Warped product submanifolds of a generalized Sasakian space form admitting nearly cosymplectic structure

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Abstract. In this paper we consider semi-invariant warped product submanifolds of the type $M = N_T \times_\psi N_\perp$ of a generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ admitting nearly cosymplectic structure and obtained a characterizing inequality ($\|\sigma_\mu(D, D^\perp\|^2 \geq f_2.\alpha.\beta + 2\|\mathcal{P}_{D^\perp}D\|^2 + 2\|\mathcal{Q}_{D^\perp}D\|^2$) for existence of warped product submanifolds. Moreover, some special cases are also discussed. The results in the present paper generalize the existing results available in the literature.

Keywords: warped product, semi-invariant submanifolds, nearly cosymplectic manifold.

1. Introduction

The conception of warped product of manifolds was instigated by R. L. Bishop and B. O. Neill in order to set up a massive velocity of manifolds of negative curvatures [9]. The formulation of warped product on submanifolds was given by B. Y. Chen (c.f., [10], [11]). Basically, B. Y. Chen [10] studied warped product of the types $N_\perp \times_\psi N_T$ and $N_T \times_\psi N_\perp$ and produced a sharp relationship between the warping function ψ and squared norm of second fundamental form for CR-warped product submanifolds in the setting of Kaehler manifolds. Later, I. Hesigawa and I. Mihai [12] investigated contact version of CR-warped product submanifolds in the setting of Sasakian manifolds. A step forward, I. Mihai [15] developed the analogous inequality for contact CR-warped products in the Sasakian space forms. Furthermore, contact CR-warped product submanifolds in Kenmotsu space form were also studied by K. Arslan et al. [2] and they achieved a sharp inequality for the squared norm of the second fundamental form and the warping function.

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M. Atceken deliberate the contact CR-warped product submanifolds in the setting of cosymplectic and Kenmotsu space forms (see [3], [4]) and compute a necessary and sufficient condition for contact CR-product. In this sequence, S. Sular and C. Özgür in [17] investigated contact CR-warped product submanifolds of trans-Sasakian generalized Sasakian space forms and obtained a more general inequality for squared norm of second fundamental form and warping function. Sufficient work has been done for contact CR-warped product submanifolds of almost contact metric manifolds (see [2], [5], [6], [18]).

Stimulated by the analysis of above researchers, in the present paper, we consider semi invariant warped product submanifolds of a generalized Sasakian space form admitting nearly cosymplectic structure and obtained some interesting results and some special cases are also discussed and correlated them with the previous works.

The paper is categorized as: section 2 is introductory which holds a brief information about almost contact metric manifolds and in particular nearly cosymplectic manifolds. Moreover, in this section the definition of a generalized Sasakian space form is given. In section 3, warped product of manifolds are discussed and we also collect some useful results for later use. Finally, we establish a characterizing inequality for squared norm of second fundamental form of contact CR-warped product submanifolds in the background of generalized Sasakian space forms admitting nearly cosymplectic structures.

2. Preliminaries

The tensorial equations of an almost contact metric manifold $\bar{M}$ with dimension $(2n+1)$ are given by

$$\phi^2 = -I + \eta \oplus \xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1. \tag{2.1}$$

where ϕ, ξ and η are the (1, 1) tensor field, a characteristic vector field and 1-form respectively.

The Riemannian metric g exists on an almost contact metric manifold $\bar{M}$ satisfies the following conditions

$$\eta(U) = g(U, \xi), \quad g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V), \tag{2.2}$$

for all $U, V \in T\bar{M}$.

An almost contact structure (ϕ, ξ, η) is said to be normal if the almost complex structure J on the product manifold $\bar{M} \times R$ given by

$$J(U, f\frac{d}{dt}) = (\phi U - f\xi, \eta(U)\frac{d}{dt}),$$

where f is $C^\infty-$function on $\bar{M} \times R$, has no torsion, that is J is integrable and the condition for normality in terms of ϕ, ξ and η is $[\phi, \phi] + 2d\eta \otimes \xi$ on $\bar{M}$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ. Finally, the *fundamental 2-form* Φ is defined by $\Phi(U, V) = g(U, \phi V)$.

An almost contact metric structure (ϕ, ξ, η, g) is said to be cosymplectic, if it is normal and both Φ and η are closed [8]. The structure is said to be nearly cosymplectic if ϕ is killing i.e.,

$$(\bar{\nabla}_U \phi)U = 0, \tag{2.3}$$

for each $U \in T\bar{M}$.

Proposition 2.1 ([7]). *On a nearly cosymplectic manifold, the vector field ξ is killing.*

From the above proposition we have $\bar{\nabla}_U \xi = 0$, for any $U \in T\bar{M}$, where $\bar{M}$ is a nearly cosymplectic manifold.

