Modification of conformable fractional derivative with classical properties

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Abstract. In this paper, a new modification of the conformable fractional derivative that uses limit approach with classical properties, linearity, product rule, semi-chain rule, quotient rule, ets. is given. Moreover, we prove that such generalization formula does not satisfy the classical chain rule.

Keywords: Conformable fractional derivative.

1. Introduction

Fractional calculus nowadays is an interesting subject in the area of mathematical analysis. The idea comes from the question asked by L'Hospital in 1695 about the meaning of $\frac{d^n f}{dx^n}$, if $n = \frac{1}{2}$, [5]. Many researchers tried to generalize the concept of the usual derivative to a fractional one. Many definitions appears and most of them use integral form, see [1, 4, 6, 7, 8, 9]. Unfortunately all the existing fractional derivatives do not satisfy the familiar product rule, quotient rule and chain rule for the derivative of two functions and most of them except Caputo derivative don't satisfy the derivative of the constant function is zero. Moreover all of them do not have a corresponding Mean Value Theorem nor Rolle's Theorem. To find a solution for some of these difficulties an interesting definition for fractional derivative that uses limit approach is given by Khalil et. all, [3] as an extension of the usual definition of derivative.

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Definition 1. ([3]). *Let $f : [0, \infty) \to \mathbb{R}$ be a function. The α'th order conformable fractional derivative of f is defined by*

$$T_\alpha(f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon} \tag{1.1}$$

for all $t > 0$ and $\alpha \in (0,1)$. If f is $\alpha-$differentiable in some $(0,a)$, $a > 0$, and $\lim_{\epsilon \to 0^+} T_\alpha(f)(t)$ exists, then $T_\alpha(f)(0) = \lim_{\epsilon \to 0^+} T_\alpha(f)(t)$.

Following Khalil, Katugampola, [2] introduce a new generalization of the conformable fractional derivative that generalizes results in [3]. Properly, the authors in [2, 3], extend the usual definition of the derivative of a function f at a point a.

Definition 2. ([2]). *Let $f : [0, \infty) \to \mathbb{R}$ and $t > 0$. Then the fractional derivative of order α is defined by*

$$D_\alpha(f)(t) = \lim_{\epsilon \to 0} \frac{f(te^{\epsilon t^{-\alpha}}) - f(t)}{\epsilon} \tag{1.2}$$

for all $t > 0$ and $\alpha \in (0,1)$. If f is α-differentiable in some $(0,a)$, $a > 0$, and $\lim_{\epsilon \to 0^+} D_\alpha(f)(t)$ exists, then $D_\alpha(f)(0) = \lim_{\epsilon \to 0^+} D_\alpha(f)(t)$.

Clearly Definition 2 generalizes Definition 1. More preciesly if we truncate the series, $te^{\epsilon t^{-\alpha}} = \sum_{k=0}^{\infty} \frac{\epsilon^k t^{1-\alpha k}}{k!}$ when $k = 1$, we get the formula in Definition 1.

A function f is said to be α-differentiable if the conformable fractional derivatives of order α exists. If $\alpha = 1$, then the first derivative

$$D_1 f(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon) - f(t)}{\epsilon} = f'(t).$$

The authors in [2, 3] as a consequence of their definitions showed that the α-fractional derivatives in (1.1) and (1.2) satisfy product rule, quotient rule and proved some results similar to that in classical calculus such as Mean Value Theorem and Rolle's Theorem.

In this paper we give a new definition for $\alpha-$fractional derivative that generalizes both Definition 1 and 2. Also a simple example is given to prove that such generalization formulas don't obey the classical chain rule.

2. New generalization of fractional derivative

In this section we present our main new definition of $\alpha-$fractional derivative and study it's classical properties. Infact we prove that such derivative satisfies product rule, quotient rule, semi chain rule. Moreover we prove some corresponding results similar to Mean Value Theorem and Rolle's Theorem. We start by the following definitions that generalizes (1.1) and (1.2).

