The non-zero divisor graph of a ring

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Abstract. Let R be a ring, we associate a simple graph $\Phi(R)$ to R, with vertices $V(R) = R\backslash\{0, 1, -1\}$, where distinct vertices $x, y \in V(R)$ are adjacent if and only if either $xy \neq 0$ *or* $yx \neq 0$. In this paper, we prove that if $\Phi(R)$ is connected such that $R \ncong Z_2 \times Z_4$ then the diameter of $\Phi(R)$ is almost 2. Also, we will pay specific attention to investigate the connectivity of certain rings such that, the ring of integers modulo n, Z_n is connected, reduced ring and matrix ring.
Keywords: ring, zero-divisor, connected graph, diameter.

Introduction

It is believed that studying the action of a ring or group on a graph is one of the best comprehensible ways of analysing the structure of the rings or groups. There are many researches appointing a graph on group or ring to study the algebraic properties of that group or ring, for example, see [1, 2, 4, 5 and 10]. Suppose that R is a ring, the non-zero divisor graph, denoted by $\Phi(R)$ has a vertex set $V(R) = R\backslash\{0, 1, -1\}$, along with vertices $x, y \in V(R)$ being connected together on the condition of $x \neq y$ and either $xy \neq 0$ *or* $yx \neq 0$. Therefore, R is domain if and only if $\Phi(R)$ is complete graph. Thus if R is finite commutative ring with one such that $\Phi(R)$ is complete then R is a filed.

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This sort of graphs is inspired by the zero divisor graph which defined as the directed graph $\Gamma(R)$ such that its vertices are all non-zero zero-divisors of R in which any two distinct vertices x and y, $x \longrightarrow y$ is an edge if and only if $xy = 0$. Several fundamental results concerning with the zero divisor graph can be seen in [2].The main object of this paper is to study how the graph theoretical properties of $\Phi(R)$ effect on the ring theoretical properties of R. We assume that all graphs are simple graphs, which means they are undirected graphs with no multiple edges or loops. By abuse of notation, we denote by $V(R)$, the vertices of a graph $\Phi(R)$. We also should mentioned that a graph $\Phi(R)$ is connected if there is a path between any two distinct vertices in $V(R)$. For vertices x and y of $V(R)$, let $d(x, y)$ be the length of a shortest path from x to y. If no such path exist we my let $d(x, y) = \infty$, also $d(x, x) = 0$. Finally, the diameter of $\Phi(R)$ is define as $Diam(\Phi(R)) = sup\{d(x, y)|x \text{ and } y \text{ are vertices of } V(R)\}$ for more details see [5]. This paper is organized as follows: In section 1, we study the non-zero divisor graph $\Phi(R)$ such that R is ring of integers modulo n, Zn. In section 2, we prove that the diameter of the graph is almost 2. Moreover, we investigate the non-zero divisor graph of reduced ring and thus of Boolean ring. In section 3, we scrutinize the connectivity of the non-zero divisor graph for the matrix ring.

1. The non-zero divisor graph of the ring of integers modulo n

This section dedicated to investigate the non-zero divisor graphs for the ring of integers modulo n, Z_n.

Lemma 1.1. $V(R)$ has an invariable element a then $\Phi(R)$ is connected.

Proof. As a is invariable then there is $b \in G$ such that $ba = ab = 1$. Let x in $V(R)$ if $ax = 0$ or $xa = 0$, then we have $x = 0$, which is a contradiction. Thus $|Deg(a)| = |V(R)|$. □

Now, for $n \notin 1, 2, 3, 6$. Let $(Z_n)^\times$ be the multiplicative group of integers modulo n. Then its order is given by Euler's phi function [9]

$\phi(n) = \prod_{i=1}^{s}(p_i^{\delta_i} - p_i^{\delta_{i-1}})$ Such that n written uniquely as $\prod_{i=1}^{s} p_i^{\delta_i}$, $\delta_i \geqslant 1$ are integers and $pi < p_{i+1}$ are prime numbers. We should note that $\phi(n)$ represent the number of invertible element in the ring of integers modulo n.

