On Lie ideals of inverse semirings

S. Sara*
House no. 638, Block F
Gulshan Ravi, Lahore
Pakistan
Postal code 54000
saro_c18@yahoo.com

M. Aslam

Abstract. The purpose of this paper is to study Lie ideals of inverse semirings, thereby extending a few well-known results of I.N Herstein and C.Lanski in the setting of inverse semirings.

Keywords: inverse semiring, Lie ideals, commutators, prime inverse semiring.

1. Introduction

Lie ring structure of an associative ring R was first considered by I.N Herstein in 1950's. He studied Lie ideals and their relationship with R in series of papers ([9],[10],[11]). In ensuing years, the study of Lie ideals generalized to prime and semiprime rings and rings with involution. Herstein's work on Lie ideals proved to be a source of motivation for many researchers to look into the Lie theory in various other directions and settings ([1],[3],[4],[5],[8],[11],[14],[18],[19],[20]). Recently, M.A Javed and M.Aslam [16] introduced the notion of Lie ideals in inverse semirings and investigated some commutativity conditions on inverse semirings with the help of Lie ideals and derivations.

Our objective is to explore and generalize Lie type results of rings in the setting of semirings, thereby we extend a few results on Lie ideals of [10],[14] and [18] to inverse semirings. These results will be helpful in investigating the theory of Lie derivation and higher derivation of Lie ideals of semirings.

Herstein [10] proved a result on Lie ideals which says that; A non-zero Lie ideal U of a 2-torsion free simple ring R is either contained in center of R or contains a non-zero ideal of R. Later, in dealing with some problems on von Neumann algebra, Herstein [14] extended this result to semiprime rings as follows; Let R be a semiprime 2-torsion free ring and V be subring of R. Suppose U is a Lie ideal of R such that $[V,U] \subset V$ then either $[V,U] = 0$ or V contains non-zero ideal of R. Lanski [18] proved the same result with prime rings (Theorem 12 of [18]). In this paper, we establish these two main results of Lie ideals for prime inverse semirings.

$*$. Corresponding author

By S, we mean a semiring with commutative addition and an absorbing zero. A semiring S is called an inverse semiring[17] if for every $a \in S$ there exists a unique element $\acute{a} \in S$ such that $a + \acute{a} + a = a$ and $\acute{a} + a + \acute{a} = \acute{a}$. Karvellas [17] proved that for all $a, b \in S$, $(a.b)' = \acute{a}.b = a.\acute{b}$ and $\acute{a}\acute{b} = ab$. Bandlet and Petrich [2] studied inverse semirings with some conditions (A1)-(A4). Throughout this paper, S will denote inverse semiring which satisfies (A-2) condition of [2] i.e; for every $a \in S$, $a + \acute{a} \in Z(S)$, where $Z(S)$ is center of S. This class of inverse semiring is known as MA semiring[15], which is indeed useful in developing the Lie type theory of semirings and studying certain additive mappings in semirings (see for example [21], [22], [23]). Commutative inverse semirings and distributive lattices are natural examples of MA-semiring. Also, if R is a ring and T is a subset of all ideals of R then $S = \{(a, I) : a \in R, I \in T\}$ is MA-semiring(see [15]). For more examples, we refer readers to [15]. A semiring S is prime if $aSb = 0$ implies either $a = 0$ or $b = 0$. S is 2-torsion free if $2x = 0$, $x \in S$ implies $x = 0$. By [15], a commutator $[.,.]$ in inverse semiring is defined as $[x, y] = xy + \acute{y}x$. Unlike a commutator of a ring, it is not necessary that $[x, x] = 0$ for every $x \in S$. We will make use of fundamental identities of commutators $[x, yz] = y[x, z] + [x, y]z$, $[xy, z] = [x, z]y + x[y, z]$ and the jacobian identity $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$ (see [15]). By [16], an additively closed subset U of S is called Lie ideal of S if $[U, S] \subset U$. By $[U, V]$, we mean an additive subsemigroup of S generated by the elements of the form $[u, v]$, where $u \in U$ and $v \in V$. A Lie ideal U is said to be commutative Lie ideal if $[U, U] = 0$. An additive mapping d on S is called derivation if $d(xy) = d(x)y + xd(y)$, $x, y \in S$.

We shall need following two results in our arguments.