Given an almost contact metric manifold $\bar{M}$, it is said to be a generalized Sasakian space form [1] if there exist three functions f_1, f_2 and f_3 on $\bar{M}$ such that

$$\begin{aligned}\bar{R}(U,V)W = {} & f_1\{g(V,W)U - g(U,W)V\} \\ & + f_2\{g(U,\phi W)\phi V - g(V,\phi W)\phi U \\ & + 2g(U,\phi V)\phi W\} + f_3\{\eta(U)\eta(W)V \\ & - \eta(V)\eta(W)U + g(U,W)\eta(V)\xi - g(V,W)\eta(U)\xi\},\end{aligned} \tag{2.4}$$

for any vector fields U, V, W on $\bar{M}$, where $\bar{R}$ denotes the curvature tensor of $\bar{M}$. If $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$ then $\bar{M}$ is Sasakian space form [8], if $f_1 = \frac{c-3}{4}$, $f_2 = f_3 = \frac{c+1}{4}$, then $\bar{M}$ is a Kenmotsu space form [13], if $f_1 = f_2 = f_3 = \frac{c}{4}$, then $\bar{M}$ is a cosymplectic space form [14].

Let M be a submanifold of an almost contact metric manifold $\bar{M}$ with induced metric g, and if ∇ and $\nabla^{\perp}$ are the induced connection on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M, respectively, then the Gauss and the Weingarten formulae are given by

$$\bar{\nabla}_U V = \nabla_U V + \sigma(U,V), \tag{2.5}$$

$$\bar{\nabla}_U N = -A_N U + \nabla^{\perp}_U N, \tag{2.6}$$

for each $U, V \in TM$ and $N \in T^{\perp}M$, where σ and A_N are the second fundamental form and the shape operator respectively, for the immersion of M in $\bar{M}$, they are related as

$$g(\sigma(U,V),N) = g(A_N U, V), \tag{2.7}$$

where g denotes the Riemannian metric on $\bar{M}$ as well as on M.

The mean curvature vector H of M is given by

$$H = \frac{1}{n}\sum_{i=1}^{n} \sigma(u_i, u_i),$$

where n is the dimension of M and $\{u_1, u_2, \ldots, u_n\}$ is a local orthonormal frame of vector fields on M. The squared norm of the second fundamental form is defined as

$$\|\sigma\|^2 = \sum_{i,j=1}^{n} g(\sigma(u_i, u_j), \sigma(u_i, u_j)). \tag{2.8}$$

A submanifold M of $\bar{M}$ is said to be a totally geodesic submanifold if $\sigma(U, V) = 0$, for each $U, V \in TM$, and totally umbilical submanifold if $\sigma(U, V) = g(U, V)H$.

For any $U \in TM$, we write

$$\phi U = PU + FU, \tag{2.9}$$

where PU is the tangential component and FU is the normal component of ϕU.

Similarly, for $N \in T^{\perp}M$, we can write

$$\phi N = tn + fN, \tag{2.10}$$

where tN and fN are the tangential and normal components of ϕN respectively.

The covariant differentiation of the tensors ϕ, P, F, t and f are defined as respectively

$$(\bar{\nabla}_U \phi)V = \bar{\nabla}_U \phi V - \phi \bar{\nabla}_U V, \tag{2.11}$$
$$(\bar{\nabla}_U P)V = \nabla_U PV - P\nabla_U V, \tag{2.12}$$
$$(\bar{\nabla}_U F)V = \nabla^{\perp}_U FV - F\nabla_U V, \tag{2.13}$$
$$(\bar{\nabla}_U t)N = \nabla_U tN - t\nabla^{\perp}_U N, \tag{2.14}$$
$$(\bar{\nabla}_U f)N = \nabla^{\perp}_U fN - f\nabla^{\perp}_U N. \tag{2.15}$$

Furthermore, for any $U, V \in TM$, the tangential and normal parts of $(\bar{\nabla}_U \phi)V$ are denoted by $\mathcal{P}_U V$ and $\mathcal{Q}_U V$ i.e.,

$$(\bar{\nabla}_U \phi)V = \mathcal{P}_U V + \mathcal{Q}_U V. \tag{2.16}$$

Since ξ is the killing vector field, then it is easy to verify the following property

$$(\bar{\nabla}_U \phi)\phi V = -\phi(\bar{\nabla}_U)\phi V). \tag{2.17}$$

On using equations (2.6)-(2.14) and (2.17), we may obtain that

$$\mathcal{P}_U V = (\bar{\nabla}_U P)V - A_{FV}U - th(U, V), \tag{2.18}$$
$$\mathcal{Q}_U V = (\bar{\nabla}_U F)V + \sigma(U, PV) - fh(U, V). \tag{2.19}$$

On a submanifold M of a nearly cosymplectic manifold by (2.3) and (2.16)

$$(a)\ \ \mathcal{P}_U V = -\mathcal{P}_V U, \quad (b)\ \ \mathcal{Q}_U V = -\mathcal{Q}_V U \tag{2.20}$$

for any $U, V \in TM$.

Now we have the following properties of $\mathcal{P}$ and $\mathcal{Q}$, which can be verified easily

(P_1) (i) $\mathcal{P}_{U+V}W = \mathcal{P}_U W + \mathcal{P}_V W$ (ii) $\mathcal{Q}_{U+V}W = \mathcal{Q}_U W + \mathcal{Q}_V W$,

(P_2) (i) $\mathcal{P}_U(V+W) = \mathcal{P}_U V + \mathcal{P}_U W$ (ii) $\mathcal{Q}_U(V+W) = \mathcal{Q}_U V + \mathcal{Q}_U W$,

(P_3) (i) $g(\mathcal{P}_U V, W) = -g(W, \mathcal{P}_U W)$ (ii) $g(\mathcal{Q}_U V, N) = -g(W, \mathcal{P}_U N)$,

for all $U, V, W \in TM$ and $N \in T^\perp M$.

