Edge maximal W_7-free graphs

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Abstract. Extremal graph theory is one of the most important subjects in graph theory. In this paper we give an upper bound to the number of edges of graphs which are W_7-free. In fact we prove that if G is a graph on n-vertices which is W_7-free, then

$$|E(G)| \leq \lfloor\frac{n^2}{4}\rfloor + \lfloor\frac{n+1}{2}\rfloor + 2.$$

Keywords: extremal graphs, wheels.

1. Introduction

In this paper we only consider finite, undirected simple graphs. The vertex set of a graph G is denoted by $V(G)$ and the edge set is denoted by $E(G)$. A cycle on n-vertices is denoted by C_n. The degree of a vertex v in $V(G)$ is denoted by $d(v)$. Moreover, we denote the minimum degree of vertices of G by $\delta(G)$ and the maximum degree by $\triangle(G)$. Also, we let $\varepsilon(G)$ and $\upsilon(G)$ denote $|E(G)|$ and $|V(G)|$, resp. The graph formed by taking a cycle C_{n-1} on $n-1$ vertices, and a vertex $u \notin V(C_{n-1})$ by joining u to each vertex of C_{n-1} is called a wheel and is denoted by W_n. We say that a graph G is H-free if G does not contain a subgraph isomorphic to the graph H as a subgraph. If $u \in V(G)$, then we denote by $N_G(u)$ or simply by $N(u)$ the set of vertices of G adjacent to u and we denote by $N[u]$ the set $N(u) \bigcup u$. Two nonadjacent vertices u, v in $V(G)$ are called twins if $N(u) = N(v)$. If H_1 and H_2 are vertex disjoint subgraphs of a graph G, then we let $E(H_1 \cup H_2) = \{xy \in E(G) : x \in V(H_1), y \in V(H_2)\}$ and $\varepsilon(H_1, H_2) = |E(H_1 \cup H_2)|$. Let $\mathcal{F}$ be a set of graphs and n be a positive integer and let $\mathcal{G}(n, \mathcal{F})$ denote the class of non-bipartite $\mathcal{F}$-free graphs on n-vertices, and

$$f(n, \mathcal{F}) = \max\{\varepsilon(G) : G \in \mathcal{G}(n, \mathcal{F})\}.$$

The branch of graph theory which aims at finding the values of the function $f(n, \mathcal{F})$ is extremal graph theory which was initiated by Turan's [16] in 1941. Many researchers have studied these problems; for example, see [1]–[5],[9],[11], [13]–[17]. Next, we state some of the results found.

Theorem 1.1 (Erdös-stone-simonovite). *Let $\mathcal{F}$ be any finite set of graphs and let r be the minimum number of chromatic number of F in $\mathcal{F}$, then*

$$f(n,\mathcal{F}) = (1 - \frac{1}{r-1})\binom{n}{2} + o(n^2).$$

Theorem 1.2 ([15]). *Let G be a graph on n-vertices such that F is wheel free, then $\varepsilon(G) \leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{4} \rfloor$.*

Al-Rhayel et al. in [1] proved that $f(n, W_5) \leq \lfloor \frac{n-2}{4} \rfloor + \lfloor \frac{s}{4} \rfloor$ for $n \geq 3$ where $s = n$ if $n \neq 4k+2$ and $s = n-1$ if $n = 4k+2$ and

$$f(n, W_6) \leq \lfloor \frac{n^2}{3} \rfloor$$

for $n \geq 6$.

Theorem 1.3 ([7]). *Let G be a graph on n-vertices with $\varepsilon(G) > \lfloor \frac{n^2}{4} \rfloor$, then G contains a cycle of every length L with $3 \leq L \leq \lfloor \frac{n+3}{2} \rfloor$.*

Haggkvist et al. [12] proved that: $f(n, C_r) \leq \lfloor \frac{(n-2)^2)}{4} \rfloor + 1$, for all r. Jia [14] proved that $f(n, C_5) \leq \lfloor \frac{(n-2)^2}{4} \rfloor + 3$ for $n \geq 9$. Bataineh et al. [5] proved that

$$f(n, \theta_7) \leq \lfloor \frac{(n-1)^2}{4} \rfloor + 3.$$

Jaradat et al. [13] proved that $f(n, \theta_{2k+1}) \leq \lfloor \frac{(n-2)^2}{4} \rfloor + 3$.

Finally, we state Turan's theorem [16].

Theorem 1.4. *Let G be a graph on n-vertices without a k-clique, then*

$$|E(G)| \leq \frac{(k-2)n^2}{2(k-1)}.$$

2. Edge maximal W_7-free graphs on $n(\leq 10)$-vertices

Throughout this section, let G be a graph on n-vertices ($n \leq 10$); clearly, if $n \leq 6$, then G has no W_7 as a subgraph, so we will only deal with the cases $n = 7, 8, 9$, and 10. Most of this section's results can be found in [2].

