Qualitative behavior of a SIRS epidemic model with vaccination on heterogeneous networks

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Abstract. This paper studies the dynamics of a SIRS epidemic model with varying population size and vaccination in a complex network. Using an analytical method, we mainly investigate the stability of the model according to the threshold $\mathcal{R}_0$. That is, if $\mathcal{R}_0$ is less than one, then the disease will die out. Alternatively, the system admits a unique endemic equilibrium which is globally asymptotically stable if $\mathcal{R}_0 > 1$. Moreover, we investigate the case when $\mathcal{R}_0 = 1$. Finally, some numerical simulations are provided to illustrate the effectiveness of the theoretical results.
Keywords: SIRS model, vaccination, stability, complex network.

1. Introduction

In order to prevent and understand the spreading of diseases, mathematical epidemic models have been developed. Based on the pioneering work by Kermark and Mckendrick [9], Many researches have studied the spread of infectious diseases in a population by compartmental models such as SIS, SIR, SIRS, SEIR or SVIS, see for instance [5, 10, 19, 11, 12]. Arino et al [2], incorporated vacci-

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nation of both newborns and susceptible individuals into an SIRS model. They shown that a backward bifurcation leading to bistability can occur depending to the efficacy of the vaccine. In [4], Onofrio studied the use of a pulse vaccination strategy to eradicate infectious diseases. However, the early models were presented on homogeneous networks which implies that all individuals are equally likely to contact each other. Obviously, this assumption is unrealistic in some sense because physical contacts between individuals vary with each individual. To deal with the effect of contact heterogeneity, another approach came to analyze the spreading of diseases using the complex network theory. In a complex network, each node represents an individual in its corresponding epidemiological state, and each edge between two nodes stands for an interaction that may allow disease transmission. Several forms of computer-generated networks have been studied in the context of disease transmission. Each of these idealized networks can be defined in terms of how individuals are distributed in space and how connections are formed. One of the most studied network is scale-free network, see for instance [3, 7, 15, 20]. Scale-free network provides a means of achieving extreme levels of heterogeneity. In such networks nodes degree followed the power-law distribution. Namely, $P(k) \sim k^{-\gamma}$, the parameter γ must be larger than zero to ensure a finite average connectivity $< k >$. One special case of scale-free networks is the Barabasi Albert (BA) model [3]. In this model $P(k) \sim k^{-3}$. It incorporates two important general concepts: growth and preferential attachment. Growth means that the number of nodes in the network increases over time. Preferential attachment means that the more connected a node is, the more likely it is to receive new links. Scale-free networks can be constructed dynamically by adding new individuals to a network one by one with a connection mechanism that imitates the natural formation of social contacts. In the preferential attachment model of Barabasi Albert [3], the existence of individuals of arbitrarily large degree means that there is no level of random vaccination that is sufficient to prevent an epidemic [1, 17, 20]. On the other hand, when there is some upper limit imposed on the degree of individuals [21], or when a scale-free network is generated by nearest neighbor attachment within a lattice [22], it becomes possible to control infection through random vaccination [8]. In addition, Li et al [13] proposed a SIRS network-based model in constant population size and studied the global dynamics through theoretical analysis and numerical simulation.

In this paper, based on the previous works, we will study a SIRS epidemic model on the scale-free networks with vaccination in a non-constant population including births and deaths, where a fraction q of the newly born individuals are vaccinated at birth . Due to the complexity of network structure, the nodes in network are divided into n classes with respect to their degrees, where n denotes the maximum degree of the network. That is to say that the nodes i and j belong to the k-th class if they both have degree k , where $k \in \{1, 2, ..., n\}$. So, the dynamical behaviour of our model can be described as

$$
(1) \qquad \begin{cases} \frac{dS_k}{dt} = & (1-q)\Lambda - (\mu_1+\nu)S_k - \beta k \Theta S_k + \gamma R_k, \\ \frac{dI_k}{dt} = & -(\mu_2+\lambda)I_k + \beta k \Theta S_k, \\ \frac{dR_k}{dt} = & q\Lambda - (\mu_3+\gamma)R_k + \nu S_k + \lambda I_k, \end{cases}
$$

where the initial states satisfy

$$
(2) \quad S_k(0), I_k(0), R_k(0) > 0 \quad \text{and} \quad S_k(0)+I_k(0)+R_k(0) \leq \frac{\Lambda}{\mu},\ k=1,2,...,n.
$$

Denote the meaningful domain for system (1) by

$$
\Delta = \{(S_1, I_1, R_1, ..., S_n, I_n, R_n) \in \mathrm{R}_+^{3n},\ S_k + I_k + R_k \leq \frac{\Lambda}{\mu},\ k = 1,2,...,n\}.
$$

The meaning of the variables and parameters in system (1) is as follows. $S_k(t)$, $I_k(t)$ and $R_k(t)$ represent the relative densities of the susceptible, infected and recovered nodes with degree k, Λ is the birth rate (and $q \in [0,1]$ is a percentage of new born vaccinated children). μ_1, μ_2, and μ_3 represents the death rates of susceptible, infected and recovered individuals, respectively. β is the infection coefficient, λ is the rate at which the infective individuals become recovered, ν is the proportional coefficient of vaccinated for susceptible, and γ is the average loss of immunity rate. Also, it is assumed that the connectivity of nodes on the network is uncorrelated, thus, the probability that an edge points to an infected node with degree k is proportional to $kP(k)I_k(t)$ such that $\Theta(t) = \sum_{k=1}^{n} \frac{kP(k)I_k(t)}{<k>}$, where $P(k)$ is the connectivity distribution and $<k> = \sum_{k=1}^{n} kP(k)$ is the average degree of the network.

The rest of this paper is organized as follows. In Section 2, we discuss the positivity and boundedness of the solutions. Then, we establish the basic reproduction number and the existence of equilibrium points. Section 3 is devoted to explore the convergence of solution of system (1) to the disease-free equilibrium and the global stability of the endemic equilibrium. Finally, conclusions and simulations are drawn in Section 4.

2. Positivity of solutions and the epidemic threshold

In this section, we will provide some basic properties of system (1). First we establish that the domain Δ is positively invariant with respect to system (1).

Lemma 2.1. *Let* $(S_1, I_1, R_1, ..., S_n, I_n, R_n)$ *be the solution of system* (1) *with initial conditions* (2) *and* $\Theta(0) > 0$. *Then, the set* Δ *is positively invariant for model* (1) *and* $\Theta(t) > 0$ *for all* $t > 0$.