Definition 1. *Let $f : [0, \infty) \to R$ and $t > 0$. Then the fractional derivative of f of order α is defined by,*

$$M^\alpha(f)(t) = \lim_{\epsilon \to 0} \frac{f(tg(\epsilon t^{-\alpha})) - f(t)}{\epsilon}, \tag{2.1}$$

for all $t > 0$ *and* $\alpha \in (0,1)$, *where,* g *is a continuously differentiable function such that* $g(0) = g'(0) = 1$. *If* f *is* α*-differentiable in some* $(0,a)$, $a > 0$, *and* $\lim_{t\to 0^+} M^{\alpha}(f)(t)$ *exists, then* $M^{\alpha}(f)(0) = \lim_{t\to 0^+} M^{\alpha}(f)(t)$.

One should notice that $\alpha-$differentiable function need not be differentiable, for example, if $f(t) = 3t^{\frac{1}{3}}$, then $M^{\frac{1}{3}}(f)(0) = \lim_{t\to 0^+} M^{\frac{1}{3}}(f)(t) = 1$. While $M^{1}(f)(0)$ does not exist. This is not the case for the known classical fractional derivatives.

The following result is a generalization of Theorem 2.1 in [3] and Theorem 2.2 in [2].

Theorem 2. *If* $f : [0,\infty) \to R$ *is* α*-differentiable at a point* $t_0 > 0$, $\alpha \in (0,1]$, *then* f *is continuous at* t_0.

Proof. Since f is α-differentiable at t_0, taking the limit as $\epsilon \to 0$ of both sides of the equation

$$f(t_0 g(\epsilon t_0^{-\alpha})) - f(t_0) = \frac{f(t_0 g(\epsilon t_0^{-\alpha})) - f(t_0)}{\epsilon}\ \epsilon,$$

we get

$$\lim_{\epsilon\to 0}[f(t_0 g(\epsilon t_0^{-\alpha})) - f(t_0)] = M^{(\alpha)}(f)(t_0)\ \lim_{\epsilon\to 0}\epsilon = 0.$$

If we put $h = \epsilon t_0^{1-\alpha} + O(\epsilon^2)$, then $\lim_{\epsilon\to 0} f(t_0 + h) = f(t_0)$, which completes the proof. □

One of the main results of this paper is the following theorem.

Theorem 3. *Let* $\alpha \in (0,1]$ *and* f, L *be* $\alpha-$*differentiable functions at a point* $t > 0$. *Then,*

(1) $M^{\alpha}(af + bL)(t) = aM^{\alpha}(f)(t) + bM^{\alpha}(L)(t)$, *for all* $a, b \in R$.
(2) $M^{\alpha}(C) = 0$, *for all constant functions,* $f(t) = C$.
(3) $M^{\alpha}(fL)(t) = gM^{\alpha}(f)(t) + fM^{\alpha}(L)(t)$.
(4) $M^{\alpha}(\frac{f}{L}) = \frac{LM^{\alpha}(f)(t) - fM^{\alpha}(L)(t)}{L(t)^2}$.
(5) *If, in addition,* f *is differentiable, then* $M^{\alpha}(f)(t) = t^{1-\alpha}\frac{df(t)}{dt}$.

Proof. Parts (1) and (2) can be proved directly from the definition. To prove (3), fix $\alpha \in (0,1]$. Then for $t > 0$,

$$\begin{aligned}
M^{\alpha}(fL)(t) &= \lim_{\epsilon\to 0}\frac{(fL)(tg(\epsilon t^{-\alpha})) - (fL)(t)}{\epsilon}\\
&= \lim_{\epsilon\to 0}\frac{f(tg(\epsilon t^{-\alpha}))L(tg(\epsilon t^{-\alpha})) - f(t)L(tg(\epsilon t^{-\alpha})) + f(t)L(tg(\epsilon t^{-\alpha})) - f(t)L(t)}{\epsilon}\\
&= \lim_{\epsilon\to 0} L(tg(\epsilon t^{-\alpha}))\frac{f(tg(\epsilon t^{-\alpha})) - f(t)}{\epsilon} + f(t)\lim_{\epsilon\to 0}\frac{L(tg(\epsilon t^{-\alpha})) - L(t)}{\epsilon}\\
&= L(tg(0))M^{\alpha}(f)(t) + f(t)M^{\alpha}(L)(t)\\
&= L(t)M^{\alpha}(f)(t) + f(t)M^{\alpha}(L)(t).
\end{aligned}$$

The proof of part (4) is similar.