Theorem 2.1. *Let G be a graph on 7-vertices if G is W_7-free then $|E(G)| \leq 17$.*

Proof. Let $\{x_1, \cdots, x_7\}$ be the vertices of G: if G contains K_6 as a subgraph, let $\{x_1, \cdots, x_6\}$ be the vertices of K_6; hence, x_7 is the seventh vertex of G, then clearly x_7 can be adjacent to at most two vertices of K_6, and hence $|E(G)| \leq 15 + 2 = 17$. Figure 1 below shows such a graph: notice that the addition of any new edge to this graph produces W_7 as a subgraph of G.

If G does not contain K_6 as a subgraph, but contains K_5: let $\{x_1, \cdots, x_5\}$ be the vertices of K_5. Now $x_6 \in V(G)$, if $|N_{K_5}(x_6)| = 5$, then G contains K_6

which is a contradiction, hence $|N_{K_5}(x_6)| \leq 4$. In fact, it is easy to check that we may assume $N_{K_5}(x_6) = \{x_1, \cdots, x_4\}$, $x_7 \in V(G)$. Figure 2 shows such a graph G, in which the addition of any new edge produces K_6 or W_7 as a subgraph of G, and clearly $|E(G)| = 16 \leq 17$. If G does not contain K_5 but contains K_4 as a subgraph of G, let $\{x_1, \cdots, x_4\}$ be the vertices of K_4, $x_5 \in V(G)$, if $N_{K_4}(x_5) = \{x_1, \cdots, x_4\}$, then G contains K_5, which is a contradiction. Hence, we may assume that $N_{K_4}(x_5) = \{x_1, x_2, x_3\}$ and let G_1 be the resulting graph on $\{x_1, \cdots, x_5\}$. If $|N_{G_1}(x_6)| = 5$, then $G[x_1, \cdots, x_6]$ contains K_5 as a subgraph, also a contradiction. Hence $|N_{G_1}(x_6)| \leq 4$, and clearly, we may assume $|N_{G_1}(x_6)| = \{x_1, x_2, x_4, x_5\}$. Let G_2 be the resulting graph on $\{x_1, \cdots, x_6\}$. Now $x_7 \in V(G)$, and we may assume $N_{G_2}(x_7) = \{x_3, x_4, x_5, x_6\}$. Figure 3 displays such a graph G, where the addition of any new edge to G produces a K_5 or a W_7 as subgraph of G, hence $|E(G)| = 6 + 3 + 8 = 17$. □

Theorem 2.2. *Let G be a graph on 8 vertices, if G is W_7–free, then $|E(G)| \leq 21$.*

Proof. Let $\{x_1, \cdots, x_8\}$ be the vertices of G. If G contains K_6 as a subgraph of G, let $\{x_1, \cdots, x_6\}$ be the vertices of K_6. $x_7, x_8 \in V(G)$ and, as above (case $n = 7$), $|N_{K_6}(x_i)| \leq 2$, $i \in \{7, 8\}$. Clearly, we may assume $N_{K_6}(x_7) = \{x_1, x_2\}$, $N_{K_6}(x_8) = \{x_3, x_4\}$, and $x_7x_8 \in E(G)$. Figure 4 shows such a graph G, where the addition of any new edge to G produces W_7 as a subgraph of G, hence $|E(G)| = 20$. If G does not contain K_6 but contains K_5 as a subgraph, let $\{x_1, \cdots, x_5\}$ be the vertices K_5. $x_6 \in V(G)$, if $|N_{k_5}(x_6)| = 5$, then we get K_6 as a subgraph of G, hence $|N_{K_5}(x_6)| \leq 4$, and as in case $n = 7$, we may assume $N_{K_5}(x_6) = \{x_1, x_2, x_3, x_4\}$ and $N_{G_1}(x_7) = \{x_1, x_6\}$, where G_1 is the resulting graph on $\{x_1, \cdots, x_6\}$. Let H be the resulting graph on $x_1, \cdots, x_7$. Now $x_8 \in V(G)$ and clearly we may assume $N_H(x_8) = \{x_1, x_6\}$, and $x_7x_8 \in E(G)$. Notice that Figure 5 displays such a graph G, where the addition of any new edge to this graph produces K_6 or W_7 as a subgraph of G; hence $|E(G)| = 19$. If G does not contain K_5 but contains K_4 as a subgraph, let $\{x_1, \cdots, x_4\}$ be the vertices of K_4. Now $x_5, x_6, x_7 \in V(G)$ and by analysis similar to that used for case $n = 7$, we may assume $N_{K_4}(x_5) = \{x_1, x_2, x_3\}$, $N_{G_1}(x_6) = \{x_1, x_2, x_4, x_5\}$ and $N_{G_2}(x_7) = \{x_3, x_4, x_5, x_6\}$, where G_1 is the resulting graph on $\{x_1, \cdots, x_5\}$, and G_2 is the resulting graph on $\{x_1, \cdots x_6\}$. Let G_3 be the resulting graph on $\{x_1, \cdots, x_7\}$ (see figure 6 below). $x_8 \in V(G)$, and one can check that we may assume $N_{G_3}(x_8) = \{x_3, x_4, x_5, x_6\}$. Figure 7 shows such a graph G, where the addition of any new edge to this graph produces K_5 or W_7 as a subgraph of G, hence $|E(G)| = 21$. □