Proof. First, we will show $\Theta(t) > 0$. In fact, from the second equation of system (1) we have

$$
(3) \qquad \frac{d\Theta(t)}{dt} = \left(-(\mu_2+\lambda) + \beta \sum_{k=1}^{n} \frac{k^2 P(k) S_k(t)}{<k>} \right) \Theta(t).
$$

Then

$$\Theta(t) = \Theta(0)\exp\Bigg(-(\mu_2+\lambda)t + \frac{\beta}{<k>}\int_0^t \sum_{k=1}^{n} k^2 P(k) S_k(s)ds\Bigg) > 0.$$

On the other hand, we have $S_k(0) > 0$ for $k = 1, ..., n$. So, by continuity there exists δ_1 such that $S_k(t) > 0$ for $t \in (0, \delta_1)$ and $k = 1, ..., n$.
Let $\delta_k = \sup\{\tau > 0 : S_k(t) > 0, \forall t \in (0, \tau)\}$. Now, we will show $S_k(t) > 0$ for all $t > 0$ and $k = 1, ..., n$. To this end, we have to proof that $\delta_k = \infty$ for $k = 1, ..., n$. Suppose not, so there exists $m \in \{1, ..., n\}$ such that $\delta_m < \infty$. Then, $S_m(\delta_m) = 0$ and $S_m(t) > 0$ for all $t \in (0, \delta_m)$. From the second equation of (1), we get $I'_m(t) + (\lambda + \mu_2)I_m(t) > 0$ for $t \in (0, \delta_m)$. Then, $I_m(t) > I_m(0)e^{-(\lambda+\mu_2)t} \geq 0$ for $t \in (0, \delta_m)$. Since $I_m(t) > 0$ and $S_m(t) > 0$ for all $t \in (0, \delta_m)$. It follows $R'_m(t) + (\mu_3 + \gamma)R_m(t) > 0$ for $t \in (0, \delta_m)$, using the similar arguments to those given for $I_m(t)$, we get $R_m(t) > 0$ for $t \in (0, \delta_m)$. By continuity of $R_m(t)$ we have $R_m(\delta_m) \geq 0$. Thus, $S'_m(\delta_m) = (1 - q)\Lambda + \gamma R_m(\delta_m) > 0$. So, there exists some $t \in (0, \delta_m)$ such that $S_m(t) < 0$. This is apparently a contradiction. Consequently $\delta_k = \infty$ for $k = 1, ..., n$, which means $S_k(t) > 0$ for all $t > 0$ and $k = 1, ..., n$. Finally, by the second and the third equation of (1), we conclude that $I_k(t) > 0$ and $R_k(t) > 0$ for all $t > 0$ and $k = 1, ..., n$.

Now, let denote $N_k(t) = S_k(t) + I_k(t) + R_k(t)$ for all $t \geq 0$ and $k = 1, ..., n$. By summing the three equations of (1), we get

$$\begin{aligned} \frac{dN_k(t)}{dt} &= \Lambda - \mu_1 S_k - \mu_2 I_k - \mu_3 R_k \\ &\leq \Lambda - \mu(S_k + I_k + R_k) \\ &\leq \Lambda - \mu N_k(t), \end{aligned} \tag{4}$$

where $\mu = \min(\mu_1, \mu_2, \mu_3)$. Using the comparison principle of ODEs we deduce

$$N_k(t) \leq \frac{\Lambda}{\mu} + (N_k(0) - \frac{\Lambda}{\mu})\exp(-\mu t).$$

Hence, $S_k(t), I_k(t), R_k(t) \leq \frac{\Lambda}{\mu}$ for all $t > 0$ and $k = 1, ..., n$, which implies that Δ is positively invariant. □

Obviously, system (1) admits the disease-free equilibrium $E^0 = (S^0, 0, R^0, ..., S^0, 0, R^0) \in \mathbb{R}^{3n}$, where

$$S^0 = \frac{((1-q)\mu_3 + \gamma)\Lambda}{\mu_1(\mu_3+\gamma) + \nu\mu_3} \quad and \quad R^0 = \frac{(q\mu_1 + \nu)\Lambda}{\mu_1(\mu_3+\gamma) + \nu\mu_3}.$$

Now, we will investigate the existence of a positive equilibrium state in terms of the number

$$\mathcal{R}_0 = \frac{<k^2>}{<k>}\frac{\beta S^0}{\mu_2 + \lambda}.$$

Lemma 2.2. *The system* (1) *admits a unique endemic equilibrium* $E^* = (S_1^*, I_1^*, R_1^*, ..., S_n^*, I_n^*, R_n^*)$ *if and only if* $\mathcal{R}_0 > 1$.

Proof. By letting the right side of system (1) equal to zero, we get the following equations

$$S_k^* = \frac{(\mu_2+\lambda)I_k^*}{\beta k \Theta^*} \tag{5}$$

$$R_k^* = \frac{q}{\mu_3+\gamma}\Lambda + \left(\frac{\nu(\mu_2+\lambda)}{(\mu_3+\gamma)\beta k\Theta^*} + \frac{\lambda}{\mu_3+\gamma}\right) I_k^* \tag{6}$$

$$I_k^* = \frac{((1-q)\mu_3+\gamma)\beta k \Theta^* \Lambda}{(\mu_2(\mu_3+\gamma)+\mu_3\lambda)\beta k\Theta^* + (\mu_2+\lambda)(\mu_1(\mu_3+\gamma)+\mu_3\nu)}, \tag{7}$$

which determine the endemic equilibrium E^* of system (1). We know that $\Theta^* = \frac{\sum kP(k)I_k^*}{<k>}$. So, from (7) we get $f(\Theta^*) = 1$, where

$$f(x) = \frac{1}{<k>}\sum_k^n \frac{((1-q)\mu_3+\gamma)\beta k^2 P(k)\Lambda}{(\mu_2(\mu_3+\gamma)+\mu_3\lambda)\beta k x + (\mu_2+\lambda)(\mu_1(\mu_3+\gamma)+\mu_3\nu)}. \tag{8}$$

Since, $f(0) = \mathcal{R}_0$ and f is a decreasing function, the equation $f(x) = 1$ has unique root if and only if $\mathcal{R}_0 > 1$. □

3. Disease-free equilibrium dynamics

In this section, we will first prove that the solution of system (1) converges in the mean to the disease-free equilibrium E^0 when $\mathcal{R}_0 < 1$, and next we show that under the same condition, the solution $(S_k(t), I_k(t), R_k(t))$ of system (1) converges to $(S^0, 0, R^0)$ for all $k \in \{1, ..., n\}$. Finally, we explore the crucial case when $\mathcal{R}_0 = 1$.