Now we prove part (5). Since f is differentiable at $t>0$, putting $tg(\epsilon t^{-\alpha}) = t+\epsilon t^{1-\alpha}+O(\epsilon^2)$ and $h=\epsilon t^{1-\alpha}(1+O(\epsilon))$ in (2.1) taking in account $g(0)=g\prime(0)=1$, we have

$$\begin{aligned} M^{\alpha}(f)(t) &= \lim_{\epsilon\to 0}\frac{f(tg(\epsilon t^{-\alpha}))-f(t)}{\epsilon}. \\ M^{\alpha}(f)(t) &= \lim_{\epsilon\to 0}\frac{f(t+\epsilon t^{1-\alpha}+O(\epsilon^2))-f(t)}{\epsilon} \\ &= \lim_{\epsilon\to 0}\frac{f(t+h)-f(t)}{\frac{1}{ht^{\alpha-1}/(1+O(\epsilon))}} \\ &= t^{1-\alpha}\frac{df}{dt}(t). \end{aligned}$$

This ends the proof. □

As an application to part (5) Theorem 3, we have

Theorem 4. *Let $\alpha\in(0,1]$ and $a,n\in R$ and $t>0$. Then,*

(1) $M^{\alpha}(t^n)=nt^{n-\alpha}$.

(2) $M^{\alpha}(e^{at})=at^{1-\alpha}e^{at}$.

(3) $M^{\alpha}(\sin(at))=at^{1-\alpha}\cos(at)$.

(4) $M^{\alpha}(\cos(at))=-at^{1-\alpha}\sin(at)$.

(5) $M^{\alpha}(\frac{1}{\alpha}t^{\alpha})=1$.

Also an unusual results is given in the following theorem.

Theorem 5. *Let $\alpha\in(0,1]$ and $t\in R$. Then,*

(1) $M^{\alpha}(\sin(\frac{1}{\alpha}t^{\alpha}))=\cos(\frac{1}{\alpha}t^{\alpha})$.

(2) $M^{\alpha}(\cos(\frac{1}{\alpha}t^{\alpha}))=-\sin(\frac{1}{\alpha}t^{\alpha})$.

(3) $M^{\alpha}(e^{\frac{1}{\alpha}t^{\alpha}})=e^{\frac{1}{\alpha}t^{\alpha}}$.

Definition 6. *Let f and L be $\alpha-$differentiable function at a point $t>0$. We say that*

(1) *f and g satisfy the semi-chain rule if*

$$M^{\alpha}(f\circ L)(t)=f'(L(t))M^{\alpha}(L)(t),$$

provided that f is continuously differentiable.

(2) *f and g satisfy the classical chain rule if*

$$M^{\alpha}(f\circ L)(t)=M^{\alpha}(f)(L(t))M^{\alpha}(L)(t).$$

In the following theorem we will show that conformable fractional derivatives do no obey the classical chain rule.

Theorem 7. *For $\alpha \in (0,1]$,*

(1) *M^α satisfy the semi-chain rule.*

(2) *M^α does not satisfy the classical chain rule.*

Proof. (1) Let f be a differentiable function for $t>0$ and L be $\alpha-$differentiable at the point $t_0>0$. Then

$$\begin{aligned} M^\alpha(f\circ L)(t_0) &= \lim_{\epsilon\to 0}\frac{f(L(t_0 g(\epsilon t_0^{-\alpha})))-f(L(t_0))}{\epsilon} \\ &= \lim_{\epsilon\to 0}\frac{f(L(t_0 g(\epsilon t_0^{-\alpha})))-f(L(t_0))}{L(t_0 g(\epsilon t_0^{-\alpha}))-L(t_0)}\frac{L(t_0 g(\epsilon t_0^{-\alpha}))-L(t_0)}{\epsilon}. \end{aligned}$$