Lemma 1.2. For $n > 6$, we have $\phi(n) > 2$.

Proof. We may write n as above, $n = \prod_{i=1}^{s} p_i^{\delta_i}$. Thus $\phi(n) = \prod_{i=1}^{s}(p_i^{\delta_i} - p_i^{\delta_{i-1}})$. And one can see immediately that $\phi(n) = 2$ if and only if there is j such that $p_j^{\delta_j} - p_j^{\delta_{j-1}} = 2$ and $\prod_{i=1_{i\neq j}}^{s}(p_i^{\delta_i} - p_i^{\delta_{i-1}})$=1. Which is impossible as $n > 6$. □

Theorem 1.3. *The ring of integers modulo* n, Z_n *is connected if and only if* $n \notin \{1, 3, 2, 6\}$

Proof. By using Lemma 1.2 we get $\phi(n) > 2$. Therefore, there is $x \in (Z_n)^{\times} \cap V(R)$. And our result follow immediately by Lemma 1.1. □

Now, by using MATLAB program [3] we create an algorithm aim to draw $\Phi(Zn)$ for all $n > 3$, and calculate the diameter of the graph. This algorithm summarized as follows :

Algorithm 1

Let zz: seat of the system
Let n: the order of the system
Let $k = 0$
Let no, Initial Conditions
For $i \longleftarrow n-3$ *to* n_f **Do**
For $j \longleftarrow n-3$ *to* n_f **Do**
Read all Seats Value $(n_1, n_2, \dot{n}_f)$
Set $ik \longleftarrow \sum_{i \longleftarrow 1}^{n} ik + 1$
Set $K(ik) \longleftarrow zz(i) * zz(j)$
If $mod(K(ik), n) \longleftarrow \quad = 0$
Set $w \longleftarrow [zz(i)zz(j)]$;
End if
End for
End for

Magma [7] is a computational algebra system support most research deal with algebra. The following algorithm aim to calculate the diameter of the non-zero divisor graph for a residue class ring $Z/nZ \cong Z_n$, $n > 3$ and the Cartesian product of $Z_n \times Z_m$. Implantations of the procedures associated with this algorithm will be via magma packages. This algorithm employed for the Cartesian product $Z_n \times Z_m$, and for Z_n, $n > 3$, we may take $Zn \cong Z_n \times Z_1$. Moreover, in [8] one can see that $Z_{nm} \cong Z_n \times Z_m$ if and only if $g.c.d(m, n) = 1$. The algorithm is as follows:

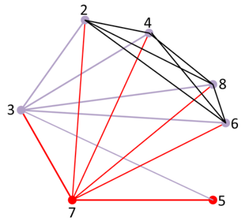
Algorithm 2

Let $Z_n \longleftarrow ResidueClassRing(n)$
Let $Z_m \longleftarrow ResidueClassRing(m)$
Let $R \longleftarrow CartesianProduct(Z_n \times Z_m)$
Let $V(R) \longleftarrow Set(R)\ diff\ \{\langle 0,0\rangle, \langle 1,1\rangle, \langle n-1, m-1\rangle\}$
Let $Gr \longleftarrow Graph\langle V(R)|\{\}\rangle$
for x, y in S **Do**
if $x \neq y$ **Then**
if $x * y \neq$ **or** $y * x \neq 0$ **Then**
Let $Gr \longleftarrow Gr + \{\{Vertices(Gr)!x, Vertices(Gr)!y\}\}$
End if
End if
End for

.

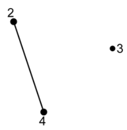
To be familiar with non-zero divisor graph, we utilize the above algorithms in the following examples:

Example 1.4. The non zero divisor graph of Z_{10} is describe as follows:

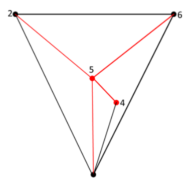


from above graph one can see that $Diam(\Phi(Z_{10}) = 2$.