3.1 Convergence in the mean

Theorem 3.1. *If* $\mathcal{R}_0 < 1$, *then for all* $k \in \{1, ..., n\}$ *we have*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t S_k(s)ds = S^0, \lim_{t\to\infty}\frac{1}{t}\int_0^t I_k(s)ds = 0, \lim_{t\to\infty}\frac{1}{t}\int_0^t R_k(s)ds = R^0. \tag{9}$$

Proof. From the second equation of (1) we get

$$\dot{\Theta} = -(\mu_2+\lambda)\Theta + \beta\Theta\frac{\sum k^2 P_k S_k}{<k>}.$$

Hence

$$\begin{aligned} d\log\Theta &= -(\mu_2+\lambda) + \beta\frac{\sum k^2 P_k S_k}{<k>} \\ &= -(\mu_2+\lambda) + \beta\frac{\sum k^2 P_k S^0}{<k>} + \beta\frac{\sum k^2 P_k (S_k - S^0)}{<k>} \\ &= -(\mu_2+\lambda)(1-\mathcal{R}_0) + \beta\frac{\sum k^2 P_k (S_k - S^0)}{<k>}. \end{aligned}$$

Using the following identities

$$\begin{aligned}(1-q)\Lambda &= (\mu_1+\nu)S^0+\gamma R^0,\\ \Lambda &= (\mu_3+\gamma)R^0+\nu S^0,\end{aligned}$$

we have

$$\dot{S}_k = -(\mu_1+\nu)(S_k-S^0)-\beta k\Theta S_k+\gamma(R_k-R^0), \tag{10}$$

$$\dot{R}_k = -(\mu_3+\gamma)(R_k-R^0)+\nu(S_k-S^0)+\lambda I_k. \tag{11}$$

So

$$\begin{aligned}\frac{\mu_3+\gamma}{\gamma}\dot{S}_k+\dot{R}_k &= \frac{-(\mu_3+\gamma)}{\gamma}(\mu_1+\nu)(S_k-S^0)-\beta\frac{\mu_3+\gamma}{\gamma}k\Theta S_k\\ &\quad +\nu(S_k-S^0)+\lambda I_k.\end{aligned}$$

Then

$$\frac{\mu_3(\mu_1+\nu)+\gamma\mu_1}{\gamma}(S_k-S^0) = -\frac{\mu_3+\gamma}{\gamma}\dot{S}_k-\dot{R}_k-\beta\frac{\mu_3+\gamma}{\gamma}k\Theta S_k+\lambda I_k. \tag{12}$$

By integrating the above equality both sides from 0 to t we get

$$\begin{aligned}\int_0^t(S_k(s)-S^0)ds &= \frac{\gamma}{\mu_3(\mu_1+\nu)+\gamma\mu_1}\Bigg(-\frac{\mu_3+\gamma}{\gamma}S_k(t)+\frac{\mu_3+\gamma}{\gamma}S_k(0)\\ &\quad -R_k(t)+R_k(0)-\beta\frac{\mu_3+\gamma}{\gamma}k\int_0^t\Theta(s)S_k(s)ds\\ &\quad +\lambda\int_0^t I_k(s)ds)\Bigg),\end{aligned}$$

which together with (2) implies

$$\begin{aligned}\int_0^t(S_k(s)-S^0)ds &\leq \frac{\gamma}{\mu_3(\mu_1+\nu)+\gamma\mu_1}\left(\frac{\Lambda}{\mu}\left(1+\frac{\mu_3+\gamma}{\gamma}\right)+\lambda\int_0^t I_k(s)ds\right)\\ &\triangleq a_0+a_1\int_0^t I_k(s)ds, \qquad (13)\end{aligned}$$

where

$$\begin{cases} a_0=\dfrac{\gamma}{\mu_3(\mu_1+\nu)+\gamma\mu_1}\left(\dfrac{\Lambda}{\mu}(1+\dfrac{\mu_3+\gamma}{\gamma})\right)\\ a_1=\dfrac{\lambda\gamma}{\mu_3(\mu_1+\nu)+\gamma\mu_1}.\end{cases}$$

From (10) and (13) we have

$$\log\Theta(t)-\log\Theta(0) \leq -(\mu_2+\lambda)(1-\mathcal{R}_0)t+\beta n a_0+\beta a_1 n\int_0^t\Theta(s)ds.$$

Hence

$$\Theta(t)\exp\left(-\beta a_1 n\int_0^t \Theta(s)ds\right) \;\leq\; \Theta(0)\exp(\beta n a_0)\exp\Big(-(\mu_2+\lambda)(1-\mathcal{R}_0)t\Big).$$

Therefore

$$\frac{d\frac{-1}{\beta a_1 n}\exp\left(-\beta a_1 n\int_0^t \Theta(s)ds\right)}{dt} \;\leq\; \Theta(0)\exp(\beta n a_0)\exp\Big(-(\mu_2+\lambda)(1-\mathcal{R}_0)t\Big).$$

By integrating the above inequality both sides from 0 to t we obtain

$$\exp\left(-\beta a_1 n\int_0^t \Theta(s)ds\right) \geq 1 + \frac{\Theta(0)\beta a_1 n\exp(\beta n a_0)}{(\mu_2+\lambda)(1-\mathcal{R}_0)}\left(\exp\Big(-(\mu_2+\lambda)(1-\mathcal{R}_0)t\Big)-1\right).$$

Which implies

$$-\beta a_1 n\int_0^t \Theta(s)ds \geq \log\left(1+\frac{\Theta(0)\beta a_1 n\exp(\beta n a_0)}{(\mu_2+\lambda)(1-\mathcal{R}_0)}\Big(\exp(-(\mu_2+\lambda)(1-\mathcal{R}_0)t)-1\Big)\right).$$

Then,

$$\frac{1}{t}\int_0^t \Theta(s)ds \leq \frac{-1}{\beta a_1 n t}\log\left(1+\frac{\Theta(0)\beta a_1 n\exp(\beta n a_0)}{(\mu_2+\lambda)(1-\mathcal{R}_0)}\Big(\exp(-(\mu_2+\lambda)(1-\mathcal{R}_0)t\Big)\right).$$

