitrary topological spaces and its rigid fibrations. Then we obtain a commutative diagram of homotopy groups and quasi-topological homotopy groups and deduce some results in this field.

Theorem 3.1. *Let $p : E \longrightarrow X$ be a rigid fibration. If A is any path component of E, then $p|_A : A \longrightarrow p(A)$ is a rigid fibration.*

Proof. It follows from [6, Lemma 2.3.1] and Definition 2.2. □

Theorem 3.2. *Let $p : E \longrightarrow X$ be a map. If E is locally path connected, then p is a rigid fibration if and only if for each path component A of E, $p(A)$ is a path component of X and $p|_A : A \longrightarrow p(A)$ is a rigid fibration.*

Proof. It follows from [6, Theorem 2.3.2] and Definition 2.2. □

The following theorems imply that if $\mathcal{C}$ is the category of connected locally path connected spaces, then every rigid fibration is a weak $\mathcal{C}$-covering map in the sense of [3].

Theorem 3.3 ([6, Lemma 2.2.4]). *A map with unique path lifting has the unique lifting property for path connected spaces.*

Theorem 3.4 ([6, Theorem 2.4.5]). *Let $p : (\tilde{X}, \tilde{x_0}) \longrightarrow (X, x_0)$ be a fibration with unique path lifting property. Let Y be a connected locally path connected space. A necessary and sufficient condition that a map $f : (Y, y_0) \longrightarrow (X, x_0)$ has a lifting $(Y, y_0) \longrightarrow (\tilde{X}, \tilde{x_0})$ is that, in $\pi_1(X, x_0)$,*

$$f_*\pi_1(Y, y_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{x_0}).$$

Remark 3.5. It follows easily from homotopy lifting property that for a fibration with unique path lifting property $p : E \longrightarrow X$, if E is a nonempty path connected space and X is connected and locally path connected, then p is a homeomorphism if and only if $p_* : \pi_1(E) \longrightarrow \pi_1(X)$ is an isomorphism [6, Corollary 2.4.6].

We know that every fibration with unique path lifting property whose base space is locally path connected and semilocally simply connected and whose total space is locally path connected is a covering map [6, Theorem 2.4.10]. In follow, we exhibit two examples of rigid fibration which are not covering map.

Example 3.6. It is known that $\pi_1^{qtop}(HE)$ has no nonconstant paths, where HE is the Hawaiian earring space [1]. Therefore there is a universal rigid fibration $p : E \longrightarrow HE$. p is not a covering map, becaouse HE is not semilocally simply connected and so it has not any universal covering map, by [5, Corollary 10.37].

Example 3.7. Let $p : \mathbb{R} \longrightarrow S^1$ be a map with $p(t) = exp(t)$. Since p is a covering map, so it is a rigid fibration, therefore $h = \prod_{n\in\mathbb{N}} p : \prod_{n\in\mathbb{N}} \mathbb{R} \longrightarrow \prod_{n\in\mathbb{N}} S^1$ is a rigid fibration by [6, Theorem 2.2.7]. Also the map $q : \mathbb{R} \times (\{\frac{1}{n}) \mid n \in \mathbb{N}\} \cup \{0\}) \longrightarrow S^1$ by $q(t, s) = p(t)$ is a rigid fibration. Note that every fiber of the maps h and q is totally path disconnected which is not discrete. Hence the rigid fibrations h and q are not covering.

We are inspired by the results of [3] and obtain the following results.

Definition 3.8. *A subgroup $H \leq \pi_1(X, x_0)$ is a rigid fibration subgroup (for simpricity, RF subgroup) if there is a rigid fibration $p : (E, e_0) \longrightarrow (X, x_0)$ with $p_*\pi_1(E, e_0) = H$.*

Remark 3.9. Let H be a subgroup of $\pi_1(X)$. If the left coset space $\pi_1^{qtop}(X)/H$ has no nonconstant paths, then H is a RF subgroup of $\pi_1(X)$, by Theorem 2.3. As an example, every subgroup of fundamental group of Hawaiian earring, $\pi_1(HE)$, is a RF subgroup. Because the left coset space $\pi_1^{qtop}(HE)/H$ has no nonconstant paths for every subgroup H of $\pi_1(HE)$[1].