An m-dimensional Riemannian submanifold M of an almost contact metric manifold $\bar{M}$, where ξ is tangent to M, is called *semi-invariant submanifold* if it admits an invariant distribution D whose orthogonal complementary distribution $D^\perp$ is anti invariant, that is

$$TM = D \oplus D^\perp \oplus \langle\xi\rangle,$$

where $\phi D \subseteq D$ and $\phi D^\perp \subseteq T^\perp M$ and $\langle\xi\rangle$ denotes 1-dimensional distribution which is spanned by ξ.

If μ is the invariant subspace of the normal bundle $T^\perp M$, then in the case of semi-invariant submanifold, the normal bundle $T^\perp M$ can be decomposed as follows

$$T^\perp M = \mu \oplus \phi D^\perp. \tag{2.21}$$

A semi-invariant submanifold M is called semi-invariant product if the distribution D and $D^\perp$ are parallel on M. In this case M is foliated by the leaves of these distributions. In general, if N_1 and N_2 are Riemannian manifolds with Riemannian metrics g_1 and g_2 respectively, then the product manifold $(N_1 \times N_2, g)$ is a Riemannian manifold with Riemannian metric g defined as

$$g(U, V) = g_1(d\pi_1 U, d\pi_1 V) + g_2(d\pi_2 U, d\pi_2 V), \tag{2.22}$$

where π_1 and π_2 are the projection maps of M onto N_1 and N_2, respectively, and $d\pi_1$ and $d\pi_2$ are their differentials.

As a generalization of the product manifold and in particular of semi-invariant product submanifold, one can consider warped product of manifolds which are defined as follows

Definition 2.2. Let (R, g_R) and (S, g_S) be two Riemannian manifolds with Riemannian metric g_R and g_S respectively and ψ be a positive differentiable function on R. The warped product of R and S is the Riemannian manifold $(R \times S, g)$, where

$$g = g_R + \psi^2 g_S.$$

For a warped product manifold $N_1 \times_f N_2$, we denote by D_1 and D_2 the distributions defined by the vectors tangent to the leaves and fibers respectively.

The warped product manifold $(R \times S, g)$ is denoted by $R \times_\psi S$. If U is the tangent vector field to $M = R \times_\psi S$ at (α, β) then

$$\|U\|^2 = \|d\pi_1 U\|^2 + \psi^2(\alpha)\|d\pi_2 U\|^2. \tag{2.23}$$

The following Lemma was proved by R. L. Bishop and B. O'Neill [9].

Theorem 2.1. *Let $M = R \times_\psi S$ be a warped product manifolds. If $X, Y \in TR$ and $V, W \in TS$ then*
(i) $\nabla_X Y \in TR$

(ii) $\nabla_X V = \nabla_V X = (\frac{X\psi}{\psi})V,$

(iii) $\nabla_V W = \frac{-g(V,W)}{\psi}\nabla\psi.$

From the above theorem, for the warped product $M = R \times_f S$ it is easy to conclude that

$$\nabla_X V = \nabla_V X = (X \ln \psi)V, \tag{2.24}$$

for any $X \in TR$ and $V \in TS$.
$\operatorname{grad}\psi$ is the gradient of ψ and is defined as

$$g(\operatorname{grad}\psi, U) = U\psi, \tag{2.25}$$

for all $U \in TM$.

Corollary 2.1. *On a warped product manifold $M = R \times_\psi S$, the following statements hold*

(i) *N_1 is totally geodesic in M,*

(ii) *N_2 is totally umbilical in M.*

In what follows, $N_\perp$ and N_T will denote a totally real and holomorphic submanifold respectively of an almost Hermitian manifold $\bar{M}$.

A warped product manifold is said to be trivial if its warping function ψ is constant.

Let M be a $m-$dimensional Riemannian manifold with Riemannian metric g and let $\{u_1, \ldots, u_m\}$ be an orthogonal basis of TM. For a smooth function ψ on M the Hessian of ψ are defined as

$$H^\psi(U, V) = UV\psi - (\nabla_U V)\psi = g(\nabla_U \operatorname{grad}\psi, V), \tag{2.26}$$

for any $U, V \in TM$. The Laplacian of ψ is defined by

$$\Delta\psi = \sum_{i=1}^{m}\{(\nabla_{u_i} u_i)\psi - u_i u_i \psi\} = -\sum_{i=1}^{m} g(\nabla_{u_i} \operatorname{grad}\psi, u_i). \tag{2.27}$$

It is evident from the above two equations that Laplacian is the negative of the Hessian. Moreover from the integration theory on manifolds, for a compact orientable Riemannian manifold M without boundary, we have

$$\int_M \Delta\psi dV = 0, \tag{2.28}$$

where dV is the volume element of M ([16]).