Theorem 2.3. *Let G be a graph on 9-vertices if G is W_7-free then*

$$|E(G)| \leq 25.$$

Proof. Let $\{x_1, \cdots, x_9\}$ be the vertices of G. If G contains K_6 as a subgraph, let $\{x_1, \cdots, x_6\}$ be the vertices of G. Notice that if $v \in \{x_7, x_8, x_9\}$, then

$|E(v, K_6)| \leq 2$, and if $B = G[x_7, x_8, x_9]$, then $|E(B)| \leq 3$, hence $|E(G)| \leq 15 + 6 + 3 = 24$. If G does not contain K_6 but contains K_5 as a subgraph, let $\{x_1, \cdots, x_5\}$ be the vertices of K_5. $x_6 \in V(G)$. Clearly, we may assume $N_{K_5}(x_6) = \{x_1, x_2, x_3, x_4\}$. Let G_1 be the resulting graph on $\{x_1, \cdots, x_6\}$. Now $\{x_7, x_8, x_9\} \subseteq V(G)$; clearly, $|N_{G_1}(x_i)| \leq 2$, for $i = 7, 8, 9$ and we may assume x_7x_8, x_7x_9 and x_8x_9 are edges in G; hence, $|E(G)| \leq 10+4+6+3 = 23$. Figure 8 below shows such a graph G, where the addition of any new edge produces K_6 or W_7 as a subgraph of G.

If G does not contain k_5 but contains K_4 as a subgraph, then by analysis as in previous cases, one can easily check that the graph G given in Figure 9 ensures that G contains K_4 and the addition of any new edge to G produces K_5 or W_7 as a subgraph of G, and $|E(G)| \leq 6 + 3 + 16 = 25$. □

Theorem 2.4. *Let G be a graph on 10-vertices, if G is W_7–free then*

$$|E(G)| \leq 30.$$

Proof. Let $\{x_1, \cdots, x_{10}\}$ be vertices of G. If G contains K_6 as a subgraph, then let $\{x_1, \cdots, x_6\}$ be the vertices of K_6. Notice that if $v \in \{x_7, x_8, x_9, x_{10}\}$, then $|E(v, K_6)| \leq 2$ and if $B = G[x_7, x_8, x_9, x_{10}]$, then $|E(B)| \leq 6$, hence $|E(G)| \leq 15 + 8 + 6 = 29$. If G does not contain K_6 but contains K_5 as a subgraph, let $\{x_1, \cdots, x_5\}$ be the vertices of K_5. $x_6 \in V(G)$, and clearly we may assume $N_{K_5}(x_6) = \{x_1, x_2, x_3, x_4\}$. Let G_1 be the resulting graph on $\{x_1, \cdots, x_6\}$, and following an analysis like for $n = 9$, we notice that $\{x_7, x_8, x_9, x_{10}\} \subseteq V(G)$, and $G[x_7, x_8, x_9, x_{10}] = K_4$ and $|N_{G_1}(x_i)| \leq 2$, $i = 7, 8, 9, 10$, hence $|E(G)| \leq 10 + 4 + 8 + 6 = 28$. If G does not contain K_5 but contains K_4 as a subgraph, let $\{x_1, \cdots, x_4\}$ be the vertices of K_4, then following an analysis as in previous cases (for more details consult [2]), we get $|E(G)| \leq 30$. □

Remark 2.5. The bound in Theorem 2.4 is sharp as illustrated in example 2.1 below.

Example 2.1. Let G_1 be the graph shown in Figure 10, where each vertex in H_1 is joined to each vertex of H_2, then clearly G_1 has exactly 30 edges and is W_7–free. and the addition of any new edge to G_1 produces W_7 as a subgraph of G_1.

3. Main results

Let G be a graph of order $n \geq 11$, following Moon [15]. Let H_n be the class of graphs obtained by partitioning the vertices of G into two sets, P and Q, such that P has $\lfloor \frac{n+1}{2} \rfloor$ vertices and Q has $\lfloor \frac{n}{2} \rfloor$ vertices, and there are as many edges joining pairs of vertices in P and Q as consistent with the requirement that no two of these edges have a vertex in common, and each vertex in P is joined to each vertex in Q.