Consequently, we have

$$(14)\quad \lim_{t\to\infty}\frac{1}{t}\int_0^t \Theta(s)ds = 0 \quad\text{and then}\quad \lim_{t\to\infty}\frac{1}{t}\int_0^t I_k(s)ds = 0, \quad k\in\{1,\dots,n\}.$$

By (3.1) we get

$$\frac{1}{t}\int_0^t (S_k(s)-S^0)ds = \frac{\gamma}{\mu_3(\mu_1+\nu)+\gamma\mu_1}\left(\frac{-(\mu_3+\gamma)}{\gamma}\frac{(S_k(t)-S_k(0))}{t} - \frac{R_k(t)-R_k(0)}{t} - \beta\frac{(\mu_3+\gamma)}{\gamma}k\frac{1}{t}\int_0^t \Theta(s)S_k(s)ds + \lambda\frac{1}{t}\int_0^t I_k(s)ds\right).$$

Combining $(S_k(t), I_k(t), R_k(t)) \in \Delta$ with (2) and (14) yields to $\lim_{t\to\infty} \Upsilon(t) = 0$, where

$$\Upsilon(t) = \frac{S_k(t) - S_k(0)}{t} + \frac{R_k(t) - R_k(0)}{t} + \frac{1}{t}\int_0^t \Theta(s) S_k(s) ds.$$

Which, together with (14), implies

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t (S_k(s) - S^0) ds = 0, \quad k \in \{1, ..., n\}.$$

Finally, from (11) one can easily deduce

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t (R_k(s) - R^0) ds = 0, \quad k \in \{1, ..., n\}.$$

□

3.2 Pointwise convergence

Theorem 3.2. *If $\mathcal{R}_0 < 1$, then for all $k \in \{1, ..., n\}$ we have*

$$\lim_{t\to\infty} S_k(t) = S^0, \quad \lim_{t\to\infty} I_k(t) = 0, \quad \textit{and} \quad \lim_{t\to\infty} R_k(t) = R^0.$$

Proof. Combining (9) and (10) leads to

$$\lim_{t\to\infty} \sup \frac{1}{t} \log \Theta(t) \leq -(\mu_2 + \lambda)(1 - \mathcal{R}_0).$$

Hence, $\lim_{t\to\infty} \Theta(t) = 0$ which leads to $\lim_{t\to\infty} I_k(t) = 0$. Using (10), we obtain

$$\begin{aligned}
&(S_k(t) - S^0)' + (\mu_1 + \nu)(S_k(t) - S^0) = -\beta k \Theta(t) S_k(t) + \gamma(R_k(t) - R^0) \\
&\Big((S_k(t) - S^0)e^{(\mu_1+\nu)t}\Big)' = \left[-\beta k \Theta(t) S_k(t) + \gamma(R_k(t) - R^0)\right] e^{(\mu_1+\nu)t}.
\end{aligned}$$

Integrating the above equality both sides from 0 to t yields to

$$\begin{aligned}
S_k(t) - S^0 &= (S_k(0) - S^0)e^{-(\mu_1+\nu)t} \\
&\quad + \left(\int_0^t \Big(- \beta k \Theta(s) S_k(s) + \gamma(R_k(s) - R^0)e^{(\mu_1+\nu)s}\Big) ds\right) e^{-(\mu_1+\nu)t}.
\end{aligned}$$

So

$$\lim_{t\to\infty} \sup(S_k(t) - S^0) \leq \Big(\frac{\gamma}{\mu_1 + \nu}\Big) \lim_{t\to\infty} \sup(R_k(t) - R^0).$$

Similarly, we also get

$$\lim_{t\to\infty} \sup(R_k(t) - R^0) \leq \Big(\frac{\nu}{\mu_3 + \gamma}\Big) \lim_{t\to\infty} \sup(S_k(t) - S^0).$$

Consequently

$$\lim_{t\to\infty} R_k(t) = R^0 \quad \text{and} \quad \lim_{t\to\infty} S_k(t) = S^0, \quad k \in \{1, ..., n\}$$

□

Theorem 3.3. *If $\nu = 0$ and $\mathcal{R}_0 = 1$, then the solution $(S_k(t), I_k(t), R_k(t))$ of system* (1) *converges to $(S^0, 0, R^0)$ for all $k \in \{1, ..., n\}$.*

Proof. Let $\varepsilon > 0$ such that $\varepsilon < \Theta(0)$. We define

$$\begin{array}{ll} \tau_1 = \inf\{t > 0, \quad \Theta(t) \leq \varepsilon\}, & \tau^{'} = \inf\{t \geq \tau_1, \quad \Theta(t) \geq \varepsilon\}, \\ \tau_m^{'} = \inf\{t \geq \tau_m, \quad \Theta(t) \geq \varepsilon^m\}, & \tau_{m+1} = \inf\{\tau_m \leq t \leq \tau_m^{'}, \quad \Theta(t) \leq \varepsilon^{m+1}\}. \end{array}$$

Let's show that $\tau_m < \infty$, $\forall m \geq 1$. We will proceed by contradiction. Suppose that $\tau_1 = \infty$, so

$$\Theta(t) \geq \varepsilon, \forall t > 0. \tag{15}$$

Which implies that

$$\Gamma_n(I(t)) \triangleq \sum \frac{k^2 P_k}{< k >} I_k(t) > \Theta(t) \geq \varepsilon, \qquad \forall t > 0. \tag{16}$$

From the differential system (1), we have

$$\frac{d \log \Theta(t)}{dt} = -(\mu_2 + \lambda)(1 - \mathcal{R}_0) + \beta \sum \frac{k^2 P_k}{< k >}(S_k(t) - S^0) \triangleq \beta\, \Gamma_n(S(t) - S^0). \tag{17}$$

The 3^{rd} equation of (1) implies that

$$R_k(t) - R^0 = (R_k(0) - R^0)e^{-(\mu_3+\gamma)t} + \lambda \int_0^t I_k(s)e^{-(\mu_3+\gamma)(t-s)} ds. \tag{18}$$

So

$$\Gamma_n(R(t) - R^0) = \Gamma_n(R(0) - R^0)e^{-(\mu_3+\gamma)t} + \lambda \int_0^t \Gamma_n(I(s))e^{-(\mu_3+\gamma)(t-s)} ds.$$