3. Semi-invariant warped product submanifolds

Recently, Siraj Uddin et al. [18] studied warped product semi-invariant submanifolds of nearly cosymplectic manifolds and prove the following theorem

Theorem 3.1. *A warped product submanifold $M = M_1 \times_f M_2$ of a nearly cosymplectic manifold $\bar{M}$ is a usual Riemannian product if the structure vector field ξ is tangent to M_2, where M_1 and M_2 are the Riemannian submanifolds of $\bar{M}$.*

In this section we consider contact CR-warped product of the type $N_T \times_\psi N_\perp$ of the nearly cosymplectic manifold $\bar{M}$, where N_T and $N_\perp$ are the invariant and anti-invariant submanifolds respectively of $\bar{M}$. Now we obtain some basic results in the following lemma.

Lemma 3.1. *Let $M = N_T \times_\psi N_\perp$ be a semi-invariant warped product submanifold of a nearly cosymplectic manifold $\bar{M}$. Then*

(i) $g(\sigma(U, W), \phi W) = -\phi U \ln \psi \|W\|^2$,

(ii) $g(\sigma(\phi U, W), \phi W) = U \ln \psi \|W\|^2$,

(iii) $g(\phi\sigma(U, W), \mathcal{P}_W U) = -\|\mathcal{P}_W U\|^2$,

(iv) $g(\sigma(\phi U, W) - \phi\sigma(U, W), \mathcal{Q}_W U) = \|\mathcal{Q}_W U\|^2$,

for any $U \in TN_T$ and $W \in TN_\perp$.

Proof. From the Gauss formula, we have

$$g(\sigma(U, W), \phi W) = -g(\bar{\nabla}_W \phi W, U).$$

By use of (2.3), we get

$$g(\sigma(U, W), \phi W) = g(\bar{\nabla}_W W, \phi U),$$

or

$$g(\sigma(U, W), \phi W) = -g(\bar{\nabla}_W \phi U, W) = -g(\nabla_W \phi U, W).$$

In view of (2.24), we get

$$g(\sigma(U,W),\phi W) = -\phi U \ln\psi g(W,W),$$

which is the part (i) of Lemma.

Part (ii) is the part (iii) of Lemma 3.4 in [18]. To prove (iii), consider

$$g(\phi\sigma(U,W),\mathcal{P}_W U) = g(\phi\bar{\nabla}_W U - \phi\nabla_W U, \mathcal{P}_W U).$$

By use of (2.5) and (2.11) the last equation gives

$$g(\phi\sigma(U,W),\mathcal{P}_W U) = g(\bar{\nabla}_W \phi U - (\bar{\nabla}_W \phi)U, \mathcal{P}_W U)).$$

From the Gauss formula and (2.16), we get

$$g(\phi\sigma(U,W),\mathcal{P}_W U) = \phi U \ln\psi g(W,\mathcal{P}_W U) - \|\mathcal{P}_W U\|^2,$$

in view of part (i) of P_3 and (2.3), we have

$$g(\phi\sigma(U,W),\mathcal{P}_W U) = -\|\mathcal{P}_W U\|^2.$$

Finally, applying the Gauss formula in the following expression

$$g(\sigma(\phi U,W),\mathcal{Q}_W U) = g(\bar{\nabla}_W \phi U - \nabla_W \phi U, \mathcal{Q}_W U),$$

or equivalently,

$$g(\sigma(\phi U,W),\mathcal{Q}_W U) = g(\mathcal{Q}_W U,\mathcal{Q}_W U) + g(\phi\bar{\nabla}_W U,\mathcal{Q}_W U),$$

again applying (2.5) and (2.11) in above equation, we get

$$g(\sigma(\phi U,W),\mathcal{Q}_W U) = \|\mathcal{Q}_W U\|^2 + g(\phi\sigma(U,W),\mathcal{Q}_W U) + U\ln\psi g(\phi W,\mathcal{Q}_W U).$$

In view of part (ii) of P_3 and (2.3) the last term in above equation becomes zero and finally, we get part (iv) of Lemma.

Now we have the following theorem

Theorem 3.2. *Let $M = N_T \times_\psi N_\perp$ be a semi-invariant warped product submanifold of a nearly cosymplectic manifold $\bar{M}$, then*

$$\begin{aligned}\|\sigma(U,W)\|^2 + \|\sigma(\phi U,W)\|^2 &= 2g(\sigma(\phi U,W),\phi\sigma(U,W)) + (U\ln\psi)^2\|W\|^2\\ &\quad + (\phi U\ln\psi)^2\|W\|^2 + \|\mathcal{P}_W U\|^2 + \|\mathcal{Q}_W U\|^2.\end{aligned} \tag{3.1}$$

Proof. From the Gauss formula, we have

$$g(\sigma(\phi U,W),\phi\sigma(U,W)) = g(\phi\sigma(U,W),\bar{\nabla}_W\phi U - \nabla_W \phi U).$$

In view of (2.11), (2.23) and part (i) of Lemma 3.1, the above equation takes the form

$$g(\sigma(\phi U,W),\phi\sigma(U,W)) = g(\phi\sigma(U,W),(\bar{\nabla}_W\phi)U - \phi\bar{\nabla}_W U) - (\phi U\ln\psi)^2\|W\|^2,$$

by (2.5) and (2.16), the above equation yields

$$g(\sigma(\phi U, W), \phi\sigma(U, W)) = g(\phi\sigma(U, W), \mathcal{P}_W U) + g(\phi\sigma(U, W), \mathcal{Q}_W U) \\ + \|\sigma(U, W)\|^2 - (\phi U \ln\psi)^2\|W\|^2. \tag{3.2}$$