Remark 3.1. 1. If two edges, say in P, have a common vertex, say x_1x_2, x_2x_3, then let y_1y_2, y_3y_4 be two edges in Q, and notice that x_2 is adjacent to every vertex of the 6–cycle $(x_1y_1y_2x_3y_4y_3x_1)$. This produces W_7 as subgraph of G.

2. Figure 11 below shows a graph G in H_n: clearly G is W_7–free and $\varepsilon(G) \geq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n}{2} \rfloor$, for all $n \geq 7$. So we conclude that:

$$f(n, W_7) \geq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n}{2} \rfloor.$$

Lemma 3.2. *Let G be a W_7–free graph on n–vertices $(n \geq 7)$ such that K_6 is a subgraph of G, then $|E(v, K_6)| \leq 2$ for all $v \in V(G - K_6)$.*

Proof. Let $\{x_1, x_2, ...x_6\}$ be the vertices of K_6, if $|N_{K_6}(v)| = 3$. Then we may assume $N_{K_6}(v) = \{x_1, x_2, x_3\}$; clearly x_2 is adjacent to every vertex of the 6–cycle $(x_1vx_3x_4x_5x_6)$, and hence W_7 is produced as a subgraph of G, which is a contradiction. □

In Remarks 3.3 and 3.4 below we construct a class of graphs F_n such that if G belongs to F_n, then G is W_7-free and $|E(G)| \geq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n}{2} \rfloor + 1$.

Remark 3.3. Let G be a graph of order $n \geq 11$. Let F_n be the class of graphs obtained by partitioning the vertices of G into two sets P, Q such that P has $\lfloor \frac{n+1}{2} \rfloor$ vertices and Q has $\lfloor \frac{n}{2} \rfloor$ vertices, and there are many edges joining vertices in P and Q, as consistent with the following requirements.

1. P has no vertices of degree 3.
2. If H has component of P, then H is one of the following: a single vertex, a P_2, a C_3 or a C_4.
3. Q has exactly one edge joining any two vertices of Q and the rest of the vertices in Q are vertices of degree 0.
4. Each vertex in P is joined to each vertex in Q; figure 12 below shows a graph G in the class F_n.

Remark 3.4. 1. Remark 3.3 above shows that a graph G in F_n is W_7–free.

2. If C_5 is a component in P, $\{x_1, x_2, x_3, x_4, x_5\}$ be this cycle, and let y_1, y_2 be the edge in G, then y_1 is adjacent to each vertex of the cycle $(y_2x_1x_2x_3x_4x_5)$; hence, W_7 is produced as a subgraph of G.

3. If P has a vertex x_1 of degree 3, then let $N(x_1) = \{x_2, x_3, x_4\}$, let y_1y_2 be the edge in G, let y_3 be a vertex of degree 0 in G and notice that under the above construction, x_1 is adjacent to every vertex of the cycle $(x_2y_3x_4y_2x_3y_1)$, and this produces W_7 as a subgraph of G.

4. The maximum number of edges in P is obtained if all components of P are either C_3's, C_4's or a combination of both, and hence this implies that each vertex of P has degree 2, thus

$$|E(G)| \geq \lfloor \frac{n+2}{2} \rfloor \lfloor \frac{n}{2} \rfloor + [\frac{n+1}{4}] + 1 \geq \lfloor \frac{n^2+2n+1}{4} \rfloor + 1 \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n}{2} \rfloor + 1.$$

Theorem 3.5. *Let G be a W_7-free graph on $n-$ vertices ($n \geq 10$). If G contains exactly one copy of K_6 as a subgraph, then $|E(G)| \leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{2} \rfloor + 2$.*

Proof. We use induction on n, for $n = 7$ see Theorem 2.1. So assume the result is true for all graphs having less than $n-$vertices, and assume G is a graph on $n-$vertices. Let $B = G - K_6$, then B is a graph with $|V(B)| < n$, hence

$$|E(B)| \leq \lfloor \frac{(n-6)^2}{4} \rfloor + \lfloor \frac{n-5}{2} \rfloor + 2,$$

notice that if $x \in V(B)$, than $E(x, K_6) \leq 2$, since otherwise W_7 is produced as a subgraph of G; Thus $|E(B, K_6) \leq 2(n-6)$, and

$$\begin{aligned} |E(G)| &\leq |E(K_6)| + |E(B, K_6)| + |E(B)|, \\ &\leq 15 + 2(n-6) + \lfloor \frac{(n-6)^2}{4} \rfloor + \lfloor \frac{n-5}{4} \rfloor + 2, \\ &\leq \frac{12 + 8n + n^2 - 12n + 36 + n - 5 + 8}{4}, \\ &= \frac{n^2 + 2n + 2 - 5n + 49}{4}, \\ &\leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{2} \rfloor + 2, \quad \text{if } n \geq 10. \end{aligned}$$