Lemma 1 (Lemma 1.2. of [21]). *If $x, y, z \in S$ then the following identities are valid:*

(1) $[xy, x] = x[y, x]$, $[x, yx] = [x, y]x$, $[x, xy] = x[x, y]$, $[yx, x] = [y, x]x$;
(2) $y[x, z] = [x, yz] + [x, y]\acute{z}$, $[x, y]z = \acute{y}[x, z] + [x, yz]$;
(3) $x[y, z] = [xy, z] + [x, z]\acute{y}$, $[x, z]y = [xy, z] + \acute{x}[y, z]$.

Theorem 2 (Theorem 2.7 of [16]). *Let S be a 2-torsion free prime inverse semiring and d be a non-zero derivation of S. If $a \in S$ and $[d(S), a] = 0$. Then $a \in Z(S)$.*

We will make frequent use of the following lemma.

Lemma 3. *Let S be an inverse semiring and $a \in S$. If $[a, x] = 0$ for all $x \in S$. Then $a \in Z(S)$.*

Proof. *We have, $0 = [a, x] = ax + x\acute{a}$. Adding xa on both sides we have, $xa = ax + x(\acute{a} + a) = ax + (\acute{a} + a)x = ax$. Thus $a \in Z(S)$.*

The main results of this paper deal with the case in which S is prime and so we assume S to be prime and 2-torsion free, in the subsequent discussions.

Theorem 4. *Let U be a non-zero Lie ideal and subsemiring of S then either $U \subset Z(S)$ or U contains a non-zero ideal of S.*

Proof. Suppose that U is commutative, that is; $[U,U] = 0$. Let $a \in U$ and $s \in S$ then $[a,s] \in U$ and $[a,[a,s]] = 0$. Replacing s by $st, t \in S$ in last equation we get, $[a,s][a,t] = 0$. Replacing t by st in last expression, we have $0 = [a,s]s[a,t]$. But S is prime so $[a,s] = 0$. Hence by Lemma 3, $U \subset Z(S)$.

Suppose that U is non-commutative. Let $a, b \in U$ such that $[a,b] \neq 0$. As U is a Lie ideal so for $s \in S$ we have,

$$[a,sb] = [a,s]b + s[a,b] \in U.$$

U is also subsemiring of S so $[a,s]\acute{b} \in U$. Thus we have, $[a,s](b+\acute{b}) + s[a,b] \in U$ or $(as + s\acute{a})(b+\acute{b}) + s[a,b] \in U$ or $(s+\acute{s})ab + s(b+\acute{b})\acute{a} + sab + s\acute{b}a \in U$ or $s[a,b] \in U$.

For $t \in S$ we have, $[s[a,b],t] \in U$ or $s[a,b]t + \acute{t}s[a,b] \in U$. From this, $s[a,b]t + \acute{t}s[a,b] + ts[a,b] \in U$ or $s[a,b]t \in U$. Clearly, $S[a,b]S$ is an ideal of S contained in U. If we say $S[a,b]S = (0)$ then either $S[a,b] = (0)$ or $[a,b] = 0$, which gives a contradiction. Hence, $S[a,b]S$ is non-zero ideal contained in U. This establishes the theorem.

Corollary 5. *Let $a \in S$ and $[a,[a,s]] = 0$ for all $s \in S$. Then $a \in Z(S)$.*

Lemma 6. *Let I be a non-zero ideal of S such that $[a,I] = 0$, $a \in S$. Then $a \in Z(S)$.*

Proof. Let $s \in S$ and $b \in I$ then $[a,sb] = 0$. Thus we have, $[a,s]I = 0$. But S is prime so $[a,s] = 0$. Hence by Lemma 3, $a \in Z(S)$.

Lemma 7. *Let U be a Lie ideal of S then either $[U,S] = 0$ or $[T(S),S] = 0$, where $T(U) = \{x \in S : [x,U] = 0\}$.*

Proof. It is easy to see that $T(U)$ is a Lie ideal and subsemiring of S so by Theorem 4 either $[T(U),S] = 0$ or $T(U)$ contains a non-zero ideal, say I. If $I \subset T(U)$ then from Lemma 6, we obtain $[U,S] = 0$.

The following corollary immediately follows from Lemma 3.