The second term on the right hand side can be solved as follows

$$g(\phi\sigma(U, W), \mathcal{Q}_{\mathcal{W}}U) = g(\phi\bar{\nabla}_W U - \phi\nabla_W U, \mathcal{Q}_W U).$$

In view of (2.11) and (2.24) the above equation takes the form

$$g(\phi\sigma(U, W), \mathcal{Q}_{\mathcal{W}}U) = -\|\mathcal{Q}_W U\|^2 + g(\sigma(\phi U, W), \mathcal{Q}_W U) - U\ln\psi g(\phi W, \mathcal{Q}_W U),$$

on using (2.17), part (ii) of P_3, the last term of above equation becomes zero and finally, we have

$$g(\phi\sigma(U, W), \mathcal{Q}_{\mathcal{W}}U) = -\|\mathcal{Q}_W U\|^2 + g(\sigma(\phi U, W), \mathcal{Q}_W U). \tag{3.3}$$

Applying part (ii) of P_3 in (3.3), we get

$$g(\phi\sigma(U, W), \mathcal{Q}_{\mathcal{W}}U) = -\|\mathcal{Q}_W U\|^2 - g((\bar{\nabla}_W\phi)\sigma(\phi U, W), U).$$

By use of (2.11), above equation gives

$$g(\phi\sigma(U, W), \mathcal{Q}_{\mathcal{W}}U) = -\|\mathcal{Q}_W U\|^2 - g(\bar{\nabla}_W\phi\sigma(\phi U, W) - \phi\bar{\nabla}_W\sigma(\phi U, W), U),$$

or

$$g(\phi\sigma(U, W), \mathcal{Q}_{\mathcal{W}}U) = -\|\mathcal{Q}_W U\|^2 - g(\phi\sigma(\phi U, W), \nabla_W U) \\ - g(\sigma(\phi U, W), \phi\sigma(U, W)) + g(\sigma(\phi U, W), \bar{\nabla}_W\phi U).$$

On using (2.5), (2.24) and part (i) of Lemma 3.1 above equation reduced to

$$g(\phi\sigma(U, W), \mathcal{Q}_{\mathcal{W}}U) = -\|\mathcal{Q}_W U\|^2 - (U\ln\psi)^2\|W\|^2 - g(\sigma(\phi U, W), \phi\sigma(U, W)) \\ + \|\sigma(\phi U, W)\|^2. \tag{3.4}$$

Using (3.4) and part (iii) of Lemma 3.1, in (3.2), we get the required result.

Now we will prove the following theorem for semi-invariant warped product submanifolds of a nearly cosymplectic generalized Sasakain space form.

Theorem 3.3. *Let $M = N_T \times_f N_\perp$ be a semi-invariant warped product submanifold of a generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ admitting nearly Cosymplectic structure, then*

$$\|\sigma(U, W)\|^2 + \|\sigma(\phi U, W)\|^2 = H^{\ln\psi}(U, U) + H^{\ln\psi}(\phi U, \phi U) + f_2.\|U\|^2.\|W\|^2 \\ + 2\|\mathcal{P}_W U\|^2 + 2\|Q_W U\|^2 + (U\ln\psi)^2 g(W, W) + (\phi U\ln\psi)^2 g(W, W), \tag{3.4}$$

for any $U \in TN_T$ and $W \in TN_\perp$.

Proof. Suppose M be a semi-invariant warped product submanifolds of a nearly Cosymplectic generalized Sasakian space form then by (2.4), we have

$$\bar{R}(U, \phi U, W, \phi W) = -f_2 g(U, U) g(W, W). \tag{3.5}$$

On the other hand by Coddazi equation

$$\begin{aligned}
&\bar{R}(U, \phi U, W, \phi W) = Ug(\sigma(\phi U, W), \phi W) \\
&- g(\bar{\nabla}_U \phi W, \sigma(\phi U, W)) - g(\sigma(\nabla_U \phi U, W), \phi W) \\
&- g(\sigma(\nabla_U W, \phi U), \phi W) - \phi U g(\sigma(U, W), \phi W) + g(\sigma(U, W), \bar{\nabla}_{\phi U} \phi W) \\
&+ g(\sigma(\nabla_{\phi U} U, W), \phi W) + g(\sigma(\nabla_{\phi U} W, U), \phi W),
\end{aligned}$$

on using (2.5), (2.23), decomposition (2.16), parts (i) and (ii) of Lemma 3.1, above equation takes the form

$$\begin{aligned}
&\bar{R}(U, \phi U, W, \phi W) = U(U \ln \psi) g(W, W) \\
&+ 2(U \ln \psi)^2 g(W, W) - g(\sigma(\phi U, W), \mathcal{Q}_U W) \\
&- g(\sigma(\phi U, W), \phi\sigma(U, W)) - U \ln \psi g(\sigma(\phi U, W), \phi W) \\
&- g(\sigma(\nabla_U \phi U, W), \phi W) - (U \ln \psi)^2 g(W, W) \\
&+ \phi U(\phi U \ln \psi) g(W, W) + 2\phi U \ln \psi g(\nabla_{\phi U} W, W) \\
&+ g(\sigma(U, W), \mathcal{Q}_{\phi U} W) + g(\sigma(U, W), \phi\sigma(\phi U, W)) \\
&+ \phi U \ln \psi g(\sigma(U, W), \phi W) + g(\sigma(\nabla_{\phi U} U, W), \phi W) \\
&- (\phi U \ln \psi)^2 g(W, W).
\end{aligned} \tag{3.6}$$