It follows from (16) that

$$\Gamma_n(R(t) - R^0) \geq \Gamma_n(R(0) - R^0)e^{-(\mu_3+\gamma)t} + \frac{\lambda\varepsilon}{\mu_3 + \gamma}\left(1 - e^{-(\mu_3+\gamma)t}\right). \tag{19}$$

On the other hand

$$d(N_k - N^0) \;\; = \;\; -\mu_1(N_k - N^0) - (\mu_2 - \mu_1)I_k - (\mu_3 - \mu_1)(R_k - R^0).$$

Which leads to

$$\begin{aligned} N_k(t) - N^0 \;\; &= \;\; (N_k(0) - N^0)e^{-\mu_1 t} - (\mu_2 - \mu_1) \int_0^t I_k(s)e^{-\mu_1(t-s)} ds \\ &\quad -(\mu_3 - \mu_1) \int_0^t (R_k(s) - R^0)e^{-\mu_1(t-s)} ds. \end{aligned} \tag{20}$$

Then we get

$$\begin{aligned}\Gamma_n(N(t)-N^0) &= \Gamma_n(N(0)-N^0)e^{-\mu_1 t}-(\mu_2-\mu_1)\int_0^t \Gamma_n(I(s))e^{-\mu_1(t-s)}ds\\ &\quad -(\mu_3-\mu_1)\int_0^t \Gamma_n(R(s)-R^0)e^{-\mu_1(t-s)}ds.\end{aligned}$$

According to (16) and (19), we have

$$\begin{aligned}\Gamma_n(N(t)-N^0) &\leq \Gamma_n(N(0)-N^0)e^{-\mu_1 t}-\frac{\mu_2-\mu_1}{\mu_1}\varepsilon\Big(1-e^{-\mu_1 t}\Big)\\ &\quad -(\mu_3-\mu_1)\Gamma_n(R(0)-R^0)\frac{e^{-\mu_1 t}\Big(1-e^{-(\mu_3+\gamma-\mu_1)t}\Big)}{\mu_3+\gamma-\mu_1}\\ &\quad -\frac{\mu_3-\mu_1}{\mu_3+\gamma}\lambda\varepsilon\left[\frac{1}{\mu_1}\Big(1-e^{-\mu_1 t}\Big)-\frac{e^{-\mu_1 t}\Big(1-e^{-(\mu_3+\gamma-\mu_1)t}\Big)}{\mu_3+\gamma-\mu_1}\right]. \qquad (21)\end{aligned}$$

In views of (17), we get

$$d\log\Theta=\beta\Big(\Gamma_n(N(t)-N^0)-\Gamma_n(I(t))-\Gamma_n(R(t)-R^0)\Big). \qquad (22)$$

Substituting (19) and (21) into (22), we obtain

$$d\log\Theta(t)\leq -\mathcal{H}\varepsilon+\ F(\varepsilon,t). \qquad (23)$$

Where

$$\mathcal{H}\varepsilon=\beta\frac{(\mu_2-\mu_1)}{\mu_1}\varepsilon+\beta\frac{(\mu_3-\mu_1)\lambda\varepsilon}{(\mu_3+\gamma)\mu_1}+\beta\varepsilon,$$

and

$$\begin{aligned}F(\varepsilon,t) &= \beta\ \Gamma(N(0)-N^0)e^{-\mu_1 t}+\frac{\beta(\mu_2-\mu_1)\varepsilon e^{-\mu_1 t}}{\mu_1}\\ &\quad -\beta(\mu_3-\mu_1)\Gamma_n(R(0)-R^0)e^{-\mu_1 t}\frac{\Big(1-e^{-(\mu_3+\gamma-\mu_1)t}\Big)}{\mu_3+\gamma-\mu_1}\\ &\quad +\left[\frac{\beta(\mu_3-\mu_1)\lambda\varepsilon}{\mu_3+\gamma}\left(\frac{1}{\mu_1}+\frac{1-e^{-(\mu_3+\gamma\mu_1)t}}{\mu_3+\gamma-\mu_1}\right)\right]e^{-\mu_1 t}\\ &\quad -\beta e^{-(\mu_3+\gamma)t}\Big(\Gamma_n(R(0)-R^0)+\frac{\lambda\varepsilon}{\mu_3+\gamma}\Big). \qquad (24)\end{aligned}$$

Since, there exists t_0 such that $t>t_0$ and $F(\varepsilon,t)\leq\frac{\mathcal{H}\varepsilon}{2}$. Then, for $t\geq t_0$ we get

$$\int_{t_0}^t d\log\Theta(s)ds\leq-\frac{\mathcal{H}\varepsilon}{2}(t-t_0).$$

Which implies that $\Theta(t) \leq \Theta(0)\, e^{-\frac{\mathcal{H}\varepsilon}{2}(t-t_0)}$ and then $\lim_{t\to\infty} \Theta(t) = 0$. This contradicts the assumption that in (15). Let's suppose that $\tau_m < \infty$ and $\tau_{m+1} = \infty$. We have $\tau_m^{'} = \infty$, which gives $\Theta(t) \geq \varepsilon^{m+1}$ for all $t > \tau_m$. Then, by using similar arguments to those given in the case when $\tau_1 = \infty$, we get

$$\begin{aligned}\Gamma_n\big(N(t) - N^0\big) \quad \leq \quad & \Gamma_n\big(N(\tau_m) - N^0\big)e^{-\mu_1 t} - \tfrac{\mu_2-\mu_1}{\mu_1}\varepsilon^{m+1}\Big(1 - e^{-\mu_1(t-\tau_m)}\Big) \\ & \frac{(-(\mu_3-\mu_2)\Gamma_n(R(\tau_m)-R^0)e^{-\mu_1 t})\Big(1-e^{-(\mu_3+\gamma-\mu_1)(t-\tau_m)}\Big)}{\mu_3+\gamma-\mu_1} \\ & -\tfrac{\mu_3-\mu_1}{\mu_3+\gamma}\lambda\varepsilon^{m+1}\Bigg[\tfrac{1}{\mu_1}\Big(1 - e^{-\mu_1(t-\tau_m)}\Big) \\ & -e^{-\mu_1 t}\frac{\Big(1-e^{-(\mu_3+\gamma-\mu_1)(t-\tau_m)}\Big)}{\mu_3+\gamma-\mu_1}\Bigg].\end{aligned}$$