Corollary 8. *Let U be a Lie ideal of S then either $U \subset Z(S)$ or $T(U) \subset Z(S)$.*

Lemma 9. *Let U_1 and U_2 be Lie ideals of S such that $[[U_1,U_2],S] = 0$. Then either $[U_1,S] = 0$ or $[U_2,S] = 0$.*

Proof. If $[U_1,U_2] = 0$ then by Lemma 7, either $[U_1,S] = 0$ or $[U_2,S] = 0$. Let $[U_1,U_2] \neq 0$. Suppose $[U_1,S] \neq 0$ and $[U_2,S] \neq 0$. Let $u \in U_1$, $v \in U_2$ and $s \in S$ then

$$[[u,[v,vs]],S] = 0.$$

Using Lemma 1 we have, $[[u,v][v,s]+v[u,[v,s]],S]=0$. In particular, $[[u,v][v,s]+v[u,[v,s]],v]=0$. From this, we obtain

$$[u,v][v,s]v+v[u,[v,s]]v+\acute{v}[u,v][v,s]+\acute{v}v[u,[v,s]]=0. \tag{1.1}$$

By Lemma 3 and the given hypothesis we have, $[U_1,U_2]\in Z(S)$ which implies that $v[[u,[v,s]],v]+[u,v][[v,s],v]=0$ and hence $[u,v][[v,s],v]=0$. Therefore, $[u,v]S[[v,s],v]=S[u,v][[v,s],v]=0$. But S is prime so $[[v,s],v]=0$ which by Lemma 7 implies that either $[v,S]=0$ or $[[v,s],S]=0$. From Corollary 5, $[v,S]=0$ which contradicts that $[U_2,S]=0$. This completes the proof.

Corollary 10. *Let U_1 and U_2 be Lie ideals of S and $[[U_1,U_2],S]=0$. Then either $U_1\subset Z(S)$ or $U_2\subset Z(S)$.*

Lemma 11. *Let U be a Lie ideal and $V\subset S$ such that $[[U,V],S]=0$. Then either $[U,S]=0$ or $[V,S]=0$.*

Proof. Consider $T=\{x\in S:[[x,U],S]=0\}$ then using jacobian identity we can show that T is a Lie ideal of S. Hence, by Lemma 9 either $[U,S]=0$ or $[T,S]=0$.

Lemma 12. *If U is a Lie ideal of S , $x\in S$ and $xUx=0$ then either $x=0$ or $U\subset Z(S)$.*

Proof. For $s\in S$ and $u\in U$, we have $x[u,s]x=0$. Replacing s by u_1xs, $u_1\in U$, we get $xuu_1xsx=0$. But S is prime so either $x=0$ or $xuu_1x=0$. Suppose $x\neq 0$. Replacing s by u_1u_2xs in $x[u,s]x=0$ we have xuu_1u_2sx. Continuing this, we arrive at $x\bar{U}x=0$ where $\bar{U}$ is subsemiring of S generated by U. If $U\not\subset Z(S)$ then $\bar{U}\not\subset Z(S)$. By Theorem 4, $\bar{U}$ contains a non-zero ideal, say I. Hence $xIx=0$ but S is prime so $x=0$, a contradiction. Hence lemma is proved.

The next lemma is an extension of Lemma 10 of [18] in a canonical fashion.

Lemma 13. *Let $U\not\subset Z(S)$ be a Lie ideal of S and V be an additive subsemigroup of S and $[U,V]\subset V$. If $v^2=0,\forall v\in V$ then $V=0$.*

Lemma 14. *Let $U\not\subset Z(S)$ be a Lie ideal of S and V be an additive subsemigroup of S such that $[V,U]\subset V$. If $[V,V]=0$ then $V\subset Z(S)$.*

Proof. Put $I=[V,U]$ then $[I,I]=0$. Let $a\in I,u\in U$ and $s\in S$ then using Lemma 1 we have, $0=[a,[a,[u,us]]=2[a,u][a,[a,s]]$. S is 2-torsion free so $[a,u][a,[a,s]]=0$. Replace s by au, we obtain

$$[a,u]^3=0, a\in I, u\in U. \tag{1.2}$$

Also,

$$[a,[a,u]]=a^2u+2au\acute{a}+ua^2=0. \tag{1.3}$$

If $a \in I$ such that $a^3 = 0$ then from (3), $a^3ua + 2a^2u\acute{a}a + aua^3 = 0$ which gives $a^2ua^2 = 0$. By Lemma 12, $a^2 = 0$. Again, by (3) we get $a = 0$. Hence, if $a \in I$ and $a^3 = 0$ then $a = 0$. So from (2), $[a, u] = 0, u \in U, a \in I$. By Lemma 7, $[I, S] = 0$ and therefore $[V, S] = 0$, by Lemma 9.

