Structures on Galois connections

A.V.S.N. Murty
VIT, Vellore-632014
Tamil Nadu
India
avsnmurthy2005@gmail.com

Abstract. In this paper it is proved that $G(Q, P)$(the set of all Galois connections of P into Q) is a meet semilattice if and only if both P and Q are meet semilattices and in this case, the meet operation in $G(Q, P)$ is point-wise and P and Q are isomorphic to meet subsemilattices of $G(Q, P)$.

Keywords: residuated mappings, Galois connections, p.o.set, lattice and sublattice.

1. Introduction

U.M. Swamy and A. V. S. N. Murty [5] studied an important properties of Residuated mappings and it is observed that, for any p.o.sets P and Q , f is a Galois connection of P into Q if and only if it is a residuated mapping of Pinto Q^d, the dual of Q (that is, $a \leq b$ in Q^d if $b \leq a$ in Q). Some observations necessitate in depth study of the abstract Galois connections [3] between general partially ordered sets and in particular, from a Heyting algebra into an algebraic lattice, in view of the well-known Birkhoff's theorem [1] and [2] which states that the lattice of subalgebras of any algebra is an algebraic lattice and that any algebraic lattice is isomorphic to the lattice of subalgebras of a suitable algebra. Here we first discuss certain properties of Galois connections and then the order structure on the set of Galois connections between certain given partially ordered sets.

2. Preliminaries

Let P and Q by any partially ordered sets (p.o.sets). A mapping $f : P \to Q$ is said to be an isotone (antitone) if, for any $a, b \in P$, $a \leq b$ implies$f(a) \leq f(b)$ $(f(a) \geq f(b))$. An antitone $f : P \to Q$ is said to be a Galois connection ([3]) of P into Q if there exists an antitone $g : Q \to P$ such that $a \leq g(f(a))$ and $x \leq f(g(x))$ for all $a \in P$ and $x \in Q$; in other words, gof and fog are extensive maps. If $f : P \to Q$ is a Galois connection, then the antitone $g : Q \to P$ satisfying the above property is unique and is called the dual of f and is denoted by f*. If f is a Galois connection, then so is f* and (f*)*= f. The following is a straight forward verification.

Theorem 2.1. Let P and Q be p.o. sets and $f : P \to Q$ an antitone. Then the following are equivalent to each other.

1. f is a Galois connection
2. There exists an antitone $g : Q \to P$ such that $a \le g(x) \Leftrightarrow x \le f(a)$,

   for any $a \in P$and $x \in Q$.
3. For each $x \in Q$, the set $\{ a \in Px \mid \ \le f(a)\}$ has the largest element.

If$f : P \to Q$ is a Galois connection, then f can be expressed in terms of its dual and vice versa as follows. For any $a \in P$ and $x \in Q$, $f^*(x)$ =The greatest in $\{b \in P \mid x \le f(b)\}$ and $f(a) =$. The greatest in $\{y \in Q \mid a \le f^*(y)\}$.

Definition 2.2. A mapping $f : P \to Q$ of p.o.sets is said to be

1. A complete join- meet homomorphism if, for any $A \subseteq P$for which $Sup_P A$ exists in P. $Inf_Q f(A)$ exists in Q and is equal to $f(Sup_P A)$
2. A complete meet- join homomorphism if, for any $A \subseteq P$for which $Inf_P A$ exists in P. $Sup_Q f(A)$ exists in Q and equal to $f(Inf_P A)$.

The above two concepts of homomorphism are independent of each other and each of them is an antitone. The following can be easily verified.

Therorem 2.3. Any Galois connection $f : P \to Q$ is a complete join – meet homomorphism.

The converse of the above theorem is not true, in the sense that a complete join-meet homomorphism need not be a Galois connection; for, if P is with the trivial ordering (that is, $a \le b$ only when $a = b$), then every mapping of P into P is a complete join-meet homomorphism, while the Galois connections of P into P are bijections of P onto P. However, if P is a complete lattice (a p.o.set in which every subset has supremum and infimum), then every complete join-meet homomorphism of P into any p.o.set Q is a Galois connection. In fact, we have the following [5].

Theorem 2.4. The following are equivalent to each other for any p.o.set P.

1. P is a complete lattice;
2. Every complete join-meet homomorphism of P into P is a Galois connection;
3. Every complete join-meet homomorphism of P into any p.o.set Q is a Galois connection.

If P and Q are p.o.sets and $R \subseteq Q$, then a map $f : P \to R$ may be a Galois connection without being a Galois connection of P into Q. In this connection, we have the following which is a consequence of theorem 3.8 of [5].

Theorem 2.5. Let R be a nonempty subset of complete lattice Q. Suppose R is complete with respect to the ordering induced by that of Q. Then the following are equivalent.