Since, f is differentiable, by the use of Mean Value Theorem there exists a point $c\in \left(L(t_0), L(t_0 g(\epsilon t_0^{-\alpha})\right)$ such that

$$f(L(t_0 g(\epsilon t_0^{-\alpha})))-f(L(t_0)) = f'(c)\left(L(t_0 g(\epsilon t_0^{-\alpha}))-L(t_0)\right).$$

Consequently,

$$M^{(\alpha)}(f\circ L)(t_0)=\lim_{\epsilon\to 0} f'(c)\frac{L(t_0 g(\epsilon t_0^{-\alpha}))-L(t_0)}{\epsilon}. \tag{2.2}$$

But $L(t_0 g(\epsilon t_0^{-\alpha}))-L(t_0) = L\left(t_0(1+\epsilon t_0^{-\alpha}+O(\epsilon^2))-(L(t_0)\right)$ which turns to zero as ϵ turns to zero. It follows that $c\to L(t_0)$ as $\epsilon\to 0$ and hence from (2.2), we get

$$M^\alpha(f\circ L)(t_0)=\lim_{\epsilon\to 0} f'(c)\lim_{\epsilon\to 0}\frac{L(t_0 g(\epsilon t_0^{-\alpha}))-L(t_0)}{\epsilon} = f'(L(t_0))M^\alpha\left(L\right)(t_0).$$

To prove (2), it is enough to give the following simple example: Put $f(t)=\sin(t)$ and $L(t)=t^3$. Then applying Theorem 3 part(5) we get,

$$M^\alpha(f\circ L)(t)=t^{1-\alpha}\frac{d(f\circ L)}{dt}(t)=t^{1-\alpha}(3t^2)\cos(t^3)=3t^{3-\alpha}\cos(t^3)$$

and

$$M^\alpha(f)(L(t))=M^\alpha(\sin t)(t^3)=(t^3)^{1-\alpha}\cos t^3,$$

$$M^\alpha(L)(t)=(t)^{1-\alpha}\frac{d}{dt}L(t)=3t^{3-\alpha}.$$

Hence

$$M^\alpha(f\circ L)(t)=3t^{3-\alpha}\cos t^3\neq 3t^{6-4\alpha}\cos t^3 = M^\alpha(f)(L(t))M^\alpha(L)(t).$$

This completes the proof of the theorem. □

3. More generalization and applications

First of all if we let the function g in Definition 2.1 to be any polynomial of degree n such that $g(0) = g\prime(0) = 1$, we get a generalization of Definition 2.7 in [2]. Furthermore the following definition generalizes the conformable fractional derivatives defined in [3, 10].

Definition 8. *Let* $k_1, k_2 : [0,\infty) \times [0,1] \to [0,\infty)$, *be continuously differentiable on a neighborhood of* (α, t), *and* $t > 0, \alpha \in (0,1], f : [0,\infty) \to \mathbb{R}$. *Then fractional derivative of* f *of order* α *is defined as*

$$D^{\alpha}(f)(t) = \lim_{\epsilon\to 0} \frac{f(t(1+\epsilon k_1(t,\alpha))) - f(t)}{\epsilon k_2(t,\alpha)}. \tag{3.1}$$

If f *is* α*-differentiable in some* $(0,a)$, $a > 0$, *and* $\lim_{t\to 0^+} D^{\alpha}(f)(t)$ *exists, then* $D^{\alpha}(f)(0) = \lim_{t\to 0^+} D^{\alpha}(f)(t)$.