□

Theorem 3.6. *Let G be a W_7-free graph on $n-$ vertices ($n \geq 8$). If G contains exactly one copy of K_5 as a subgraph, but it does not contain K_6, then $|E(G)| \leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{2} \rfloor + 2$.*

Proof. we use induction on n. The result being proved for $n = 7, 8, 9, 10$ (see Theorems 2.1, 2.2, 2.3 and 2.4). Assume the result holds for all graphs having less than $n-$vertices, and let G is a graph on $n-$vertices. Let G_1 be the graph on $6-$vertices shown in Figure 13. Let $B_1 = G - G_1$, then $|V(B_1)| < n$, and hence $|E(B_1)| \leq \lfloor \frac{(n-6)^2}{4} \rfloor + \lfloor \frac{n-5}{2} \rfloor + 2$.

If $x \in V(B_1)$, then clearly $E(x, K_6)| \leq 2$, as otherwise either K_6 or W_7 is produced as a subgraph of G, hence $|E(B_1, K_6)| \leq 2(n-6)$; therefore

$$\begin{aligned} |E(G)| &\leq |E(G_1)| + |E(B_1)| + |E(B_1, K_6)|, \\ &\leq 14 + \lfloor \frac{(n-6)^2}{4} \rfloor + \lfloor \frac{n-5}{2} \rfloor + 2 + 2(n-6), \\ &\leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{2} \rfloor + 2, \quad \text{if } n \geq 8. \end{aligned}$$

□

Theorem 3.7. *Let G be a W_7-free graph on $n-$ vertices ($n \geq 10$). If G contains K_4 but it does not contain K_5 as a subgraph, then $|E(G)| \leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{2} \rfloor + 2$.*

Proof. we use induction on $n\,(n \geq 10)$. Let $\{x_1, \ldots, x_n\}$ be the vertices of G, and $\{x_1, \ldots, x_4\}$ be the vertices of K_4. $x_5 \in V(G)$, hence $|N_{K_4}(x_5)| \leq 3$, otherwise we get K_5 as a subgraph of G. Clearly we may assume that $N_{K_4}(x_5) = \{x_1, x_2, x_3\}$. Let G_1 be resulting graph on $\{x_1, x_2, x_3, x_4, x_5\}$, note that $x_6 \in V(G)$. If $|N_{G_1}(x_6)| = 5$, then K_5 is a produced as a subgraph of G, hence $|N_{G_1}(x_6)| \leq 4$, and clearly we may assume $N_{G_1}(x_6) = \{x_1, x_2, x_4, x_5\}$. Let G_2 be a resulting graph on $\{x_1, \ldots, x_6\}$, now $x_7 \in V(G)$, if $|N_{G_2}(x_7)| = 5$, then K_5 is produced as a subgraph of G, hence $|N_{G_2}(x_7)| \leq 4$, and we may assume that $N_{G_2}(x_7) = \{x_3, x_4, x_5, x_6\}$. Let G_3 be the resulting graph on $\{x_1 \ldots, x_7\}$. Notice that the addition of any new edge to G_3 either produces K_5 or W_7 as a subgraph of G. Now $x_8 \in V(G)$, and we may assume that $N_{G_3}(x_8) = \{x_3, x_4, x_5, x_6\}$. Let G_4 be the resulting graph on $\{x_1, \ldots, x_8\}$, and notice that the addition of any new edge to G_4 either produces K_5 or W_7 as a subgraph of G. Also $x_9 \in V(G)$, and by analysis as above we conclude that $|N_{G_4}(x_9)| \leq 4$, and may assume that $N_{G_4}(x_9) = \{x_1, x_2, x_7, x_8\}$. Let G_5 be the resulting graph on $\{x_1, \ldots, x_9\}$, and notice that the addition of any new edge to G_5 (see Figure 14) produces either K_5 or W_7 as a subgraph of G, hence $|E(G_5)| \leq 25$. Let $B = G - G_5$, hence $|V(B)| = n - 9 < n$, and by induction hypothesis we have:

$$|E(B)| \leq \lfloor \frac{(n-9)^2}{4} \rfloor + \lfloor \frac{n-8}{2} \rfloor + 2.$$