1. For any complete$P$, every Galois connection of P into R is a Galois connection of P into Q.
2. For any subset A of R, $Inf_R A = Inf_Q A$.

3. Structures on Galois connections

For any p.o.sets P and Q, let $G(P,Q)$ denote the set of all Galois connections of P into Q. Naturally, we are interested only in the case when $G(P,Q)$ is non-empty. In this context, it can be seen that, a p.o.set Q has largest element (denoted by 1) if and only if $G(P,Q)$ is non-empty for any p.o.set P with largest element; for, the constant map f, defined by$f(a) = 1$ for all $a \in P$, is a Galois connection. For this reason, we shall consider only non-trivial bounded p.o.sets; that is, p.o.sets with the largest element and smallest element demoted by 1 and 0 respectively and $1 \neq 0$. For any f, $g \in G(P,Q)$ we shall define$f \leq g$ if $f(a) \leq g(a)$for all $a \in P$. It can be easily proved that $f \leq g$if and only if $f^* \leq g^*$and hence $f \mapsto f^*$ is an order isomorphism of $G(P,Q)$ onto $G(Q,P)$. In the following, we shall embed P and Q in $G(P,Q)$.

Theorem 3.1. Let $P$and Q be bounded p.o.sets. For any $a \in P$ and $x \in Q$, define f_a , $g_x : P \to Q$ by

$$f_a(b) = \begin{cases} 1, & \text{if } b \leq a \\ 0, & \text{if } b \nleq a \end{cases} \quad \text{and} \quad g_x(b) = \begin{cases} x, & \text{if } b \neq 0 \\ 1, & \text{if } b = 0 \end{cases}$$

Then f_a and g_x are Galois connections and $a \to f_a$ and $x \to g_x$ are embeddings of P and Q respectively in $G(P,Q)$.

Proof. It can be easily verified that f_a and g_x are antitones. For any $a \in P$ and $x \in Q$, define$f_a^*, g_x^* : Q \to P$ by

$$f_a^*(y) = \begin{cases} a, & \text{if } y \neq 0 \\ 1, & \text{if } y = 0 \end{cases} \text{ and } g_x^*(y) = \begin{cases} 1, & \text{if } y \leq x \\ 0, & \text{if } y \nleq x \end{cases}$$

Then f_a^* and g_x^* are antitones, $b \leq f_a^*(f_a(b))$ and $y \leq f_a(f_a^*(y))$ for all $b \in P$ and $y \in Q$. Therefore f_a is a Galois connection and its dual if f_a^* . Similarly g_x is a Galois connection and g_x^* is its dual. It can be easily verified

that $a \leq b \Leftrightarrow f_a \leq f_b$ and $x \leq y \Leftrightarrow g_x \leq g_y$ for any a, $b \in P$ and x, $y \in Q$. Thus $a \mapsto f_a$ is an embedding of P into $G(P,Q)$ and $x \mapsto g_x$ is an embedding of Q into $G(P,Q)$.
A p.o.set is called a meet (join) semilattice if any two of its elements have the infimum (supremum, respectively) in it. The infimum and supremum of two elements a and b will be denoted by $a \wedge b$ and $a \vee b$ respectively when they exist.

Theorem 3.2. $G(P,Q)$ is a meet semilattice if and only if both $P$and Q are meet semilattices and, in this case, the meet operation in $G(P,\ Q)$ is point-wise and P and Q are isomorphic to meet subsemilattices of $G(P,\ Q)$.

Proof. Suppose that P and Q are meet semilattices. For any $f, g \in G(P,\ Q)$, define $f \wedge g$ and $f^* \wedge g^* :\ Q \to P$ point-wise; that is,$(f \wedge g)(a) = f(a) \wedge g(a)$ and $(f^* \wedge g^*)(x) = f^*(x) \wedge g^*(a)$ for all $a \in P$and $x \in Q$. Then, clearly $f \wedge g$and $f^* \wedge g^*$ are antitones. Also for any $a \in P$ and $x \in Q$,

$$\begin{aligned}(f \wedge g)(f^* \wedge g^*)(x) &= f(f^*(x) \wedge g^*(x)) \wedge g(f^*(x) \wedge g^*(x)) \\ &\geq f(f^*(x)) \wedge g(g^*(x)) \\ &\geq x\end{aligned}$$

and, similarly, $(f^* \wedge g^*)\,((f \wedge g)\,(x)) \geq x$.

Therefore, $(f \wedge g)$ is a Galois connection and its dual is$f^* \wedge g^*$. Thus $G(P,\ Q)$ is a meet sublattice and the meet operation $\wedge$ is point wise.