Example 3.10. Let HA be the harmonic archipelago space. Any proper subgroup of $\pi_1(HA)$ is not a RF subgroup. Indeed, $\pi_1^{qtop}(HA)$ is indiscrete and therefore, for any subgroup $H \leq \pi_1(HA)$, the left coset space $\pi_1^{qtop}(HA)/H$ is not totally path disconnected.

Theorem 3.11. *If $\{H_i\}_{i\in I}$ is any set of RF subgroup of $\pi_1(X, x_0)$, then:*
(i) $\prod_i H_i$ is a rigid fibration subgroup.
(ii) $\bigcap_i H_i$ is a rigid fibration subgroup.

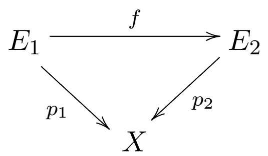
Proof. (*i*) If H_i is a RF subgroup of $\pi_1(X, x_0)$, then there is a rigid fibration $p_i : (E_i, e_i) \longrightarrow (X, x_0)$ with $p_*\pi_1(E_i, e_i) = H_i$ for all $i \in I$. By putting $(E, e) = (\prod_i E_i, (e_i))$, the product $p = \prod p_i : (E, e) \longrightarrow (\prod_i X, x_0)$ is a rigid fibration by [6, Theorem 2.2.7] and

$$p_*\pi_1(E, e) = p_*\pi_1(\prod_i E_i, (e_i)) = \prod_i p_{i*}\pi_1(E_i, e_i) = \prod_i H_i.$$

Thus $\prod_i H_i$ is a RF subgroup.

(*ii*) It follows from a similar argument of [3, Theorem 2.36] by applying [6, Theorem 2.2.7]. □

For a given space X, let $RF(X)$ denote the category of rigid fibrations over X which is the category whose objects are rigid fibrations $p : E \longrightarrow X$ and morphisms are commutative triangles of the form



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Let $DCov(X)$ denote the category of disk-coverings over X and $GSet$ denote the category of G-Sets and G-equivariant functions. One can see that for a topological space X, $RF(X) \subseteq DCov(X)$. Recall that the functor $\mu : DCov(X) \longrightarrow \pi_1(X, x_0)Set$ was defined as follows: On objects, μ is defined as the fiber $\mu(p) = p^{-1}(x_0)$. If $q : E_0 \longrightarrow X$ is a disk-covering and $f : E \longrightarrow E_0$ is a map such that $q \circ f = p$, then $\mu(f)$ is the restriction of f to a $\pi_1(X, x_0)$-equivariant function $p^{-1}(x_0) \longrightarrow q^{-1}(x_0)$. μ is a faithful functor [3, Lemma 2.5], and so the restriction on $RF(X)$ is also faithful. With the same argument of [3, Theorem 2.11], we have the following theorem.

Theorem 3.12. *The functor $\mu : RF(X) \longrightarrow \pi_1(X, x_0)Set$ is fully faithful.*

The functor μ in Theorem 3.12 is not necessarily an equivalence of categories. Because subgroups $H \leq \pi_1(X, x_0)$ exist which are not RF subgroups(see Example 3.10).

Every morphism in $RF(X)$ is itself a rigid fibration [1, lemma 4.5]. By this fact and [6, Theorem 2.2.6] we have the following result.

Proposition 3.13. *Suppose that $p : E \longrightarrow X$ and $q : X \longrightarrow Y$ are maps:*
(i) If p and q are rigid fibrations, then so is $q \circ p$,
(ii) If q and $q \circ p$ are rigid fibrations, then so is p.

Remark 3.14. Let $g : X \longrightarrow Y$ be a rigid fibrations of Y; then there is a functor $\mathcal{G} : RF(X) \longrightarrow RF(Y)$ that is defined identity on morphisms and on objects as follows. If $p : E \longrightarrow X$ is a rigid fibration of X, we define $\mathcal{G}(p) = g \circ p$.

The following theorem is one of the main results of this paper.