By (23), we have

$$d\log\Theta(t) \leq -\mathcal{H}\varepsilon^{m+1} + F(\varepsilon^{m+1}, t-\tau_m),$$

there exists $t_0^{'}$ such that $t > t_0^{'} \vee \tau_m$ and $F(\varepsilon^{m+1}, t-\tau_m) \leq \frac{\mathcal{H}\varepsilon^{m+1}}{2}$. Which yields to

$$\int_{t_0^{'}\vee\tau_m}^{t} d\log(\Theta(s))ds \leq -\frac{\mathcal{H}\varepsilon^{m+1}}{2} \times (t - (t_0^{'} \vee \tau_m)), \tag{25}$$

thus

$$\Theta(t) \leq \Theta(0)\; e^{-\frac{\mathcal{H}\varepsilon^{m+1}}{2}\times(t-(t_0^{'}\vee\tau_m))}. \tag{26}$$

Hence, $\Theta(t) \longrightarrow 0$ as $t \longrightarrow \infty$, which contradicts the assumption that in (15). Beside, $\tau_m < \infty$ for all m in $\mathbb{N}$. By construction, the sequence $(\tau_m)_{m\in\mathbb{N}}$ is increasing. Hence, τ_m converges to τ_∞ . Also, We have $\tau_\infty = \infty$ (otherwise $\Theta(\tau_m) = \varepsilon^m$ which leads to $\Theta(\tau_\infty) = 0$, contradiction with $\Theta(t) > 0$ for all $t > 0$).

Finally, let $\eta > 0$ and $m_0 = \left[\frac{\log\eta}{\log\varepsilon}\right] + 1$, where $[.]$ denotes the integer part. For all $t \geq \tau_{m_0}$, there exists $m \geq m_0$ such that $\tau_m \leq t \leq \tau_{m+1}$ and $\Theta(t) \leq \varepsilon^m \leq \varepsilon^{m_0} \leq \varepsilon^{\log\eta/\log\varepsilon} = \eta$. So, $\Theta(t)$ converges to 0 and automatically $I_k(t)$ converges to 0 for all $k \in \{1, .., n\}$. It follows that

$$\lim_{t\to\infty}\int_0^t I_k(s)e^{-(\mu_3+\gamma)(t-s)}ds = 0.$$

Which implies by (18) that $R_k(t)$ converges to R^0 for all $k \in \{1, .., n\}$. Similarly we obtain

$$\lim_{t\to\infty}\int_0^t (R_k(s) - R^0)e^{-\mu_1(t-s)}ds = 0.$$

Then, From (20), one can deduce that $N_k(t)$ converges to N^0 and it immediately yields $S_k(t)$ converges to S^0. Finally, we have shown that $(S_k(t), I_k(t), R_k(t))$ converge towards to $(S^0, 0, R^0)$ for all $k \in \{1, .., n\}$. □

4. Stability of the endemic equilibrium

In this section, We show the global asymptotical stability of the equilibrium E^* of the system (1), by means of a suitable Lyapunov function.

Theorem 4.1. *If $\mathcal{R}_0 > 1$ and the following assumptions hold*

$$\lambda > \nu\frac{\mu_2+\mu_3}{\mu_1+\mu_3} \quad and \quad \gamma < \frac{4(\mu_1+\nu)\big(\mu_1+\mu_3-\frac{\nu}{\lambda}(\mu_2+\mu_3)\big)\mu_2}{(\mu_1+\mu_2)^2}.$$

Then E^* is globally asymptotically stable.

Proof. We consider the following Lyapunov function $\mathcal{W}=\sum_{i=1}^{4}\mathcal{W}_i$, where

$$\begin{aligned}
\mathcal{W}_1 &= \frac{a_1}{2}\sum_k \frac{kP_k}{<k>S_k^*}(S_k-S_k^*)^2,\ \mathcal{W}_2 = a_1\left(\Theta-\Theta^*-\Theta^*\log\frac{\Theta}{\Theta^*}\right),\\
\mathcal{W}_3 &= \frac{a_3}{2}\sum_k \frac{kP_k}{<k>S_k^*}(R_k-R^*)^2, \mathcal{W}_4 = \frac{a_4}{2}\sum_k \frac{kP_k}{<k>S_k^*}(S_k-S_k^*+I_k-I_k^*\\
&\qquad +R_k-R_k^*)^2,
\end{aligned}$$

and a_1, a_3, a_4 are positive constants to be determined suitably. We now give the derivative of each of the previous functions.

$$\begin{aligned}
\mathcal{W}_1'+\mathcal{W}_2' &= a_1\textstyle\sum_k \frac{kP_k}{<k>S_k^*}(S_k-S_k^*)\Big[-\mu_1(S_k-S_k^*)-\beta k\Theta(S_k-S_k^*)\\
&\quad -\beta kS_k^*(\Theta-\Theta^*)+\gamma(R_k-R_k^*)\Big]\\
&\quad +a_1(\Theta-\Theta^*)\beta\textstyle\sum_k\frac{k^2P_k}{<k>}(S_k-S_k^*)\\
&= -a_1(\mu_1+\nu)\textstyle\sum_k\frac{kP_k}{<k>S_k^*}(S_k-S_k^*)^2-a_1\beta\Theta\sum_k\frac{kP_k}{<k>S_k^*}(S_k-S_k^*)^2\\
&\quad +a_1\gamma\textstyle\sum_k\frac{kP_k}{<k>S_k^*}(S_k-S^*)(R_k-R_k^*).\\
\mathcal{W}_3' &= a_3\textstyle\sum_k\frac{kP_k}{<k>S_k^*}(R_k-R_k^*)\Big[-(\mu_3+\gamma)(R_k-R_k^*)+\nu(S_k-S_k^*)\\
&\quad +\lambda(I_k-I_k^*)\Big]\\
&= -a_3(\mu_3+\gamma)\textstyle\sum_k\frac{kP_k}{<k>S_k^*}(R_k-R_k^*)^2\\
&\quad +a_3\nu\textstyle\sum_k\frac{kP_k}{<k>S_k^*}(R_k-R_k^*)(S_k-S_k^*)\\
&\quad +a_3\lambda\textstyle\sum_k\frac{kP_k}{<k>S_k^*}(R_k-R_k^*)(I_k-I_k^*).\\
\mathcal{W}_4' &= a_4\textstyle\sum_k\frac{kP_k}{<k>S_k^*}\Big(S_k-S_k^*+I_k-I_k^*+R_k-R_k^*\Big)\times\\
&\quad \Big(-\mu_1(S_k-S_k^*)-\mu_2(I_k-I_k^*)-\mu_3(R_k-R_k^*)\Big).
\end{aligned}$$