Example 1.5. $\Phi(Z_6)$ is disconnect:



Example 1.6. Another example of connected non-zero divisor graph is $\Phi(Z_8)$

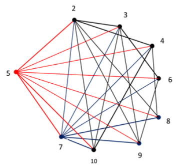


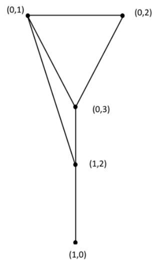
Also, we should note that $Diam(\Phi(Z_8) = 2$.

Example 1.7. The non-zero divisor of $\Phi(Z_{12})$ is describe as follows: Furthermore, $Diam(\Phi(Z_{12}) = 2$.

On the other hand, easy calculating by using Algorithm 2 one can obtain the next example which shows that $Diam(\Phi(Z_2 \times Z_4)) = 3$. Indeed. in the coming section we going to prove that the only connected non-zero divisor graph of diameter 3 is $\Phi(Z_2 \times Z_4)$.

Example 1.8. The non-zero divisor graph of $Z_2 \times Z_4$. This graph with diameter 3 as we see below:





2. Connectivity of the non-zero divisor graph

The results of this section related to connectivity of the graph. The first result shows that diameter of the connected non-zero divisor graph is almost 2.

Theorem 2.1. *Let R be a ring. If $\Phi(R)$ is a connected non-zero divisor graph and $R \ncong Z_2 \times Z_4$. Then $Diam(\Phi(R)) \leqslant 2$.*

Proof. Let $x, y \in V(R)$, such that $x \neq y$. If $xy \neq 0$, then we done. So we may assume that $xy = 0 = yx$. As $\Phi(R)$ is connected then there is $a, b \in V(R)$, such that $d(x, a) = d(y, b) = 1$. Which means either ($ax \neq 0$ or $xa \neq$) and ($by \neq 0$ or $yb \neq 0$). Thus we have the following subcases:

1. If $d(y, a) = 1$ or $d(x, b) = 1$, so in both cases we have $Diam(\Phi(R)) \leqslant 2$.

2. If $bx = xb = ay = ya = 0$, let $w = a \mp b$. Then it's obvious that $d(x, w) = 1 = d(y, w)$. Thus we may assume that both $a - b$ and $a + b$ not in $V(R)$. Hence we need to consider the following subcases:

I.If $a - b = 0$ or $a + b = 0$. Then $ax = bx = 0$. Which is a contradiction.

II. If $a + b = 1$ and $a - b = 1$. Note that with this conditions we have $ax = x$, $2b = 0$, $2a = 2$ and $by = y$. Also we consider the following:

i. If $x^2 = 0$. Then we consider the following cases:

- if $y^2 = 0$, then we take $q = y + 1$, so that $d(q, x) = d(q, y) = 1$. Furthermore, this is not true if $y = \mp 2$ multiply by a we get $0 = ay = 2a = 2$, but this means $y = 0$ which is a contradiction.

- if $y^2 \neq 0$, then $x + 1$, $x - 1$, $y + a$ and $y - a$ connect x with y and this is not true if $R \ncong Z_2 \times Z_4$. Indeed, in this we have $Diam(\Phi(R)) = 3$ as we showed in Example 1.8. Else, we may take $z \in V(R)$ such that $z \notin \{x, y, a, -a, b\}$. if $d(z, x) = d(y, z) = 1$ we done, if $d(x, z) = 1, d(y, z) = 0$ take $z + b$, and take $z+a$ if $d(x,z)=0, d(y,z)=1$. Finally take $z+1$ to connect x with y in other cases.

ii. If $x^2 \neq 0$. Then we consider the following cases:

- if $y^2=0$, then $d(y+1,x)=d(y+1,y)=1$, also this not true if $y=\mp 2$, multiply by a we get $0=ay=2a=2$, but this mean $y=0$ which is a contradiction.
- if $y^2 \neq 0$, then $x+y \in V(R)$ connect x with y.