And, similarly, $(f^* \wedge g^*)\,((f \wedge g)\,(x)) \geq x$. Therefore, $f \wedge g$is a Galois correction and its dual is $f^* \wedge g^*$. Thus $G(P,\ Q)$ is a meet sublattice and the meet operation $\wedge$ is point-wise.

Conversely, suppose that$G(P,\ Q)$, is a meet semilattice. Since $G(Q,\ P) \cong G(P,\ Q)$, it follows that $G(Q,\ P)$ is also a meet semilattice. Now, let $x,\ y \in Q$ and $g_x$and g_y be the corresponding (as in theorem 2.1) Galois connections from P into Q. Let $g = g_x \wedge g_y$ in $G(P,\ Q)$ and choose $0 \neq a \in P$. Then $g \leq g_x$ and $g \leq g_y$ and hence $g(a) \leq g_x(a) = x$ and $g(a) \leq g_y(a) = y$ so that g(a) is a lower bound of $x$and$y$. If z is any lower bound of x and y, then $g_z \leq g_x$ and $g_z \leq g_y$and hence $g_z \leq g_x \wedge g_y = g$so that $z = g_z(a) \leq g(a)$. Thus $g(a)$is the infimum of x and y and, infact, $g = g_{g(a)}$. Therefore, Q is a meet semilattice and $x \mapsto g_x$is an isomorphism of Q onto a meet subsemilattice of$G(P,\ Q)$. Similarly it can be verified that P is a meet semilattice which is isomorphic to a meet subsemilattice of $G(Q,\ P)\,(\cong\ G(P,\ Q))$. Note that, even if P and Q are complete lattices, the supremum $f \vee g$ in $G(P,\ Q)$ may not be the point-wise; for, consider the following.

Example 3.3. Let P be the interval [0,1] of real numbers with usual ordering and Q the set of all subgroups of the group (Z, +)of integers with the inclusion ordering. Define $f, g : P \to Q$ by

$$f(a) = \begin{cases} 0, & \text{if}\, a = 0 \\ \mathrm{Z}, & \text{if } \ a = 1 \\ 2^n\mathrm{Z}, & \text{if } \ \frac{n-1}{n} < a \ \leq \ \frac{n}{n+1} \text{ for some}\, n \in \mathrm{Z}^+ \end{cases}$$

and

$$g(a) = \begin{cases} 0 & \text{if } \ a = 1 \\ \mathrm{Z}, & \text{if } \ a = 0 \\ 3^n\mathrm{Z}, & \text{if } \ \frac{n-1}{n} < a \ \leq \ \frac{n}{n+1} \text{ for some}\, n \in \mathrm{Z}^+ \end{cases}$$

Then f and g are Galois connections and the dual f^* of $f$is given by

$$f^*(H) = \begin{cases} 1, & \text{if } \ H = \{0\} \\ 0, & \text{if } \ H \not\subseteq 2^n\mathrm{Z} \ \text{ for any } \ n \in \mathrm{Z}^+ \\ \frac{n}{n+1}, & \text{if } \ H \subseteq 2^n\mathrm{Z} \ \text{ and } \ H \not\subseteq 2^{n+1}\mathrm{Z} \end{cases}$$

and analogously g^*. If h is the point wise supremum of f and g, then

$$h(a) = f(a) \vee g(a) \begin{cases} \{0\}, & \text{if } \ a = 1 \\ \mathrm{Z}, & \text{if } \ a < 1 \end{cases}$$

for any $a \in P$.

Note that h is not a Galois connection; for, if so, and if h^* is the dual of$h$, then

$$\frac{n}{n+1} \leq h^*(h\,(\frac{n}{n+1})\,) = h^*(\mathrm{Z}),$$

for all $n \in \mathrm{Z}^+$ and hence $h^*(\mathrm{Z}) = 1$ so that $\mathrm{Z} \subseteq h(h^*(\mathrm{Z})) = h(1) = \{0\}$, which is an absurd. Thus the point-wise supremum of $f$and g is not a Galois connection.

References

[1] G. Birkhoff, *Lattice theory*, Amer. Math. Colloq. Publ., Vol. XXV (AMS Providence, RI, 1967.

[2] J. Derderian, *Residuated mappings,* Pacipic Journal of Mathematics, 20 (1967), 34-43.

[3] Ore Oystein, *Galois connections*, Trans. Amer. Math. Soc., 55 (1944), 494-513.

[4] G. Gratzer, *General lattice theory,* Academic Press, New York, 1978.

[5] U.M. Swamy, A.V.S.N. Murty, *On residuated mappings,* Southeast Asian Bulletin of Mathematics, 32 (2008), 371-377.

[6] U.M. Swamy, A.V.S.N. Murty, *Algebra-abstract and modern*, Pearson Education, 2012.

Accepted: 29.11.2018