From part (ii) of Lemma 3.1 and (2.7)

$$g(A_{\phi W} W, \phi U) = U \ln \psi g(W, W).$$

Since $\nabla_U U \in TN_T$, then we can replace U by $\nabla_U U$ as

$$g(A_{\phi W} W, \phi \nabla_U U) = \nabla_U U \ln \psi g(W, W).$$

Applying the Gauss formula, we find

$$g(A_{\phi W} W, \phi \bar{\nabla}_U U - \phi\sigma(U, U)) = \nabla_U U \ln \psi g(W, W). \tag{3.7}$$

On using (2.11), (2.3) and (2.24) in the following equation

$$g(\sigma(U, U), \phi W) = -g(\phi \bar{\nabla}_U U, W),$$

one can conclude that $\sigma(X, X) \in \mu$, using this fact in (3.7), we get

$$g(A_{\phi W} W, \phi \bar{\nabla}_U U) = \nabla_U U \ln \psi g(W, W).$$

By use of (2.5) and (2.3), the above expression reduced to

$$g(A_{\phi W}W, \nabla_U \phi U) = \nabla_U U \ln\psi g(W, W),$$

or

$$g(\sigma(\nabla_U \phi U, W), \phi W) = \nabla_U U \ln\psi g(W, W). \tag{3.8}$$

Similarly, we can get

$$g(\sigma(\nabla_{\phi U} U, W), \phi W) = \nabla_{\phi U} \phi U \ln\psi g(W, W). \tag{3.9}$$

Moreover, using (2.20)(b) as follows

$$\begin{aligned}&- g(\sigma(\phi U, W), \mathcal{Q}_U W) + g(\sigma(U, W), \mathcal{Q}_{\phi U} W) = g(\sigma(\phi U, W), \mathcal{Q}_W U)\\ &- g(\sigma(U, W), \mathcal{Q}_W \phi U),\end{aligned}$$

or

$$\begin{aligned}&- g(\sigma(\phi U, W), \mathcal{Q}_U W) + g(\sigma(U, W), \mathcal{Q}_{\phi U} W) = g(\sigma(\phi U, W), \mathcal{Q}_W U)\\ &- g(\sigma(U, W), (\bar{\nabla}_W \phi)\phi U),\end{aligned}$$

using (2.17), it is easy to see that

$$\begin{aligned}&- g(\sigma(\phi U, W), \mathcal{Q}_U W) + g(\sigma(U, W), \mathcal{Q}_{\phi U} W)\\ &= g(\sigma(\phi U, W) - \phi\sigma(U, W), \mathcal{Q}_W U) - g(\phi\sigma(U, W), \mathcal{P}_W U).\end{aligned} \tag{3.10}$$

By making use of (3.8), (3.9) and (3.10) in (3.6), we get

$$\begin{aligned}&\bar{R}(U, \phi U, W, \phi W) = \{U(U \ln\psi) + \phi U(\phi U \ln\psi)\\ &- (\nabla_U U \ln\psi + \nabla_{\phi U} \phi U \ln\psi)\}\|W\|^2\\ &- 2g(\phi\sigma(U, W), \sigma(\phi U, W)) - g(\phi\sigma(U, W), \mathcal{P}_W U)\\ &+ g(\sigma(\phi U, W) - \phi\sigma(U, W), \mathcal{Q}_W U).\end{aligned}$$

By parts (iii) and (iv) of Lemma 3.1 and (2.26), we have

$$\begin{aligned}&\bar{R}(U, \phi U, W, \phi W) = H^{\ln\psi}(U, U) + H^{\ln\psi}(\phi U, \phi U) + \|\mathcal{P}_W U\|^2\\ &+ \|\mathcal{Q}_W U\|^2 - 2g(\phi\sigma(U, W), \sigma(\phi U, W)).\end{aligned} \tag{3.11}$$

Applying (3.1) and (3.5) in (3.11), we get

$$\begin{aligned}&- f_2.\|U\|^2.\|W\|^2 = H^{\ln\psi}(U, U) + H^{\ln\psi}(\phi U, \phi U) + 2\|\mathcal{P}_W U\|^2 + 2\|\mathcal{Q}_W U\|^2\\ &- \|\sigma(U, W)\|^2 - \|\sigma(\phi U, W\|^2 + (U \ln\psi)^2\|W\|^2 + (\phi U \ln\psi)^2\|W\|^2,\end{aligned}$$

which is the required result.

Note 3.1. If for semi-invariant warped product submanifolds $M = N_T \times_\psi N_\perp$ the ambient manifold is cosymplectic space form $\bar{M}(c)$, then $f_2 = \frac{c}{4}$ and $\mathcal{P} = \mathcal{Q} = 0$.