Indeed if $k_1(t,\alpha) = t^{-\alpha}$ and $k_2(t,\alpha) = 1$, we have Khalil Definition [3], while if If $k_1(t,\alpha) = t^{-\alpha-1}(t-a)$, and $k_2(t,\alpha) = 1 - at^{-\alpha}$, (3.1) turns to be preciely

$$D^{\alpha}(f) = \lim_{\epsilon\to 0} \frac{f(t(1+\epsilon t^{-\alpha-1}(t-a)) - f(t)}{\epsilon(1-at^{-\alpha})} = D^{a}_{\alpha}(f)(t),$$

which coincide with the definition of Sarikaya, budak, and Usta in [10].

Theorem 9. *Let* $k_1, k_2 : [0,\infty) \times [0,1] \to [0,\infty)$, *be continuously differentiable on a neighborhood of* (α, t), *and* $t > 0, \alpha \in (0,1], f : [0,\infty) \to \mathbb{R}$. *Then fractional derivative* $D^{\alpha}(f)$ *of order* α *defined as*

$$D^{\alpha}(f)(t) = \lim_{\epsilon\to 0} \frac{f(t(1+\epsilon k_1(t,\alpha))) - f(t)}{\epsilon k_2(t,\alpha)}$$

doesn't satisfy the classical chain rule unless $\alpha = 1$.

Proof. Note that: for $\alpha \neq 1, tk_1(t,\alpha) \neq k_2(t,\alpha)$. Other wise

$$\begin{aligned} D^{\alpha}(f)(t) &= \lim_{\epsilon\to 0} \frac{f(t(1+\epsilon k_1(t,\alpha)) - f(t)}{\epsilon k_2(t,\alpha)} \\ &= \lim_{\epsilon\to 0} \frac{f(t+h) - f(t)}{h} = f'(t), \end{aligned}$$

where $h = \epsilon t k_1(t,\alpha)$. Hence $tk_1(t,\alpha) \neq k_2(t,\alpha)$.

To prove the theorem it is enough to show that if f and L are differentiable non constant functions, then for $\alpha \neq 1, D^{\alpha}$ doesn't satisfy the classical chain rule.

Since f and L are differentiable, then

$$\begin{aligned} D^{\alpha}(L)(t) &= \lim_{\epsilon\to 0} \frac{L(t(1+\epsilon k_1(t,\alpha)) - L(t)}{\epsilon k_2(t,\alpha)} \\ &= \lim_{\epsilon\to 0} \frac{L(t(1+\epsilon k_1(t,\alpha)) - L(t)}{\epsilon t k_1(t,\alpha)} . \frac{tk_1(t,\alpha)}{k_2(t,\alpha)} \\ &= \frac{tk_1(t,\alpha)}{k_2(t,\alpha)} L'(t). \end{aligned} \tag{3.2}$$

Similarly,

$$D^{\alpha}(f)(L(t)) = \frac{L(t)k_1(L(t),\alpha)}{k_2(L(t),\alpha)} f'(L(t)) \tag{3.3}$$

and

$$D^{\alpha}(f \circ L)(t) = \frac{tk_1(t,\alpha)}{k_2(t,\alpha)} (f \circ L)'(t) = \frac{tk_1(t,\alpha)}{k_2(t,\alpha)} f'(L(t))L'(t). \tag{3.4}$$

Now suppose that D^{α} does satisfy the classical chain rule. Then using (3.2) and (3.3) we have:

$$\begin{aligned} D^{\alpha}(f \circ L)(t) &= D^{\alpha}(f)(L(t))\ D^{\alpha}(L)(t) \\ &= \frac{L(t)k_1(L(t),\alpha)}{k_2(L(t),\alpha)} f'(L(t))\ \frac{tk_1(t,\alpha)L'(t)}{k_2(t,\alpha)} \\ &= \frac{tk_1(t,\alpha)}{k_2(t,\alpha)} f'(L(t))L'(t). \qquad (3.5) \end{aligned}$$

Thus from (3.4) and (3.5) , we get

$$f'(L(t))L'(t)\frac{tk_1(t,\alpha)}{k_2(t,\alpha)}\left(1 - \frac{L(t)k_1(L(t),\alpha)}{k_2(L(t),\alpha)}\right) = 0.$$