Note that if $x \in V(B)$, then $|E(x, G_5)| \leq 4$, hence $|E(B, K_5)| \leq 4(n-9)$, and we conclude that:

$$\begin{aligned}
|E(G)| &\leq |E(G_5)| + |E(B)| + |E(B, G_5)|, \\
&\leq 25 + \lfloor \frac{(n-9)^2}{4} \rfloor + \lfloor \frac{n-8}{2} \rfloor + 2 + 4(n-9), \\
&\leq 25 + \lfloor \frac{(n-9)^2}{4} \rfloor + \lfloor \frac{n-8}{2} \rfloor + 2 + 4n + 36, \\
&\leq \lfloor \frac{n^2 - 18n + 81}{4} \rfloor + \lfloor \frac{2n-16}{4} \rfloor + 4n - 9, \\
&\leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{-18n + 81 + 2n - 16 + 16n - 36}{4} \rfloor, \\
&\leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{2} \rfloor + \frac{27 - 2n}{4}, \\
&\leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{2} \rfloor + 2, \quad \text{if } n \geq 10.
\end{aligned}$$

□

Theorem 3.8. *Let G be a graph of order $n \geq 11$, if G is W_7–free, then*

$$|E(G)| \leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{2} \rfloor + 2.$$

Proof. To prove this theorem we use strong mathematical induction on m, where m is the number of vertex disjoint K_5 subgraphs contained in G. If $m = 0$, then done by Theorem 3.7, hence we assume that the result holds for all values less than m. We need to prove that the result holds for m, let G be a graph on n–vertices having m vertex disjoint K_5 subgraphs. Let K_5 be a subgraph of G, $\{x_1, x_2, x_3, x_4, x_5\}$ be the vertices of K_5, and $B = G - K_5$.

Case 1. There exist $x_6 \in V(B)$ such that x_6 is adjacent to all vertices of K_5, hence K_6 is produced as a subgraph of G. let $B_1 = G - K_6$; then $\varepsilon(B_1, K_6) \leq 2$, for all $x \in B_1$, hence $\varepsilon(B_1, K_6) \leq 2(n-6)$, and by induction hypothesis

$$\varepsilon(B_1) \leq \lfloor\frac{(n-6)^2}{4}\rfloor + \lfloor\frac{n-5}{2}\rfloor + 1,$$

hence

$$\begin{aligned}
\varepsilon(G) &\leq \varepsilon(B_1) + \varepsilon(K_6, B_1) + \varepsilon(K_6) \\
&\leq \lfloor\frac{(n-6)^2}{4}\rfloor + \lfloor\frac{n-5}{2}\rfloor + 1 + 2(n-6) + 15 \\
&\leq \lfloor\frac{n^2 - 12n + 36}{4}\rfloor + \lfloor\frac{2n-10}{4}\rfloor + 2n + 4 \\
&\leq \lfloor\frac{n^2}{4}\rfloor - 3n + 9 + \lfloor\frac{n+1}{2}\rfloor - 3 + 2n + 4 \\
&\leq \lfloor\frac{n^2}{4}\rfloor + \lfloor\frac{n+1}{2}\rfloor + 1 \qquad \text{if } -n + 10 \leq 1 \\
&\leq \lfloor\frac{n^2}{4}\rfloor + \lfloor\frac{n+1}{2}\rfloor + 1 \qquad \text{if } n \geq 9 \\
&\leq \frac{n^2}{4}\rfloor + \lfloor\frac{n+1}{2}\rfloor + 2 \qquad \text{if } n \geq 9.
\end{aligned}$$

Case 2. There exist $x_6 \in V(B)$ such that x_6 is adjacent to four vertices of K_5; clearly, we may assume $N_{K_5}(x_6) = \{x_1, x_2, x_3, x_4\}$. Let H be the resulting graph on $\{x_1, \cdots, x_6\}$, and let $B_2 = G - H$. We claim that $\varepsilon(y, H) \leq 2$ for all y in $V(B_2)$. To prove this claim, assume $|N_H(y)| = 3$, since $N_H(x_5) = N_H(x_6)$ and $N_H(x_1) = N_H(x_2) = N_H(x_3) = N_H(x_4)$; then $N_H(y)$ must be one of the following sets: $\{x_1, x_2, x_3\}$, $\{x_1, x_2, x_5\}$, $\{x_1, x_5, x_6\}$, if $N_H(y) = \{x_1, x_2, x_3\}$. Then, x_2 is adjacent to every vertex of the 6–cycle $(yx_3x_6x_4x_5x_1)$. If $N_H(y) = \{x_1, x_2, x_5\}$, then x_2 is adjacent to every vertex of the 6–cycle $(yx_1x_6x_3x_4x_5)$. If $N_H(y) = \{x_1, x_5, x_6\}$, then x_1 is adjacent to every vertex of the 6–cycle $(yx_6x_2x_3x_4x_5)$. In any case, W_7 is produced as a subgraph of G and this proves our claim. Now