In view of Note 3.1 we have the following corollary

Corollary 3.1. *Let $M = N_T \times_f N_\perp$ be a semi-invariant warped product submanifold of a cosymplectic space form $\bar{M}(c)$, then we have*

$$\|\sigma(U,W)\|^2 + \|\sigma(\phi U,W)\|^2 = H^{\ln\psi}(U,U) + H^{\ln\psi}(\phi U,\phi U) + \frac{c}{4}\|U\|^2.\|W\|^2 + \{(U\ln\psi)^2 + (\phi U\ln\psi)^2\}g(W,W),$$

for any $U \in TN_T$ and $W \in TN_\perp$.

Let $\{u_0 = \xi, u_1, u_2, \ldots, u_\alpha, \phi u_1, \phi u_2, \ldots, \phi u_\alpha, u^1, u^2, \ldots, u^\beta\}$ be an orthonormal frame of TM such that $\{u_1, u_2, \ldots, u_\alpha, \phi u_1, \phi u_2, \ldots, \phi u_\alpha\}$ are tangential to TN_T and $\{u^1, u^2, \ldots, u^\beta\}$ are tangential to $TN_\perp$. Moreover assume that $\{\phi u^1, \phi u^2, \ldots, \phi u^\beta\}$ and $\{n_1, n_2, \ldots, n_{2r}\}$ are tangential to $\phi N_\perp$ and μ respectively.

Finally, we will prove the main theorem

Theorem 3.4. *Let $M = N_T \times_\psi N_\perp$ be a compact orientable semi-invariant warped product submanifold of a nearly cosymplectic generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$. Then M is a semi-invariant warped product if*

$$\|\sigma_\mu(D, D^\perp\|^2 \geq f_2.\alpha.\beta + 2\|\mathcal{P}_{D^\perp}D\|^2 + 2\|\mathcal{Q}_{D^\perp}D\|^2,$$

where σ_μ denotes the component of σ in μ, $2\alpha + 1$ and β are the dimensions of N_T and $N_\perp$.

Proof. By the definition of Laplacian of $\ln\psi$

$$-\Delta\ln\psi = \sum_{i=1}^{\alpha} g(\nabla_{u_i}\,\mathrm{grad}\,\ln\psi, u_i) + \sum_{i=1}^{\alpha} g(\nabla_{\phi u_i}\,\mathrm{grad}\,\ln\psi, \phi u_i) + \sum_{j=1}^{\beta} g(\nabla_{u^j}\,\mathrm{grad}\,\ln\psi, u^j) + g(\nabla_\xi\,\mathrm{grad}\,\ln\psi, \xi).$$

Since, the ambient manifold $\bar{M}$ is nearly cosymplectic manifold and moreover, the structure vector field ξ is killing on $\bar{M}$, the induced connection is Levi-Civita and $\mathrm{grad}\,\psi \in TN_T$, then by some straight forward calculations it is easy to see that $g(\nabla_\xi\,\mathrm{grad}\,\ln\psi, \xi) = 0$.
Hence, by use of (2.29), the last equation can be written as

$$-\Delta\ln\psi = \sum_{i=1}^{\alpha}\{H^{\ln\psi}(u_i,u_i) + H^{\ln\psi}(\phi u_i,\phi u_i)\} + \sum_{j=1}^{\beta} g(\nabla_{u^j}\,\mathrm{grad}\,\ln\psi, u^j).$$

Further by a simple calculations, we can get the following

$$-\Delta \ln \psi = \sum_{i=1}^{\alpha} \{H^{\ln \psi}(u_i, u_i) + H^{\ln \psi}(\phi u_i, \phi u_i)\} + \beta \| \operatorname{grad} \ln \psi \|^2. \tag{3.12}$$

In (3.4) using local frame of vector fields on N_T and $N_\perp$ and summing both sides over $i = 1, 2, \ldots, \alpha$ and $j = 1, 2, \ldots, \beta$, we have

$$\begin{aligned} &\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta}(\|\sigma(u_i, u^j)\|^2 + \|\sigma(\phi u_i, u^j)\|^2) \\ &= \sum_{i=1}^{\alpha}\{H^{\ln \psi}(u_i, u_i) + H^{\ln \psi}(\phi u_i, \phi u_i)\}\beta \\ &+ f_2.\alpha.\beta + \beta \sum_{i=1}^{\alpha}\{(u_i \ln \psi)^2 + (\phi u_i \ln \psi)^2\}\|Z\|^2 \\ &+ 2\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta}(\|\mathcal{P}_{u^j} u_i\|^2 + \|\mathcal{Q}_{u^j} u_i\|^2). \end{aligned} \tag{3.13}$$

Furthermore, we can write the second fundamental form σ as follows

$$\sigma(u_i, u^j) = \sum_{k=1}^{\beta} g(\sigma(u_i, u^j), \phi u^k)\phi u^k + \sum_{t=1}^{2r} g(\sigma(u_i, u^j), n_t)n_t,$$

for each $1 \leq i \leq \alpha$ and $1 \leq j \leq \beta$. Taking the inner product of the above equation with $\sigma(u_i, u^j)$ we get

$$\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta} \|\sigma(u_i, u^j)\|^2 = \sum_{i=1}^{\alpha}\sum_{j,k=1}^{\beta} g(\sigma(u_i, u^j), \phi u^k)^2 + \sum_{i=1}^{\alpha}\sum_{j=1}^{\beta}\sum_{t=1}^{2r} g(\sigma(u_i, u^j), n_t)^2,$$

then making use of part (i) of Lemma 3.1, the last equation takes the form

$$\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta}(\|\sigma(u_i, u^j)\|^2 = \beta \sum_{i=1}^{\alpha}(\phi u_i \ln \psi)^2 + \sum_{i=1}^{\alpha}\sum_{j=1}^{\beta} \|\sigma_\mu(u_i, u^j)\|^2. \tag{3.14}$$