So

$$\begin{aligned}\mathcal{W}' &= \sum_k \frac{kP_k}{<k>S_k^*}\Bigg[-a_1(\mu_1+\nu)(S_k-S_k^*)^2 - a_4\mu_2(I_k-I_k^*)^2 \\ &\quad -a_4(\mu_1+\mu_2)(S_k-S_k^*)(I_k-I_k^*)\Bigg] - a_1\beta\Theta\sum_k \frac{kP_k}{<k>S_k^*}(S_k-S_k^*)^2 \\ &\quad -\Big((\mu_3+\gamma)+a_4\mu_3\Big)\sum_k \frac{kP_k}{<k>S_k^*}(R_k-R_k^*)^2 \\ &\quad +\Big[a_1\gamma+a_3\nu-a_4(\mu_1+\mu_3)\Big]\sum_k \frac{kP_k}{<k>S_k^*}(S_k-S_k^*)(R_k-R_k^*) \\ &\quad +\Big[a_3\lambda-a_4(\mu_2+\mu_3)\Big]\sum_k \frac{kP_k}{<k>S_k^*}(R_k-R_k^*)(I_k-I_k^*).\end{aligned}$$

Then

$$\begin{aligned}\mathcal{W}' &\leq -a_1(\mu_1+\nu)\sum \frac{kP_k}{<k>S_k^*}\Big(S_k-S_k^*+\frac{a_4(\mu_1+\mu_2)}{2a_1(\mu_1+\nu)}(I_k-I_k^*)\Big)^2 \\ &\quad -\Big[\frac{4a_1a_4(\mu_1+\nu)\mu_2-\big(a_4(\mu_1+\mu_2)\big)^2}{4a_1(\mu_1+\nu)}\Big]\sum_k \frac{kP_k}{<k>S_k^*}(I_k-I_k^*)^2 \\ &\quad -\Big((\mu_3+\gamma)+a_4\mu_3\Big)\sum_k \frac{kP_k}{<k>S_k^*}(R_k-R_k^*)^2 \\ &\quad +\Big[a_1\gamma+a_3\nu-a_4(\mu_1+\mu_3)\Big]\sum_k \frac{kP_k}{<k>S_k^*}(S_k-S_k^*)(R_k-R_k^*) \\ &\quad +\Big[a_3\lambda-a_4(\mu_2+\mu_3)\Big]\sum_k \frac{kP_k}{<k>S_k^*}(R_k-R_k^*)(I_k-I_k^*).\end{aligned} \tag{27}$$

Consequently, in order to get $\mathcal{W}' \leq 0$, the parameters a_1, a_3, and a_4 should satisfy

$$a_1\gamma+a_3\nu-a_4(\mu_1+\mu_3) = 0, \tag{28}$$

$$a_3\lambda-a_4(\mu_2+\mu_3) = 0, \tag{29}$$

$$\frac{4a_1a_4(\mu_1+\nu)\mu_2-\big(a_4(\mu_1+\mu_2)\big)^2}{4a_1(\mu_1+\nu)} < 0. \tag{30}$$

Therefore, from (28) and (29) we can choose $a_3 = a_4\frac{\mu_2+\mu_3}{\lambda}$ and $a_1 = \frac{a_4}{\gamma}\big(\mu_1+\mu_3-\frac{\nu}{\lambda}(\mu_2+\mu_3)\big)$. Thus (30) holds when

$$\gamma < \frac{4(\mu_1+\nu)\big(\mu_1+\mu_3-\frac{\nu}{\lambda}(\mu_2+\mu_3)\big)\mu_2}{(\mu_1+\mu_2)^2}. \tag{31}$$

So, it follows from (27) that $\mathcal{W}' \leq 0$. Also, we have $\mathcal{W}' = 0$ if and only if $S_k = S_k^*$, $I_k = I_k^*$ and $R_k = R_k^*$ for $k = 1, 2, ..., n$. According to the LaSalle invariant principle [6], the unique endemic equilibrium state E^* is globally asymptotically stable. This completes the proof. □

5. Simulation and discussion

In this section, several numerical examples are designed to illustrate the dynamics of system (1). Using a preferential attachment algorithm, a BA network can be generated following the methods in [3]. The schema of the scale-free network with different sizes is illustrated in Figure 1.

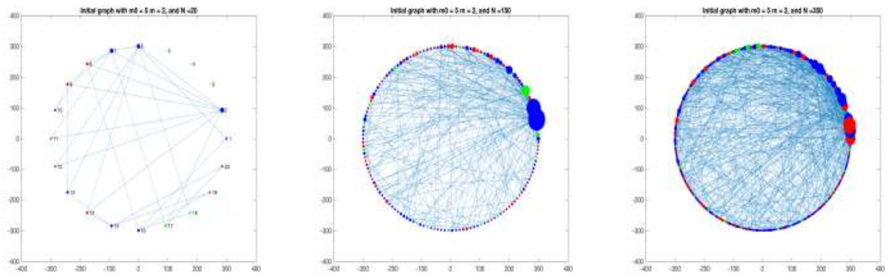

Figure 1: A Barabasi-Albert scale-free network of respectively 20, 150 and 350 nodes, it starts with $m_0 = 5$ fully connected nodes, and then each time a new node is added to the network with $m = 2$ links until the network size is reached.

Example 1. Consider a scale-free network with 20 nodes, and the parameters values $\Lambda = 0.03$, $\lambda = 0.2$, $\beta = 0.01$, $\mu_1 = 0.08$, $\mu_2 = 0.08$, $\mu_3 = 0.05$,, $\gamma = 0.6$ and $\nu = 0.015$. In this situation $\mathcal{R}_0 = 0.72 < 1$. Hence, according to Theorem (3.2) the solution of system (1) converges to E^0, (see Figure 2).