III. If $a+b=-1$ and $a-b=-1$, note that with this conditions we get $ax=-x$, $2b=0$, $2a=-2$, $by=-y=y$ and $2y=0$. Then we may apply similar argument as in case II to show that $Diam(\Phi(R))2$.

IV. If $a+b=-1$ and $a-b=1$, with these conditions we obtain $ax=x$, $2b=b$, $2a=0$, $by=-y$ and $2x=0$. Also we consider the following cases:

i. If $x^2=0$. Then we consider the following cases:

- if $y^2=0$, then we take $w=x+1$ clearly that $d(w,x)=d(w,y)=1$, also this not true if $x=\mp 2$, multiply by a we get $0=bx=2b=2$, but this mean $x=0$ which is a contradiction.
- if $y^2 \neq 0$, then $x+1$ and $x-1$ connect x with y, and this mean $ax=2a=0$, again we get a contradiction.

ii. If $x^2 \neq 0$, then we consider the following:

- if $y^2=0$ then $y+1, y-1, x+b$ and $x-b$ connect x with y. Moreover, this is not true if $R \ncong Z_2 \times Z_4$, which has diameter 3. Otherwise, we may take $z \in V(R)$ such that $z \notin \{x,y,a,-a,b\}$. Now if $d(z,x)=d(y,z)=1$ we done, if $d(x,z)=1, d(y,z)=0$ take $z+b$ in this case, and take $z+a$ if $d(x,z)=0, d(y,z)=1$. Finally take $z+1$ to connect x and y in other cases.
- if $y^2 \neq 0$, then $x+y \in V(R)$ connect x and y.

V. If $a+b=1$ and $a-b=-1$, with this conditions we obtain $ax=x, 2b=b, 2a=0, by=y$ and $2x=0$. Then we may apply similar argument as in case IV to show that $Diam(\Phi(R)) \leqslant 2$. □

The next theorem shows the connectivity of the reduced rings:

Theorem 2.2. *Let R be a reduced ring such that $|V(R)|>3$. Then $\Phi(R)$ is connected.*

Proof. Let $x,y \in V(R)$, then if $d(x,y)=1$, then we done. So we may let $d(x,y) \neq 1$, thus $xy=yx=0$. In this, we need to consider the following subcases:

I. If R without one then $x+y$ connected x with y (if $x+y=0$ then $x^2=0$ which is a contradiction as R is reduced)

II. If R with one, then if $x+y=1$. Since $|V(R)|>3$, then there is $z \neq w \in V(R) \setminus \{x,y\}$. Again we have to consider the following subcases:

1. if $zx=0=zy$ then $z-1$ connect x with y, and this not true if $z=2$, in this case $z+1$ will connect x with y also this is not true if $z=-2$. If both cases not true, then we have 4=0, and this means $z^2=0$, which a contradiction.
2. if $zx \neq 0 \neq zy$, then z connected x with y.
3. if $zx \neq 0$, $zy=0$, then $z+y$ connect x with y only if $z+y \neq -1$. If $z+y=-1$, then we get $2y=0, x^2=x, z^2=-z$ and $x=-z$. Now if w satisfy 1 or 2 then we done, else we have the following subcases:
   i. if $wy \neq 0$, $wx=0$, then $w+x$ connect x with y, and it is not true if $w+x=-1$, and this lead to $2x=0, y^2=y, w^2=-w$ and $y=-w$, this means $x=z$, a contradiction.
   ii. if $wy=0$, $wx \neq 0$, then we get $w+y$ connect x with y, and it is not true if $w+y=-1$, which yields that $w+y=z+y$, thus $w=z$, a contradiction.
4. if $zx=0$, $zy \neq 0$, then similar as case II, 3, one can find a path between x and y.

III. If R with one such that $x+y=-1$. Then the proof is the same as in case II.

□

The above theorem not always correct. Especially when $|V(R)|=3$. A very obvious example can be seen in Example 1.5.