Similarly, we can get

$$\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta} \|\sigma(\phi u_i, u^j)\|^2 = \beta \sum_{i=1}^{\alpha}(u_i \ln \psi)^2 + \sum_{i=1}^{\alpha}\sum_{j=1}^{\beta} \|\sigma_\mu(\phi u_i, u^j)\|^2. \tag{3.15}$$

By use of (3.12), (3.14) and (3.15) in (3.13), we have

$$\begin{aligned}
&\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta}(\|\sigma_\mu(u_i,u^j)\|^2+\|\sigma_\mu(\phi u_i,u^j)\|^2)\\
&=(-\Delta\ln\psi-\beta\|\operatorname{grad}\,\ln\psi\|^2)\beta+f_2.\alpha.\beta\\
&+2\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta}(\|\mathcal{P}_{u^j}u_i\|^2+\|\mathcal{Q}_{u^j}u_i\|^2).
\end{aligned}\tag{3.16}$$

Now, we use the following notations

$$\|\sigma_\mu(D,D^\perp)\|^2=\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta}\|\sigma_\mu(u_i,u^j)\|^2+\|\sigma_\mu(\phi u_i,u^j)\|^2),$$

$$\|\mathcal{P}_{D^\perp}D\|^2+\|\mathcal{Q}_{D^\perp}D\|^2=\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta}(\|\mathcal{P}_{u^j}u_i\|^2+\|\mathcal{Q}_{u^j}u_i\|^2).$$

In view of above notations, (3.16) can be written as

$$\begin{aligned}
&-\beta\Delta\ln\psi=\|\sigma_\mu(D,D^\perp)\|^2+\beta^2\|\operatorname{grad}\,\ln\psi\|^2-f_2.\alpha.\beta\\
&-2\|\mathcal{P}_{D^\perp}D\|^2-2\|\mathcal{Q}_{D^\perp}D\|^2.
\end{aligned}\tag{3.17}$$

From (2.28), we can conclude that

$$\begin{aligned}
&\int_M\{\|\sigma_\mu(D,D^\perp)\|^2+\beta^2\|\operatorname{grad}\,\ln\psi\|^2-f_2.\alpha.\beta\\
&-2\|\mathcal{P}_{D^\perp}D\|^2-2\|\mathcal{Q}_{D^\perp}D\|^2)\}dV=0.
\end{aligned}\tag{3.18}$$

Here, if

$$\|\sigma_\mu(D,D^\perp)\|^2\geq f_2.\alpha.\beta+2\|\mathcal{P}_{D^\perp}D\|^2+2\|\mathcal{Q}_{D^\perp}D\|^2,$$

(3.17) and the above inequality implies that $\|\operatorname{grad}\,\ln\psi\|=0$ i.e., ψ is constant, since $\beta\neq 0$, which proves the Theorem completely.

Now, we have some consequences of above findings as follows

Corollary 3.2. *Let $M=N_T\times_\psi N_\perp$ be a compact orientable semi-invariant warped product submanifold of a cosymplectic space form $\bar{M}(c)$, if*

$$\|\sigma_\mu(D,D^\perp)\|^2\geq\frac{c.\alpha.\beta}{4},$$

then M is contact semi-invariant product

Note 3.2. Almost similar result was also proved by M. Atceken in [3] in the form of the Theorem 3.4.

Corollary 3.3. *Let $M = N_T \times_\psi N_\perp$ be a compact orientable semi-invariant warped product submanifold of a generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ admitting nearly cosymplectic structure. Then M is semi invariant product if and only if*

$$\|\sigma_\mu(D, D^\perp)\|^2 = f_2.\alpha.\beta + 2\|\mathcal{P}_{D^\perp}D\|^2 + 2\|\mathcal{Q}_{D^\perp}D\|^2. \tag{3.19}$$

Proof. Assume that M is compact semi-invariant warped product submanifold of a generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ admitting nearly cosymplectic structure satisfying (3.19), then from (3.18), ψ is constant i.e., M is contact semi-invariant product.

Conversely, if M is semi-invariant product, i.e., ψ is constant and from part (i) of Lemma 3.1 $\sigma(X, Z) \in \mu$, for all $X \in TN_T$ and $Z \in TN_\perp$ then it is easy to see that (3.19) holds.

Corollary 3.4. *Let $M = N_T \times_\psi N_\perp$ be a compact orientable semi-invariant warped product submanifold of a cosymplectic space form $\bar{M}(c)$. Then M is semi-invariant product if and only if*

$$\|\sigma_\mu(D, D^\perp)\|^2 = \frac{c.\alpha.\beta}{4}.$$

Note 3.3. Proposition 3.5 in [3] is approximate similar to Corollary 3.4.

4. Acknowledgments

The authors are highly thankful to anonymous referees for their valuable suggestions and comments which have improved the paper.

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Accepted: 21.04.2018