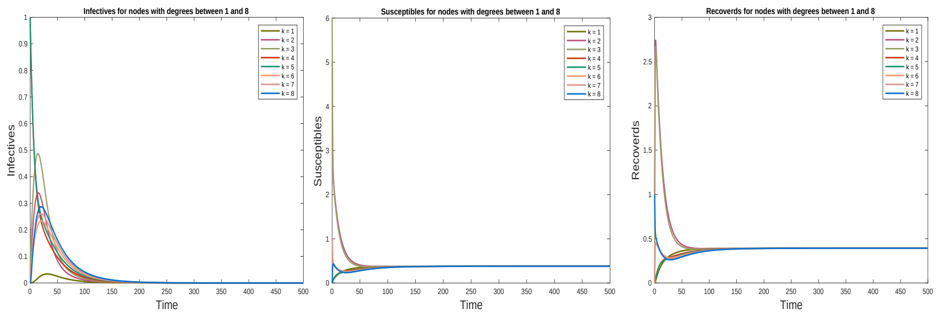

Figure 2: The time evolution of the densities of each state

Example 2. Figure (3) shows the evolution of infectives with several different values of ν, respectively $0.1, 0.2, 0.25, 0.3, 0.35$ and 0.4. We observe that the values of $I(t)$ eventually converge to corresponding equilibrium points at higher speeds as the parameter ν grows, which reveal the important role of vaccination in the stability of system (1).

Example 3. Figure (4) manifest the influence of network size in the time evolution of $I(t)$ of system (1). It is observed that the values of $I(t)$ eventually converge to corresponding equilibrium points at faster rates as the network size increases.

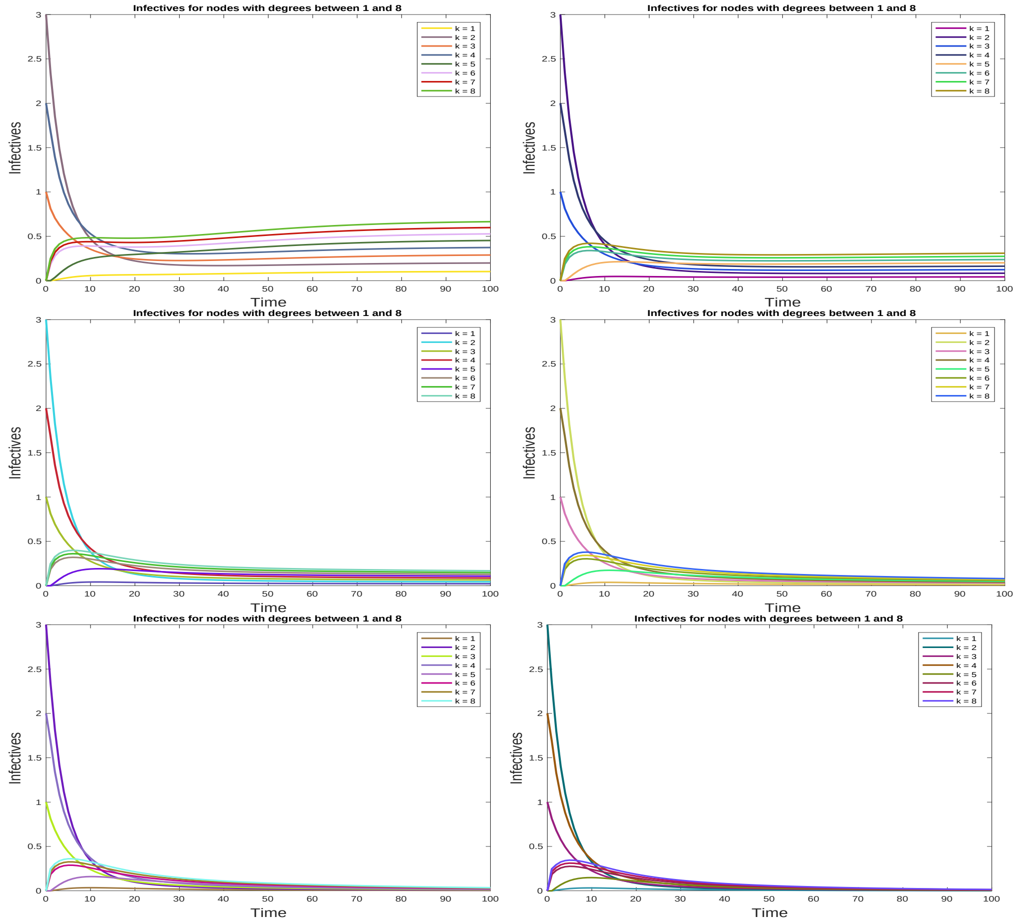

Figure 3: Time evolution of infectives with different values of ν.

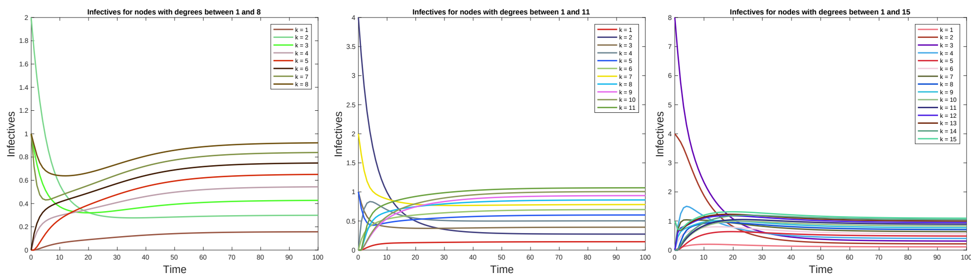

Figure 4: Time evolution of infectives with different network sizes (20, 30, and 50 nodes).

6. Conclusion

In this paper, we have studied an SIRS epidemic model with vaccination in complex heterogeneous networks and where contacts between human are treated as a scale-free social network. We obtain a specific expression of the threshold $\mathcal{R}_0$ through the existence of the endemic equilibrium. It is concluded that the solution of the system (1) converges to the disease free equilibrium E^0 if $\mathcal{R}_0 < 1$, which means from the biological point of view, the disease always dies out eventually. Otherwise the system admits a unique endemic equilibrium, which is globally asymptotically stable if $\mathcal{R}_0 > 1$. We have also treated the crucial case when $\mathcal{R}_0 = 1$ and we have shown that $(S_1, I_1, R_1, ..., S_n, I_n, R_n)$ converges consecutively to the disease free equilibrium $E^0 = (S^0, 0, R^0, ..., S^0, 0, R^0)$. To confirm the accuracy of the theoretical analysis, several numerical simulations are performed. Namely, We have found that the percentages of infectives will increase in the early time and then decrease until achieve a steady state as the parameter of vaccination ν increases. Also, we have shown the impact of network size in the convergence of infectives to the steady states.

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