$$\begin{aligned}
\varepsilon(G) &\leq \varepsilon(B_2) + \varepsilon(H, B_1) + \varepsilon(H) \\
&\leq \lfloor\frac{(n-6)^2}{4}\rfloor + \lfloor\frac{n-5}{2}\rfloor + 1 + 2(n-6) + 14 \\
&\leq \lfloor\frac{n^2}{4}\rfloor - 3n + 9 + \lfloor\frac{n+1}{2}\rfloor - 3 + 1 + 2n + 2
\end{aligned}$$

$$
\begin{aligned}
&\leq \lfloor\frac{n^2}{4}\rfloor + \lfloor\frac{n+1}{2}\rfloor + 1 \qquad \text{if } -n+9 \leq 1 \\
&\leq \lfloor\frac{n^2}{4}\rfloor + \lfloor\frac{n+1}{2}\rfloor + 1 \qquad \text{if}\, n \geq 8 \\
&\leq \lfloor\frac{n^2}{4}\rfloor + \lfloor\frac{n+1}{2}\rfloor + 2 \qquad \text{if}\, n \geq 8.
\end{aligned}
$$

Case 3**.** There exists a vertex $x_6 \in V(B)$ such that x_6 is adjacent to 3-vertices of K_5; clearly, we may assume these vertices to be $\{x_1, x_2, x_3\}$. Let H_1 be the resulting graph on $\{x_1, \cdots, x_6\}$, $B_3 = G - H_1$ and $y \in V(B_3)$. We claim that $\varepsilon(y, H_1) \leq 3$ for all $y \in V(B_3)$. To prove this claim, notice that if $|N_{H_1}(y)| = 4$, then $x_6 \in N_{H_1}(y)$, since otherwise y is adjacent to 4 vertices of K_5, and we go back to case 2 above and since $N_{H_1}(x_1) = N_{H_1}(x_2) = N_{H_1}(x_3)$ and$N_{H_1}(x_4) = N_{H_1}(x_5)$ then we may assume that $N_{H_1}(y)$ to be one of the following sets: $\{x_1, x_2, x_3, x_6\}$ or $\{x_1, x_2, x_5, x_6\}$. If $N_{H_1}(y) = \{x_1, x_2, x_5, x_6\}$, then x_2 is adjacent to every vertex of the 6–cycle $(yx_1x_5x_4x_3x_6)$. If $N_{H_1}(y) = \{x_1, x_2, x_3, x_6\}$, then again x_2 is adjacent to every vertex of 6–cycle $(yx_1x_5x_4x_3x_6)$. In any case, W_7 is produced as a subgraph of G, hence $|N_{H_1}(y)| \leq 3$. Clearly, if $N_{H_1}(y) = \{x_1, x_2, x_3\}$, then this bound is achieved, and we conclude that $\varepsilon(y, H_1) \leq 3$ for all $y \in V(B_3)$, and hence

$$
\begin{aligned}
\varepsilon(G) &\leq \varepsilon(H_1) + \varepsilon(B_3) + \varepsilon(B_3, H_1) \\
&\leq 13 + \lfloor\frac{(n-6)^2}{4}\rfloor + \lfloor\frac{n-5}{2}\rfloor + 1 + 3(n-6) \\
&\leq \lfloor\frac{n^2 - 12n + 36}{4}\rfloor + \lfloor\frac{n+1}{2}\rfloor - 3 + 1 + 3n - 18 \\
&\leq \lfloor\frac{n^2}{4}\rfloor - 3n + 9 + \lfloor\frac{n+1}{2}\rfloor - 3 + 14 + 3n - 18 \\
&\leq \lfloor\frac{n^2}{4}\rfloor + \lfloor\frac{n+1}{2}\rfloor + 2, \qquad \text{for all}\, n.
\end{aligned}
$$

□

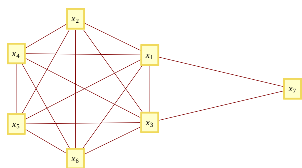
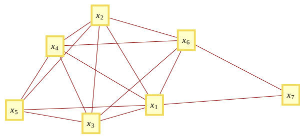


Figure 1:



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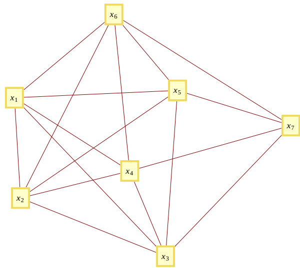

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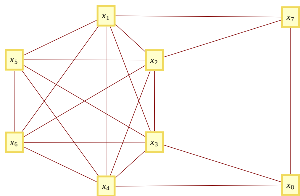

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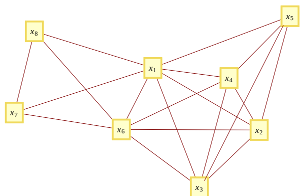

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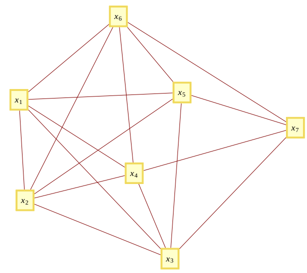