Corollary 2.3. The non-zero divisor graph of a Boolean rings is connected.

3. The non-zero divisor graph of a matrix ring

Let R be a ring. The next theorem illustrate the condition on R in order to get the connectivity of matrix ring $M_n(R)$, such that $n \geq 2$.

Theorem 3.1. *Let $A,B \in V(M)$ such that $d(A,B) \neq 1$. We aim to find $W \in V(M)$ connect A with B. First since $A,B \in V(M)$, thus there are i,j,d and k such that $a_{ij} \neq 0$ and $b_{dk} \neq 0$.To do that we consider the following subcases:*

i. if $a_{ij}=b_{dk}$, then $W \in V(M)$, such that $w_{ji}=1$ and zero otherwise, connect A with B.

ii. *if $n \neq 2$, and no there exist $0 \neq a_{ij} = b_{ij} \neq 0$. Since $A \neq 0$ and $B \neq 0$. Then $W \in V(M)$, connected A with B such that $w_{ji} = 1$, $w_{kd} = 1$ and zero otherwise.*

iii. *if $n = 2$, and no there exist $0 \neq aij = bij \neq 0$, similar as case ii, but this not work if $ij = 11$, and $dk = 22$ or conversely, then $W \in V(M)$, connected A with B such that $w_{11} = 1$, $w_{12} = 1$ and zero otherwise.*

The next theorem shows under what circumstances $\Phi(R)$ become connected when R ring without one.

Theorem 3.2. *Let R be a ring without one, such that $deg(x) > 0$ for all $x \in V(R)$. Then the graph of $M_n(R)$ is connected.*

Proof. Let $A, B \in V(M)$. If $d(A,B) = 1wedone.$ so we may let $d(A,B) \neq 1$. Then since A, B are not equal to zero then there are i, j, k and $d \in n$ such that $a_{ij} \neq 0$ and $b_{dk} \neq 0$. Because of $deg(x) > 0$ for all $x \in V(R)$, then there are $w_1, w_2 V(R)$, such that either $a_{ij}w_1 \neq 0$ and $b_{kh}w_2 \neq 0$ or the other way around. We aim to find $W \in V(R)$ connect A with B. To do that we consider the following cases:

i. if $a_{ij}w_1 \neq 0$ and $b_{ij}w_2 \neq 0$ then let $W = \{w_{js} = w_1, w_{ht} = w_2\}$, such that $t \neq s$ and zero otherwise.

ii. if $w_1 a_{ij} \neq 0$ and $w_2 b_{ij} \neq 0$ then $W = \{w_{is} = w_1, w_{kt} = w_2\}$, such that $t \neq s$ and zero otherwise.

iii. if $w_1 a_{ij} \neq 0$ and $b_{ij} w_2 \neq 0$ then $W = \{w_{is} = w_1, w_{ht} = w_2\}$, such that $t \neq s$ and zero otherwise.

iv. if $a_{ij} w_1 \neq 0$ and $w_2 b_{ij} \neq 0$ then $W = \{w_{js} = w_1, w_{kt} = w_2\}$, such that $t \neq s$ and zero otherwise.

In all cases we can find a path between A and B.

□

It can be shown that Theorem 3.2 not always true. To illustrate that we give the following Exampe 4.1: Let $R = \{0, 2, 4, 6\} \subseteq Z8$, then the non-zero divisor graph of $M = M_n(R)$, such that $n \geqslant 2$, is disconnected. Indeed, if we take $A \in V(M)$ such that $a_{11} = 4$ and $a_{ij} = 0$ for all $i = 2, 3 \ldots, n$ and $j = 2, 3 \ldots, n$. Then $AX = XA = 0$ for all $X \in V(M)$.

Corollary 3.3. *Let R be a ring such that $\Phi(R)$ is connected then $\Phi(M)$ is connected.*

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Accepted: 31.08.2019