Theorem 15. *Let U be a Lie ideal and $V \subset U$ be an additive subsemigroup of S such that $[V, U] \subset V$. If $[[V, V], S] = 0$ then $[V, U] = 0$.*

Proof. If $[V, V] = 0$ then by above Lemma $[V, S] = 0$ and hence $[V, U] = 0$, as desired. Suppose $v_1, v_2 \in V$ such that $\alpha = [v_1, v_2] \neq 0$ then $[\alpha, S] = 0$ or $\alpha \in Z(S)$, by Lemma 3. For $s \in S$, let $d(s) = [v_1, s]$ then by given hypothesis we have

$$[d^2(u), S] = 0, \ \forall u \in U.$$

For $x \in S$, let $u_1 = [v_2, x] \in U$ then $v_2u_1 \in U$. Thus we have $[d^2(v_2u_1), S] = 0$. But $d^2(v_2u_1) = v_2d^2(u_1) + 2\alpha d(u_1)$. Therefore, $[v_2d^2(u_1) + 2\alpha d(u_1), S] = 0$. In particular, $0 = [v_2d^2(u_1) + 2\alpha d(u_1), v_2] = 2\alpha[d(u_1), v_2]$ or $\alpha[d(u_1), v_2] = 0$.

As S is prime so non-zero elements of $Z(S)$ cannot be zero divisors in S, thus we have $[d(u_1), v_2] = 0$. Hence $[[v_2, d(x)], v_2] = 0$. Replacing x by v_1x in last expression and using it again, we have $0 = [[v_2, d(v_1x)], v_2] = [[v_2, v_1d(x)], v_2] = \alpha[v_2, d(x)] + v_1[[v_2, d(x)], v_2] + [\acute{\alpha}, v_2]d(x) + \acute{\alpha}[d(x), v_2] = 2\alpha[v_2, d(x)]$.

Therefore, $\alpha[v_2, d(x)] = 0$. Primeness of S implies that $[v_2, d(x)] = 0, \ x \in S$. By Theorem 2 , $v_2 \in Z(S)$. As $[[v_1, v_2], S] = 0$ so $[[v_1, v_2], v_2] = 0$ or $[v_1, v_2]v_2 = 0$. This implies that either $\alpha = 0$ or $v_2 = 0$, a contradiction. Hence theorem is proved.

Theorem 16. *Let U be a Lie ideal of S and V a subsemiring of S such that $[V, U] \subset V$. Then either $[U, S] = 0$, $[V, S] = 0$ or V contains a non-zero ideal of S.*

Proof. If $[V, U] = 0$ then by Lemma 7, either $[V, S] = 0$ or $[U, S] = 0$. Suppose $[V, U] \neq 0$. As U is Lie ideal we have $[V, U] \subset V \cap U$. Let $s \in S, \ t \in [V, U]$ and $v \in V$ then $[v, [t, ts]] \in [V, U] \subset V$ or $[v, t[t, s]] \in V$, by Lemma 1. Also, $t[v, [t, s]] \in V$. Thus by Lemma 1 we have $[v, t][t, s] = [v, [t, ts]] + \acute{t}[v, [t, s]] \in V$, that is;

$$[v, t][t, s] \in V, \quad t \in [V, U], v \in V. \tag{1.4}$$

Again, using Lemma 1 we obtain

$$[v, t][t, w]s = [v, t][t, ws] + [v, t]w[t, s]. \tag{1.5}$$

Using the fact that $[V, U] \subset V$ and V is subsemiring of S we have, $[v, t]w[t, s] = [v, t](w + \acute{w} + w)[t, s] = [v, t][t, s](w + \acute{w}) + [v, t]w[t, s] = [v, t][t, s]w + [v, t][w, [t, s]] \in V$. This together with (4) and (5) gives $[v, t][t, w]s \in V, \ \forall w, v \in V, t \in [V, U], s \in S$. Hence $[u, [v, t][t, w]s] \in V$ for all $u \in U, \ s \in S$. So we have,