The case $f'(L(t)) = 0$, $L'(t) = 0$ or $tk_1(t,\alpha) = 0$, is impossible. Hence

$$\frac{L(t)k_1(L(t),\alpha)}{k_2(L(t),\alpha)} = 1$$

and this implies that $L(t)k_1(L(t),\alpha) = k_2(L(t),\alpha)$ which is impossible since in this case α turns to be 1. This completes the proof. □

The following two theorems extend the Rolle's and the Mean Value Theorem for $\alpha-$fractional derivative proved in [2, 3] .

Theorem 10 (Roll's theorem for $\alpha-$fractional differentiable functions)**.** *Let $a, b > 0$ and f be a real valued continuous function on $[a, b]$ with $f(a) = f(b)$. If f is $\alpha-$differentiable on (a, b) for some $\alpha \in (0, 1)$, then there exists $c \in (a, b)$, such that $M^{\alpha}(f)(c) = 0$.*

Proof. Since f is continuous on $[a, b]$ and $f(a) = f(b)$, there is a point $c \in (a, b)$, at which the function f has local extrema. But f is $\alpha-$differentiable at c. So

$$M^{\alpha}(f)(c) = \lim_{\epsilon \to 0^-} \frac{f(cg(\epsilon c^{-\alpha})) - f(c)}{\epsilon} = \lim_{\epsilon \to 0^+} \frac{f(cg(\epsilon c^{-\alpha})) - f(c)}{\epsilon}.$$

Since the two limits have opposite sings it follows that $M^{\alpha}(f)(c) = 0$. □

Theorem 11 (Generalized Mean Value Theorem for $\alpha-$fractional differentiable functions)**.** *Let f and h be two continuous functions on $[a,b]$ with $a>0$. If f and h are both $\alpha-$differentiable functions on (a,b), then there exists $c\in(a,b)$ such that*

$$(f(b)-f(a))M^{\alpha}(h)(c)=(h(b)-h(a))M^{\alpha}(f)(c).$$

Proof. Let $u(x)=(f(b)-f(a))h(x)-(h(b)-h(a))f(x)$. Then u satisfies the conditions of Roll's Theorem on $[a,b]$.

Hence there exists $c\in(a,b)$ such that $M^{\alpha}(u)(c)=0$. Consequently, $(f(b)-f(a))M^{\alpha}(h)(c)=(h(b)-h(a))M^{\alpha}(f)(c)$. □

Corollary 12. *(Mean Value Theorem for $\alpha-$fractional differentiable functions).*

Let f be a continuous function on $[a,b]$ with $a>0$. If f is $\alpha-$differentiable on (a,b) for some $\alpha\in(0,1)$, then there exists $c\in(a,b)$ such that

$$M^{(\alpha)}(f)(c)=\frac{f(b)-f(a)}{\frac{1}{\alpha}b^{\alpha}-\frac{1}{\alpha}a^{\alpha}}.$$

Proof. Taking $h(x)=\frac{x^{\alpha}}{\alpha}$ in Theorem 11, we get the result. □

For the case $\alpha\in(n,n+1]$, we have the following generalization:

Definition 13. *Let $\alpha\in(n,n+1]$, for some $n\in N$ and f be an $n-$differentiable at $t>0$. Then the $\alpha-$fractional dirivative of f defined by*

$$M^{\alpha}(f)(t)=\lim_{\epsilon\to 0}\frac{f^{(n)}(tg(\epsilon t^{-\alpha}))-f^{(n)}(t)}{\epsilon}$$

if the limit exists.

As a direct consequence of Definition 13 and part (5) of Theorem 3, one can show that

$$M^{\alpha}(f)(t)=t^{n+1-\alpha}f^{(n+1)}(t),$$

where $\alpha\in(n,n+1]$ and f is $(n+1)-$differentiable at $t>0$.

4. Fractional integral

According to results obtained in [2] and [3] due to our similar results in Theorem 3 and Theorem 4, it is nice to say that we can still have the same definition of fractional integral. We start by the following definition.