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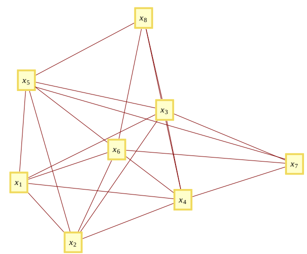

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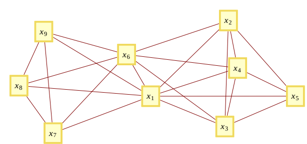

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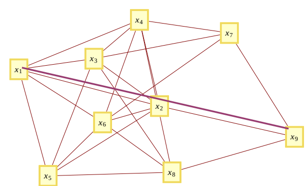

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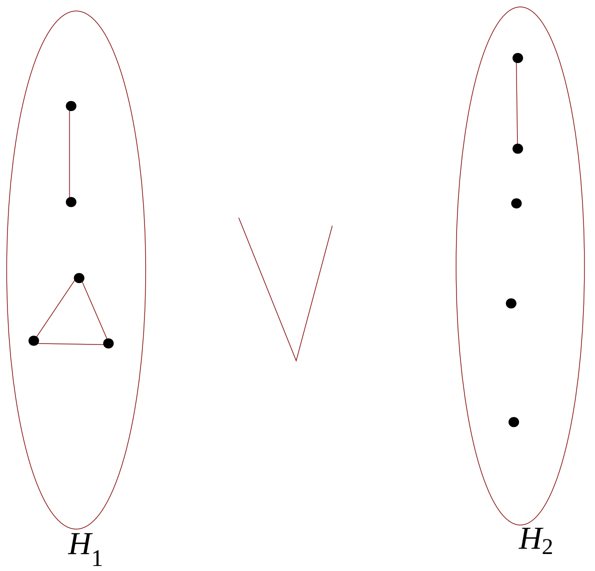

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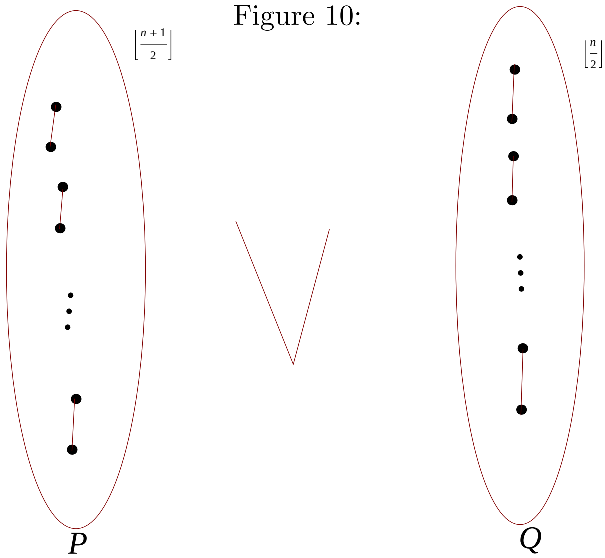

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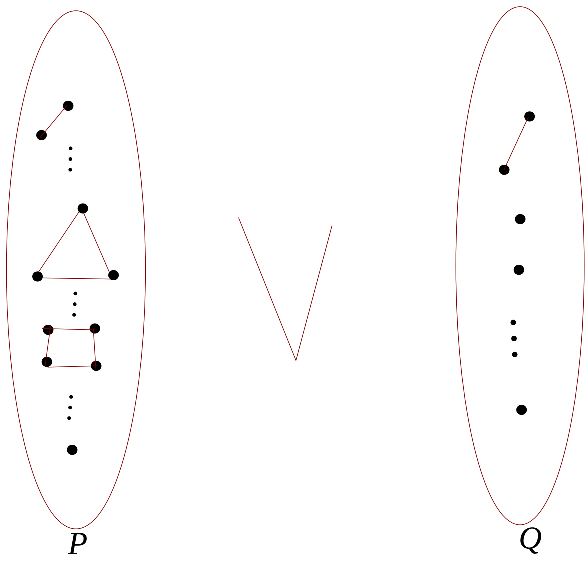

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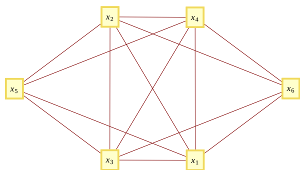

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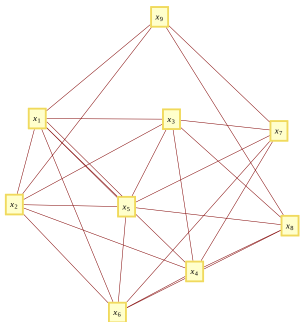

Figure 14:

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