Theorem 3.15. *Let $p : E_H \longrightarrow X$ and $q : E_G \longrightarrow Y$ be two rigid fibrations of X and Y, respectively, such that $p_*\pi_1(E_H) = H$ and $q_*\pi_1(E_G) = G$.*

(i) If $g : X \longrightarrow Y$ is a continuous map such that $g_*(H) \leq G$, then there is a map $f : E_H \longrightarrow E_G$ such that $q \circ f = g \circ p$.

(ii) If $g : X \longrightarrow Y$ is a rigid fibration such that $g_*(H) \leq G$, then $f : E_H \longrightarrow E_G$ defined in part (i) is a rigid fibration.

(iii) If $g : X \longrightarrow Y$ is a continuous map and there is a map $f : E_H \longrightarrow E_G$ such that $q \circ f = g \circ p$, then $g_*(H) \leq G$.

Proof. (i) Consider the map $g \circ p : E_H \longrightarrow Y$. Since q is a rigid fibration, it has lifting property for path connected spaces. Since

$$(f \circ p)_*(\pi_1(H)) = g_*(H) \leq G = q_*\pi_1(E_G),$$

there exist a map $f : E_H \longrightarrow E_G$ such that $q \circ f = g \circ p$.

(ii) Since $g \circ p$ and q are two rigid fibrations of Y, f is a rigid fibration of E_H by Proposition 3.13.

(iii) Since $q \circ f = g \circ p$, by applying the functor π_1, $q_* \circ f_* = g_* \circ p_*$ and then

$$g_*(H) = g_* \circ p_*(\pi_1(E_H)) = q_* \circ f_*(\pi_1(E_H)) \leq q_*(\pi_1(E_G)) = G.$$

□

Theorem 3.16. *Let H and G be two RF subgroups of $\pi_1(X)$ and $\pi_1(Y)$, respectively. If $g : X \longrightarrow Y$ is a rigid fibration such that $g_*(H) \leq G$, then $g_*(H)$ is a RF subgroup of $\pi_1(Y)$.*

Proof. By hypothises, there exist two rigid fibrations $p : E_H \longrightarrow X$ and $q : E_G \longrightarrow Y$ of X and Y, respectively, such that $p_*\pi_1(E_H) = H$ and $q_*\pi_1(E_G) = G$. Using Theorem 3.15 part (ii), there is a rigid fibration $f : E_H \longrightarrow E_G$ with $q \circ f = g \circ p$. The composition $q \circ f : E_H \longrightarrow Y$ is a rigid fibration of Y such that

$$(q \circ f)_*(\pi_1(E_H)) = (g \circ p)_*(\pi_1(E_H)) = g_*(H).$$

□

The following result can be concluded from part (i) of Theorem 3.15.

Theorem 3.17. *Let $p : E_H \longrightarrow X$ and $q : E_G \longrightarrow Y$ be two rigid fibrations of X and Y with fibers F_1 and F_2, respectively, such that $p_*\pi_1(E_H) = H$ and $q_*\pi_1(E_G) = G$. Let $g : X \longrightarrow Y$ be a continuous map such that $g_*(H) \leq G$; then* (*i*) *there is a commutative diagram in* Set_* *with exact rows:*

(1)

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \pi_n(F_1) & \xrightarrow{i_*} & \pi_n(E_H) & \xrightarrow{p_*} & \pi_n(X) & \xrightarrow{d} & \pi_{n-1}(F_1) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle (f|_{F_1})_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle (f|_{F_1})_*} & & \\ \cdots & \longrightarrow & \pi_n(F_2) & \xrightarrow{j_*} & \pi_n(E_G) & \xrightarrow{q_*} & \pi_n(Y) & \xrightarrow{d'} & \pi_{n-1}(F_2) & \longrightarrow & \cdots, \end{array}$$

where the existence of the map $f : E_H \longrightarrow E_G$ *follows from Theorem 3.15.*

(*ii*) *there is a commutative diagram in* Top_* *with exact rows:*

(2)