$$u[v, t][t, w]s = [u, [v, t][t, w]s] + [v, t][t, w]su \in V.$$

Continuing in this manner, we arrive at $\bar{U}[v,t][t,w]S \subset V$ where $\bar{U}$ is subsemiring generated by U. Because $[V,U] \neq 0$, hence by Lemma 3 $U \not\subset Z(S)$ so $\bar{U} \not\subset Z(S)$. By Theorem 4, $\bar{U}$ contains a non-zero ideal say, I of S. Then $I[v,t][t,w]S$ is an ideal of S in V. Our theorem is proved if we show $I[v,t][t,w] \neq 0$. Suppose $I[v,t][t,w] = 0$ then primeness of S implies that

$$[v,t][t,w] = 0, \forall v, w \in V, t \in [V,U]. \tag{1.6}$$

Replacing v by vv_1 in (6) we have

$$[v,t]v_1[t,w] = 0, \forall w, v, v_1 \in V, t \in [V,U]. \tag{1.7}$$

Let $s \in S, a \in V$ and $t_1 \in [V,U]$ then $u = [[a,t_1],s] \in U$ which implies that $[u,[a,t_1]] \in [V,U] \subset V$. But $[u,[a,t_1]] = 2[a,t_1]s[a,t_1]$. Thus $2[a,t_1]s[a,t_1] \in V$. From this and (7) we get

$$[v,t][a,t_1]S[a,t_1][t,w] = 0. \tag{1.8}$$

Primeness of S implies that either $[v,t][a,t_1] = 0$ or $[a,t_1][t,w] = 0$. Suppose for all $v, a \in V$ and $t, t_1 \in [V,U]$

$$[v,t][a,t_1] = 0. \tag{1.9}$$

Put $V_1 = [V,[V,U]]$ then V_1 is a Lie ideal of U i.e; $[V_1,U] \subset V_1$ and from (9), we have

$$[v,t]V_1 = 0. \tag{1.10}$$

Thus $[v,t][v_1,u] = 0, u \in U, v_1 \in V_1$. From this, we have $[v,t]UV_1 = 0$. In particular, $[v,t]U[v,t] = 0$. By Lemma 12, either $[v,t] = 0$ or $U \subset Z(S)$. But $[V,U] \neq 0$ so $[v,t] = 0, v \in V, t \in [V,U]$. Hence $V_1 = [V,[V,U]] = 0$. Put $I = [V,U]$ then $[I,U] \subset I$ and $[I,I] \subset [V,[V,U]] = 0$. It follows from Lemma 14 that $[I,S] = 0$. By Corollary 10, we get $[V,S] = 0$. Thus assuming V does not contains non-zero ideal, we obtain the desired result. Similarly, the case with $[a,t_1][t,w] = 0$. This completes the proof.

Corollary 17. *Let U be a Lie ideal of S and V a subsemiring of S such that $[V,U] \subset V$. Then either $U \subset Z(S)$, $V \subset Z(S)$ or V contains a non-zero ideal of S.*

Theorem 18. *Let U be a Lie ideal and V an additive subsemigroup of S such that $[V,U] \subset V$. Then either $V \subset Z(S), U \subset Z(S)$ or $0 \neq [M,S] \subset V$ for an ideal $0 \neq M$ of S.*

Proof. Put $I = [V,U]$ and $T = \{x \in S : [x,S] \subset I\}$. It is easy to see that T is subsemiring of S, since for $t_1, t_2 \in T$ and $x \in S$ we have, $[t_1t_2, x] = t_1t_2x + t_1(t_2 + t_2')x + x't_1t_2 = t_1t_2x + (t_2 + t_2')xt_1 + x't_1t_2 = [t_1, t_2x] + [t_2, xt_1] \in I$.

Moreover, by jacobian identity we have

$$[[I,I],S] \subset [I,[I,S]] \subset [I,[U,S]] \subset [V,U] = I.$$

which gives that $[I,I] \subset T$. Consider a subsemiring T_0 of T which is generated by $[I,I]$. As $[[I,I],U] \subset [I,[I,U]] \subset [I,I]$ therefore, $[T_0,U] \subset T_0$. By Theorem 16, either $[T_0,S] = 0, [U,S] = 0$ or T_0 contains a non zero ideal say M of S. If $[T_0,S] = 0$ then $[[I,I],S] = 0$ and hence $[I,U] = 0$, by Theorem 15. Set $T_1 = \{x \in S; [x,U] = 0\}$. Then T_1 is Lie ideal of S, because $I \subset T_1$. So by Lemma 9 and Lemma 11 we have either $[U,S] = 0$ or $[V,S] = 0$. If $M \subset T_0 \subset T$ then $[M,S] \subset I \subset V$. Hence proved the theorem.

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