Definition 14. *Let $a\geq 0, t\geq a$ and let f be a function defined on $(a,t]$. Then for $\alpha\in R$, the $\alpha-$fractional Integral of f is defined by $I_a^{\alpha}(f)(t)=\int_a^t\frac{f(x)}{x^{1-\alpha}}dx$ provided the Riemann improper integral exists.*

The following theorem shows an interesting property of the above definition that the $\alpha-$fractional integral and $\alpha-$derivatives are inverse each others.

Theorem 15. *Let $a \geq 0, t \geq a$ and let f be a continuous function defined on $(a,t]$. Then for $\alpha \in R$ such that $I_a^\alpha f$ exists. Then $M^\alpha(I_a^\alpha f)(t) = f(t), for\ t \geq a$.*

Proof. Since f is continuous, $I_a^\alpha f$ is clearly differentiable. Therefore, using part (5) of Theorem 3, we have

$$M^\alpha(I_a^\alpha f)(t) = t^{1-\alpha}\frac{d}{dt}(I_a^\alpha(f)(t)) = t^{1-\alpha}\frac{d}{dt}\int_a^t \frac{f(x)}{x^{1-\alpha}}dx = t^{1-\alpha}\frac{f(t)}{t^{1-\alpha}} = f(t).$$

□

Using Definition 14 one can easily prove the following Theorem.

Theorem 16. *Let $\alpha \in (0,1]$ and f, g be $\alpha-$Fractional Integrable on $[0,t]$. Then,*
(1) $I_a^\alpha(bf + cg)(t) = bI_a^\alpha(f)(t) + cI_a^\alpha(g)(t)$, *for all* $a, b \in R$.
(2) $I_a^\alpha(t^n) = \frac{1}{(n+\alpha)}[t^{n+\alpha} - a^{n+\alpha}]$, *for all* $n \in R$.
(3) $I_a^\alpha(c) = \frac{c}{\alpha}[t^\alpha - a^\alpha]$, *for all constant functions,* $f(t) = c$.

The following theorem shows that the usual integration by parts can be applicable in the case of conformable fractional derivative.

Theorem 17. *(Integration by parts) Let f, g be $\alpha-$integrable functions. Then* $I_a^\alpha(fM^\alpha(g)) = fg - I_a^\alpha(gM^\alpha(f))$.

Proof. Using Theorem 3 part 3, we have

$$M^{(\alpha)}(fg) = gM^{(\alpha)}(f) + fM^{(\alpha)}(g).$$

Integrate both sides of the above equation we get:

$$I_a^\alpha(M^{(\alpha)}(fg)) = I_a^\alpha(gM^{(\alpha)}(f)) + I_a^\alpha(fM^{(\alpha)}(g)).$$

Consequently,

$$I_a^\alpha(fM^{(\alpha)}(g)) = fg - I_a^\alpha(gM^{(\alpha)}(f)).$$

This completes the proof. □

We end this paper by the following example:

Example 18. Evaluate $I_0^{\frac{1}{2}}(t^{\frac{3}{2}}e^t)$

Solution 19. *Using integration by parts we have*

$$\begin{aligned} I_0^{\frac{1}{2}}(t^{\frac{3}{2}}e^t) &= t[e^t - 1] - I_0^{\frac{1}{2}}(t^{\frac{1}{2}}\left(e^t - 1\right)) \\ &= t[e^t - 1] - \int_0^t \frac{x^{\frac{1}{2}}\left(e^x - 1\right)}{x^{\frac{1}{2}}}dx = (t-1)e^t + 1. \end{aligned}$$

The same alternative solution can be found as

$$I_0^{\frac{1}{2}}(t^{\frac{3}{2}}e^t) = \int_0^t \frac{x^{\frac{3}{2}}e^x}{x^{\frac{1}{2}}}dx = \int_0^t xe^x dx = \int_0^t xe^x dx = (t-1)e^t + 1.$$

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