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \pi_n^{qtop}(F_1) & \xrightarrow{i_*} & \pi_n^{qtop}(E_H) & \xrightarrow{p_*} & \pi_n^{qtop}(X) & \xrightarrow{d} & \pi_{n-1}^{qtop}(F_1) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle (f|_{F_1})_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle (f|_{F_1})_*} & & \\ \cdots & \longrightarrow & \pi_n^{qtop}(F_2) & \xrightarrow{j_*} & \pi_n^{qtop}(E_G) & \xrightarrow{q_*} & \pi_n^{qtop}(Y) & \xrightarrow{d'} & \pi_{n-1}^{qtop}(F_2) & \longrightarrow & \cdots. \end{array}$$

(*iii*) *there is a long exact sequence of homotopy groups.*

(3) $$\cdots \longrightarrow \pi_n(E_H) \longrightarrow \pi_n(E_G) \oplus \pi_n(X) \longrightarrow \pi_n(Y) \longrightarrow \pi_{n-1}(E_H) \longrightarrow \cdots.$$

(*iv*) *there is a long exact sequence of topological homotopy groups.*

(4)

$$\cdots \longrightarrow \pi_n^{qtop}(E_H) \longrightarrow \pi_n^{qtop}(E_G) \oplus \pi_n^{qtop}(X) \longrightarrow \pi_n^{qtop}(Y) \longrightarrow \pi_{n-1}^{qtop}(E_H) \longrightarrow \cdots.$$

Proof. (*i*) Since $g_*(H) \leq G$, there is a map $f : E_H \longrightarrow E_G$ such that $q \circ f = g \circ p$, by Theorem 3.15. Since $f \circ i = j \circ f|_{F_1}$, $q \circ f = g \circ p$ and π_n is a functor, the first two squares commute. To see commutativity of the last squares, consider the following diagram:

(5)

$$\begin{array}{ccccc} \Omega X & \xrightarrow{k} & Mp & \xleftarrow{\lambda} & F_1 \\ \downarrow{\scriptstyle g_\sharp} & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle f|_{F_1}} \\ \Omega Y & \xrightarrow{k'} & Mq & \xleftarrow{\lambda'} & F_2, \end{array}$$

where the maps $g_\sharp$ and l are induced maps. It is easy to see that Diagram (5) is commutative. Therefore the induced diagram by the functor π_n, that is the last square in Diagram (2), is commutative.

(*ii*) It follows from a similar argument of part (*i*) by applying the functor π_n^{qtop}.

(*iii*) The result holds from [5, Lemma 6.2] and part (*i*).

(*iv*) The result holds from [5, Lemma 6.2] and part (*ii*). □

Remark 3.18. Let $H < \pi_1(X)$, let $G < \pi_1(Y)$, and let the left coset spaces $\pi_1^{qtop}(X)/H$ and $\pi_1^{qtop}(Y)/G$ have no nonconstant paths. By Theorem 2.3, Diagrams (1) and (2) are commutative in Set$_*$ and Top$_*$, respecively, where $g : X \longrightarrow Y$ is a continuous map with $g_*(H) \leq G$.

The following result follows from using Diagrams (3), (4) and Theorem 2.3.

Corollary 3.19. *Let* $\pi_1^{qtop}(X)$ *and* $\pi_1^{qtop}(Y)/G$ *have no nonconstant paths, where* $G < \pi_1(Y)$*. If there is a continuous map* $g : X \longrightarrow Y$*, then*

$$\pi_n(E_G) \oplus \pi_n(X) \cong \pi_n(Y)$$

and

$$\pi_n^{qtop}(E_G) \oplus \pi_n^{qtop}(X) \cong \pi_n^{qtop}(Y).$$

References

[1] D. K. Biss, *The topological fundamental group and generalized covering spaces,* Topology and Appl., 124 (2002), 355-371.

[2] J. Brazas, *The topological fundamental group and free topological groups,* Topology and Appl., 158 (2011), 779-802.

[3] J. Brazas, *Generalized covering space theories,* Theory Appl. Categ., 30 (2015), 1132-1162.

[4] H. Ghane, Z. Hamed, B. Mashayekhy, and H. Mirebrahimi, *Topological homotopy groups*, Bull. Belg. Math. Soc. Simon Stevin, 15 (2008), 455-464.

[5] Joseph J. Rotman, *An introduction to algebraic topology*, Springer-Verlag New York, 1988.

[6] Edwin H. Spanier, *Algebraic topology*, Springer-Verlag New York, 1960.

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